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HANDBOOK OF DIFFERENTIAL EQUATIONS

Evolutionary Equations

VOLUME 2

Edited by

C.M. Dafermos

E. Feireisl

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VOLUME II

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EVOLUTIONARY EQUATIONS

Volume II

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Preface

This is the second volume in the series *Evolutionary Equations*, part of the *Handbook of Differential Equations* project. Whereas Volume I was intended to provide an overview of diverse abstract approaches, the guiding philosophy of the present volume is to offer a representative sample of the most challenging specific equations and systems arising in scientific applications.

Three chapters are devoted to the modern mathematical theory of fluid dynamics: Chapter 1 deals with the Euler equations, Chapter 5 provides a general introduction to the theory of incompressible viscous fluids, and Chapter 3 discusses the asymptotic limits of discrete mechanical systems described by the Boltzmann equation.

In a different direction, Chapter 2 introduces the blow-up phenomena of solutions of general parabolic equations and systems.

Chapters 4 and 6 are closely related and deal with mathematical problems arising in materials science.

Finally, Chapter 7 explores the topic of nonlinear wave equations.

We have deliberately chosen diverse topics as well as styles of presentation in order to expose the reader to the enormous variety of problems, methodology and potential applications.

We should like to express our thanks to the authors who have contributed to the present volume, to the referees who have generously spent time reading the papers, and to the editors and staff of Elsevier.

Constantine Dafermos
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CHAPTER 1

Euler Equations and Related Hyperbolic Conservation Laws

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Abstract

Some aspects of recent developments in the study of the Euler equations for compressible fluids and related hyperbolic conservation laws are analyzed and surveyed. Basic features and phenomena including convex entropy, symmetrization, hyperbolicity, genuine nonlinearity, singularities, *BV* bound, concentration and cavitation are exhibited. Global well-posedness for discontinuous solutions, including the *BV* theory and the L^∞ theory, for the one-dimensional Euler equations and related hyperbolic systems of conservation laws is described. Some analytical approaches including techniques, methods and ideas, developed recently, for solving multidimensional steady problems are presented. Some multidimensional unsteady problems are analyzed. Connections between entropy solutions of hyperbolic conservation laws and divergence-measure fields, as well as the theory of divergence-measure fields, are discussed. Some further trends and open problems on the Euler equations and related multidimensional conservation laws are also addressed.

Keywords: Adiabatic, Clausius–Duhem inequality, Compensated compactness, Compressible fluids, Conservation laws, Divergence-measure fields, Entropy solutions, Euler equations, Finite difference schemes, Free boundary approaches, Gauss–Green formula, Genuine nonlinearity, Geometric fluids, Glimm scheme, Hyperbolicity, Ill-posedness, Isentropic, Isothermal, Lax entropy inequality, Potential flow, Self-similar, Singularity, Supersonic shocks, Supersonic vortex sheets, Traces, Transonic shocks, Multidimension, Well-posedness

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1. Introduction

Hyperbolic conservation laws, quasilinear hyperbolic systems in divergence form, are one of the most important classes of nonlinear partial differential equations, which typically take the following form:

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^d, \tag{1.1}$$

where $\nabla_{\mathbf{x}} = (\partial_{x_1}, \ldots, \partial_{x_d})$ and

$$\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_d) : \mathbb{R}^n \to \big(\mathbb{R}^n\big)^d$$

is a nonlinear mapping with $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, \ldots, d$.

Consider plane wave solutions

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \mathbf{x} \cdot \omega) \quad \text{for } \omega \in \mathcal{S}^{d-1}.$$

Then $\mathbf{w}(t, \xi)$ satisfies

$$\partial_t \mathbf{w} + \big(\nabla \mathbf{f}(\mathbf{w}) \cdot \omega\big)\, \partial_\xi \mathbf{w} = 0,$$

where $\nabla = (\partial_{w_1}, \ldots, \partial_{w_n})$.

In order that there is a stable plane wave solution, it requires that, for any $\omega \in \mathcal{S}^{d-1}$,

$$\big(\nabla \mathbf{f}(\mathbf{w}) \cdot \omega\big)_{n \times n} \text{ have } n \text{ real eigenvalues } \lambda_i(\mathbf{w}; \omega) \text{ and be diagonalizable,}$$
$$1 \leqslant i \leqslant n. \tag{1.2}$$

Based on this, we say that system (1.1) is hyperbolic in a state domain $\mathcal{D}$ if condition (1.2) holds for any $\mathbf{w} \in \mathcal{D}$ and $\omega \in \mathcal{S}^{d-1}$.

The simplest example for multidimensional hyperbolic conservation laws is the following scalar conservation law

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d, \tag{1.3}$$

with $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ nonlinear. Then

$$\lambda(u, \omega) = \mathbf{f}'(u) \cdot \omega.$$

Therefore, any scalar conservation law is hyperbolic.

As is well known, the study of the Euler equations in gas dynamics gave birth to the theory of hyperbolic conservation laws so that the system of Euler equations is an archetype of this class of nonlinear partial differential equations. In general, the Euler equations for compressible fluids in $\mathbb{R}^d$ are a system of $d+2$ conservation laws

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot \mathbf{m} = 0 & \text{(Euler 1755–1759)}, \\ \partial_t \mathbf{m} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho}\right) + \nabla_{\mathbf{x}} p = 0 & \text{(Cauchy 1827–1829)}, \\ \partial_t E + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m}}{\rho}(E + p)\right) = 0 & \text{(Kirchhoff 1868)} \end{cases} \tag{1.4}$$

for $(t, \mathbf{x}) \in \mathbb{R}^{d+1}_+, \mathbb{R}^{d+1}_+ = \mathbb{R}_+ \times \mathbb{R}^d := (0, \infty) \times \mathbb{R}^d$. System (1.4) is closed by the constitutive relations

$$p = p(\rho, e), \qquad E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e. \tag{1.5}$$

In (1.4) and (1.5), $\tau = 1/\rho$ is the deformation gradient (specific volume for fluids, strain for solids), $\mathbf{v} = (v_1, \ldots, v_d)^\top$ is the fluid velocity with $\rho \mathbf{v} = \mathbf{m}$ the momentum vector, p is the scalar pressure and E is the total energy with e the internal energy which is a given function of (τ, p) or (ρ, p) defined through thermodynamical relations. The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The other two thermodynamic variables are temperature θ and entropy S. If (ρ, S) are chosen as the independent variables, then the constitutive relations can be written as

$$(e, p, \theta) = \big(e(\rho, S), p(\rho, S), \theta(\rho, S)\big) \tag{1.6}$$

governed by

$$\theta \, \mathrm{d}S = \mathrm{d}e + p \, \mathrm{d}\tau = \mathrm{d}e - \frac{p}{\rho^2} \, \mathrm{d}\rho. \tag{1.7}$$

For a polytropic gas,

$$p = R\rho\theta, \qquad e = c_{\mathrm{v}}\theta, \qquad \gamma = 1 + \frac{R}{c_{\mathrm{v}}} \tag{1.8}$$

and

$$p = p(\rho, S) = \kappa \rho^\gamma \mathrm{e}^{S/c_{\mathrm{v}}}, \quad e = \frac{\kappa}{\gamma - 1}\rho^{\gamma - 1}\mathrm{e}^{S/c_{\mathrm{v}}}, \tag{1.9}$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_{\mathrm{v}} > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent and $\kappa > 0$ can be any positive constant by scaling.

As shown in Section 2.4, no matter how smooth the initial data is, the solution of (1.4) generally develops singularities in a finite time. Then system (1.4) is complemented by the Clausius–Duhem inequality

$$\partial_t(\rho S) + \nabla_{\mathbf{x}} \cdot (\mathbf{m}S) \geqslant 0 \quad \text{(Clausius 1854, Duhem 1901)} \tag{1.10}$$

in the sense of distributions in order to single out physical discontinuous solutions, so-called *entropy solutions*.

When a flow is isentropic, that is, entropy S is a uniform constant S_0 in the flow, then the Euler equations for the flow take the following simpler form

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho}\right) + \nabla p = 0, \end{cases} \tag{1.11}$$

where the pressure is regarded as a function of density, $p = p(\rho, S_0)$, with constant S_0. For a polytropic gas,

$$p(\rho) = \kappa \rho^{\gamma}, \quad \gamma > 1, \tag{1.12}$$

where $\kappa > 0$ can be any positive constant under scaling. This system can be derived from (1.4) as follows. It is well known that, for smooth solutions of (1.4), entropy $S(\rho, \mathbf{m}, E)$ is conserved along fluid particle trajectories, i.e.,

$$\partial_t(\rho S) + \nabla_{\mathbf{x}} \cdot (\mathbf{m} S) = 0. \tag{1.13}$$

If the entropy is initially a uniform constant and the solution remains smooth, then (1.13) implies that the energy equation can be eliminated, and entropy S keeps the same constant in later time. Thus, under the constant initial entropy, a smooth solution of (1.4) satisfies the equations in (1.11). Furthermore, it should be observed that solutions of system (1.11) are also a good approximation to solutions of system (1.4) even after shocks form, since the entropy increases across a shock to third order in wave strength for solutions of (1.4), while in (1.11) the entropy is constant. Moreover, system (1.11) is an excellent model for the isothermal fluid flow with $\gamma = 1$ and for the shallow water flow with $\gamma = 2$.

In the one-dimensional case, system (1.4) in Eulerian coordinates is

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p\right) = 0, \\ \partial_t E + \partial_x \left(\frac{m}{\rho}(E + p)\right) = 0 \end{cases} \tag{1.14}$$

with $E = \frac{1}{2}\frac{m^2}{\rho} + \rho e$. The system above can be rewritten in Lagrangian coordinates in one-to-one correspondence as long as the fluid flow stays away from vacuum $\rho = 0$,

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \left(e + \frac{v^2}{2}\right) + \partial_x (p v) = 0 \end{cases} \tag{1.15}$$

with $v = m/\rho$, where the coordinates (t, x) are the Lagrangian coordinates, which are different from the Eulerian coordinates for (1.14); for simplicity of notation, we do not distinguish them. For the isentropic case, systems (1.14) and (1.15) reduce to

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p\right) = 0 \end{cases} \tag{1.16}$$

and

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \end{cases} \tag{1.17}$$

respectively, where pressure p is determined by (1.12) for the polytropic case, $p = p(\rho) = \tilde{p}(\tau)$ with $\tau = 1/\rho$. The solutions of (1.16) and (1.17), even for entropy solutions, are equivalent (see [52,332]).

This chapter is organized as follows. In Section 2 we exhibit some basic features and phenomena of the Euler equations and related hyperbolic conservation laws such as convex entropy, symmetrization, hyperbolicity, genuine nonlinearity, singularities and *BV* bound. In Section 3 we describe some aspects of a well-posedness theory and related results for the one-dimensional isentropic, isothermal and adiabatic Euler equations, respectively. In Sections 4–7 we discuss some samples of multidimensional models and problems for the Euler equations with emphasis on the prototype models and problems that have been solved or expected to be solved rigorously at least for some cases. In Section 8 we discuss connections between entropy solutions of hyperbolic conservation laws and divergence-measure fields, as well as the theory of divergence-measure fields to construct a good framework for studying entropy solutions. Some analytical approaches including techniques, methods, and ideas, developed recently, for solving multidimensional problems are also presented.

2. Basic features and phenomena

In this section we exhibit some basic features and phenomena of the Euler equations and related hyperbolic conservation laws.

2.1. *Convex entropy and symmetrization*

A function $\eta : \mathcal{D} \to \mathbb{R}$ is called an entropy of system (1.1) if there exists a vector function $\mathbf{q} : \mathcal{D} \to \mathbb{R}^d$, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_d)$, satisfying

$$\nabla \mathbf{q}_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \quad i = 1, \ldots, d. \tag{2.1}$$

An entropy $\eta(\mathbf{u})$ is called a convex entropy in $\mathcal{D}$ if

$$\nabla^2 \eta(\mathbf{u}) \geqslant 0 \quad \text{for any } \mathbf{u} \in \mathcal{D}$$

and a strictly convex entropy in $\mathcal{D}$ if

$$\nabla^2\eta(\mathbf{u}) \geqslant c_0 I$$

with a constant $c_0 > 0$ uniform for $\mathbf{u} \in \mathcal{D}_1$ for any $\mathcal{D}_1 \subset \overline{\mathcal{D}_1} \Subset \mathcal{D}$, where I is the $n \times n$ identity matrix. Then the correspondence of (1.10) in the context of hyperbolic conservation laws is the Lax entropy inequality

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \leqslant 0 \tag{2.2}$$

in the sense of distributions for any C^2 convex entropy–entropy flux pair $(\eta, \mathbf{q})$.

THEOREM 2.1. *A system in* (1.1) *endowed with a strictly convex entropy* η *in a state domain* $\mathcal{D}$ *must be symmetrizable and hence hyperbolic in* $\mathcal{D}$.

PROOF. Taking ∇ of both sides of the equations in (2.1) with respect to $\mathbf{u}$, we have

$$\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}) + \nabla\eta(\mathbf{u})\nabla^2\mathbf{f}_i(\mathbf{u}) = \nabla^2\mathbf{q}_i(\mathbf{u}), \quad i = 1, \ldots, d.$$

Using the symmetry of the matrices

$$\nabla\eta(\mathbf{u})\nabla^2\mathbf{f}_i(\mathbf{u}) \quad \text{and} \quad \nabla^2\mathbf{q}_i(\mathbf{u})$$

for fixed $i = 1, 2, \ldots, d$, we find that

$$\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}) \text{ is symmetric.} \tag{2.3}$$

Multiplying (1.1) by $\nabla^2\eta(\mathbf{u})$, we get

$$\nabla^2\eta(\mathbf{u})\,\partial_t\mathbf{u} + \sum_{i=1}^{d} \nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u})\nabla_{x_i}\mathbf{u} = 0. \tag{2.4}$$

The fact that the matrices $\nabla^2\eta(\mathbf{u}) > 0$ and $\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}), i = 1, 2, \ldots, d,$ are symmetric implies that system (1.1) is symmetrizable. Notice that any symmetrizable system must be hyperbolic, which can be seen as follows.

Since $\nabla^2\eta(\mathbf{u}) > 0$ for $\mathbf{u} \in \mathcal{D}$, then the hyperbolicity of (1.1) is equivalent to the hyperbolicity of (2.4), while the hyperbolicity of (2.4) is equivalent to that, for any $\omega \in \mathcal{S}^{d-1}$,

$$\text{all zeros of the determinant } \left|\lambda\nabla^2\eta(\mathbf{u}) - \nabla^2\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u}) \cdot \omega\right| \text{ are real.} \tag{2.5}$$

Since $\nabla^2\eta(\mathbf{u})$ is real symmetric and positive definite, there exists a matrix $C(\mathbf{u})$ such that

$$\nabla^2\eta(\mathbf{u}) = C(\mathbf{u})C(\mathbf{u})^\top.$$

Then the hyperbolicity is equivalent to that, for any $\omega \in S^{d-1}$, the eigenvalues of the following matrix

$$C(\mathbf{u})^{-1}\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u})\cdot\omega\big(C(\mathbf{u})^{-1}\big)^\top \tag{2.6}$$

are real, which is true since the matrix is real and symmetric. This completes the proof. □

REMARK 2.1. This theorem is particularly useful to determine whether a large physical system is symmetrizable and hence hyperbolic, since most of physical systems from continuum physics are endowed with a strictly convex entropy. In particular, for system (1.4),

$$(\eta_*, \mathbf{q}_*) = (-\rho S, -\mathbf{m}S) \tag{2.7}$$

is a strictly convex entropy–entropy flux pair when $\rho > 0$ and $p > 0$; while, for system (1.11), the mechanical energy and energy flux

$$(\eta_*, \mathbf{q}_*) = \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho), \frac{\mathbf{m}}{\rho}\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho) + p(\rho)\right)\right) \tag{2.8}$$

is a strictly convex entropy–entropy flux pair when $\rho > 0$ for polytropic gases. For multidimensional hyperbolic systems of conservation laws without a strictly convex entropy, it is possible to enlarge the system so that the enlarged system is endowed with a globally defined, strictly convex entropy. See [29,111,113,275,295].

REMARK 2.2. The observation that systems of conservation laws endowed with a strictly convex entropy must be symmetrizable is due to Godunov [155–157], Friedrich and Lax [140] and Boillat [22]. See also [284].

REMARK 2.3. This theorem has many important applications in the energy estimates. Basically, the symmetry plays an essential role in the following situation: For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\begin{aligned} &2\mathbf{u}^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\,\partial_{x_k}\mathbf{u} \\ &\quad = \partial_{x_k}\big(\mathbf{u}^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\mathbf{u}\big) - \mathbf{u}^\top\,\partial_{x_k}\big(\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\big)\mathbf{u} \end{aligned} \tag{2.9}$$

for $k = 1, 2, \ldots, d$. This is very useful to make energy estimates for various problems.

There are several direct, important applications of Theorem 2.1 based on the symmetry property of system (1.1) endowed with a strictly convex entropy such as (2.9). We list three of them below.

2.1.1. *Local existence of classical solutions.* Consider the Cauchy problem for a general hyperbolic system (1.1) with a strictly convex entropy η whose Cauchy data is

$$\mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{2.10}$$

THEOREM 2.2. *Assume that* $\mathbf{u}_0 : \mathbb{R}^d \to \mathcal{D}$ *is in* $H^s \cap L^\infty$ *with* $s > d/2 + 1$. *Then, for the Cauchy problem* (1.1) *and* (2.10), *there exists a finite time* $T = T(\|\mathbf{u}_0\|_s, \|\mathbf{u}_0\|_{L^\infty}) \in (0, \infty)$ *such that there is a unique bounded classical solution* $\mathbf{u} \in C^1([0, T] \times \mathbb{R}^d)$ *with*

$$\mathbf{u}(t, \mathbf{x}) \in \mathcal{D} \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^d$$

and

$$\mathbf{u} \in C\big([0, T]; H^s\big) \cap C^1\big([0, T]; H^{s-1}\big).$$

Kato [184,185] first formulated and applied a basic idea in the semigroup theory to yield the local existence of smooth solutions to (1.1).

The proof of this theorem in [241] relies solely on the elementary linear existence theory for symmetric hyperbolic systems with smooth coefficients via a classical iteration scheme (cf. [101]) by using the symmetry of system (1.1), especially (2.9). In particular, for all $\mathbf{u} \in \mathcal{D}$, there is a positive definite symmetric matrix $A_0(\mathbf{u}) = \nabla^2 \eta(\mathbf{u})$ that is smooth in $\mathbf{u}$ and satisfies

$$c_0 \mathbf{I} \leqslant A_0(\mathbf{u}) \leqslant c_0^{-1} \mathbf{I} \tag{2.11}$$

with a constant $c_0 > 0$ uniform for $\mathbf{u} \in \mathcal{D}_1$, for any $\mathcal{D}_1 \subset \overline{\mathcal{D}}_1 \Subset \mathcal{D}$, such that $A_i(\mathbf{u}) = A_0(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u})$ is symmetric. Moreover, a sharp continuation principle was also provided: For $\mathbf{u}_0 \in H^s$ with $s > d/2 + 1$, the interval $[0, T)$ with $T < \infty$ is the maximal interval of the classical H^s existence for (1.1) if and only if either

$$\big\|(\mathbf{u}_t, D\mathbf{u})(t, \cdot)\big\|_{L^\infty} \to \infty \quad \text{as } t \to T,$$

or

$$\mathbf{u}(t, \mathbf{x}) \text{ escapes every compact subset } K \Subset \mathcal{D} \quad \text{as } t \to T.$$

The first catastrophe in this principle is associated with the formation of shock waves and vorticity waves, among others, in the smooth solutions, and the second is associated with a blow-up phenomenon such as focusing and concentration.

In [246], Makino, Ukai and Kawashima established the local existence of classical solutions of the Cauchy problem with compactly supported initial data for the multidimensional Euler equations, with the aid of the theory of quasilinear symmetric hyperbolic systems; in particular, they introduced a symmetrization which works for initial data having either compact support or vanishing at infinity. There are also discussions in [48] on the local existence of smooth solutions of the three-dimensional Euler equations (1.4) by using an identity to deduce a time decay of the internal energy and the Mach number.

The local existence and stability of classical solutions of the initial–boundary value problem for the multidimensional Euler equations can be found in [182,189,191] and the references cited therein.

2.1.2. *Stability of Lipschitz solutions, rarefaction waves, and vacuum states in the class of entropy solutions in L^∞*

THEOREM 2.3. *Assume that system* (1.1) *is endowed with a strictly convex entropy η on compact subsets of $\mathcal{D}$. Suppose that $\mathbf{v}$ is a Lipschitz solution of* (1.1) *on $[0,T)$, taking values in a convex compact subset K of $\mathcal{D}$, with initial data $\mathbf{v}_0$. Let $\mathbf{u}$ be any entropy solution of* (1.1) *on $[0,T)$, taking values in K, with initial data $\mathbf{u}_0$. Then*

$$\int_{|\mathbf{x}|<R} \left|\mathbf{u}(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leqslant C(T) \int_{|\mathbf{x}|<R+Lt} \left|\mathbf{u}_0(\mathbf{x}) - \mathbf{v}_0(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}$$

holds for any $R>0$ and $t \in [0,T)$, with $L>0$ depending solely on K and the Lipschitz constant of $\mathbf{v}$.

The main point for the proof of Theorem 2.3 is to use the relative entropy–entropy flux pair (cf. [105])

$$\alpha(\mathbf{u},\mathbf{v}) = \eta(\mathbf{u}) - \eta(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u}-\mathbf{v}), \tag{2.12}$$

$$\beta(\mathbf{u},\mathbf{v}) = \mathbf{q}(\mathbf{u}) - \mathbf{q}(\mathbf{v}) - \nabla\eta(\mathbf{v})\big(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\big) \tag{2.13}$$

and to calculate and find

$$\partial_t \alpha(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \beta(\mathbf{u},\mathbf{v}) \leqslant -\big\{\partial_t\big(\nabla\eta(\mathbf{v})\big)(\mathbf{u}-\mathbf{v}) + \nabla_{\mathbf{x}}\big(\nabla\eta(\mathbf{v})\big)\big(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\big)\big\}.$$

Since $\mathbf{v}$ is a classical solution, we use the symmetry property of system (1.1) with the strictly convex entropy η to have

$$\begin{aligned}
\partial_t\big(\nabla\eta(\mathbf{v})\big) &= (\partial_t \mathbf{v})^\top \nabla^2\eta(\mathbf{v}) \\
&= -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \big(\nabla \mathbf{f}_k(\mathbf{v})\big)^\top \nabla^2\eta(\mathbf{v}) \\
&= -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \nabla^2\eta(\mathbf{v}) \nabla \mathbf{f}_k(\mathbf{v}).
\end{aligned}$$

Therefore, we have

$$\partial_t \alpha(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \beta(\mathbf{u},\mathbf{v}) \leqslant -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \nabla^2\eta(\mathbf{v})\, Q\mathbf{f}_k(\mathbf{u},\mathbf{v}),$$

where

$$Q\mathbf{f}_k(\mathbf{u},\mathbf{v}) = \mathbf{f}_k(\mathbf{u}) - \mathbf{f}_k(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u}-\mathbf{v}).$$

Integrating over a set

$$\big\{(\tau, \mathbf{x})\colon\ 0 \leqslant \tau \leqslant t \leqslant T, |\mathbf{x}| \leqslant R + L(t-\tau)\big\}$$

with the aid of the Gauss–Green formula in Section 8 and choosing $L > 0$ large enough yields the expected result.

Some further ideas have been developed to show the stability of planar rarefaction waves and vacuum states in the class of entropy solutions in L^∞ for the multidimensional Euler equations by using the Gauss–Green formula in Section 8.

THEOREM 2.4. *Let $\omega \in \mathcal{S}^{d-1}$. Let*

$$\mathbf{R}(t, \mathbf{x}) = (\hat{\rho}, \hat{\mathbf{m}})\left(\frac{\mathbf{x}\cdot\omega}{t}\right)$$

be a planar solution, consisting of planar rarefaction waves and possible vacuum states, of the Riemann problem

$$\mathbf{R}|_{t=0} = \begin{cases} (\rho_-, \hat{\mathbf{m}}_-), & \mathbf{x}\cdot\omega < 0, \\ (\rho_+, \hat{\mathbf{m}}_+), & \mathbf{x}\cdot\omega > 0, \end{cases}$$

with constant states $(\rho_\pm, \hat{\mathbf{m}}_\pm)$. Suppose $\mathbf{u}(t,\mathbf{x}) = (\rho, \mathbf{m})(t, \mathbf{x})$ is an entropy solution in L^∞ of (1.11) *that may contain vacuum. Then, for any $R > 0$ and $t \in [0, \infty)$,*

$$\int_{|\mathbf{x}|<R} \alpha(\mathbf{u}, \mathbf{R})(t, \mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \int_{|\mathbf{x}|<R+Lt} \alpha(\mathbf{u}, \mathbf{R})(0, \mathbf{x})\,\mathrm{d}\mathbf{x},$$

where $L > 0$ depends solely on the bounds of the solutions $\mathbf{u}$ and $\mathbf{R}$, and

$$\alpha(\mathbf{u}, \mathbf{R}) = (\mathbf{u}-\mathbf{R})^\top\left(\int_0^1 \nabla^2\eta_*\big(\mathbf{R} + \tau(\mathbf{u}-\mathbf{R})\big)\,\mathrm{d}\tau\right)(\mathbf{u}-\mathbf{R})$$

with $\eta_*(\mathbf{u}) = E \equiv \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho)$.

REMARK 2.4. Theorem 2.3 is due to Dafermos [110] (also see [111]). Theorem 2.4 is due to Chen and Chen [56], where a similar theorem was also established for the adiabatic Euler equations (1.4) with appropriate chosen entropy; also see [55] and [70].

REMARK 2.5. For multidimensional hyperbolic systems of conservation laws with partially convex entropies and involutions, see [111]; also see [24,106].

REMARK 2.6. For distributional solutions to the Euler equations (1.4) for polytropic gases, it is observed in Perthame [269] that, under the basic integrability condition

$$\rho, E, \rho\mathbf{v}\cdot\mathbf{x}, |\mathbf{v}|E \in L^1_{\mathrm{loc}}\big(\mathbb{R}_+; L^1\big(\mathbb{R}^d\big)\big)$$

and the condition that entropy $S(t,\mathbf{x})$ has an upper bound, the internal energy decays in time and, furthermore, the only time-decay on the internal energy suffices to yield the time-decay of the density. Also see [48].

2.1.3. *Local existence of shock front solutions.* Shock front solutions, the simplest type of discontinuous solutions, are the most important discontinuous nonlinear progressing wave solutions in compressible Euler flows and other systems of conservation laws. For a general multidimensional hyperbolic system of conservation laws (1.1), shock front solutions are discontinuous piecewise smooth entropy solutions with the following structure:

(i) there exist a C^2 time–space hypersurface $\mathcal{S}(t)$ defined in $(t,\mathbf{x})$ for $0 \leqslant t \leqslant T$ with time–space normal $(\mathbf{n}_t, \mathbf{n}_{\mathbf{x}}) = (\mathbf{n}_t, \mathbf{n}_1, \dots, \mathbf{n}_d)$ and two C^1 vector-valued functions, $\mathbf{u}^+(t,\mathbf{x})$ and $\mathbf{u}^-(t,\mathbf{x})$, defined on respective domains $\mathcal{D}^+$ and $\mathcal{D}^-$ on either side of the hypersurface $\mathcal{S}(t)$, and satisfying

$$\partial_t \mathbf{u}^{\pm} + \nabla \cdot \mathbf{f}\big(\mathbf{u}^{\pm}\big) = 0 \quad \text{in } \mathcal{D}^{\pm}; \tag{2.14}$$

(ii) the jump across the hypersurface $\mathcal{S}(t)$ satisfies the Rankine–Hugoniot condition

$$\big\{\mathbf{n}_t\big(\mathbf{u}^+ - \mathbf{u}^-\big) + \mathbf{n}_{\mathbf{x}} \cdot \big(\mathbf{f}\big(\mathbf{u}^+\big) - \mathbf{f}\big(\mathbf{u}^-\big)\big)\big\}\big|_{\mathcal{S}} = 0. \tag{2.15}$$

For the quasilinear system (1.1), the surface $\mathcal{S}$ is not known in advance and must be determined as a part of the solution of the problem; thus the equations in (2.14) and (2.15) describe a multidimensional, highly nonlinear, free-boundary value problem for the quasilinear system of conservation laws.

The initial data yielding shock front solutions is defined as follows. Let $\mathcal{S}_0$ be a smooth hypersurface parametrized by α, and let $\mathbf{n}(\alpha) = (\mathbf{n}_1, \dots, \mathbf{n}_d)(\alpha)$ be a unit normal to $\mathcal{S}_0$. Define the piecewise smooth initial data for respective domains $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$ on either side of the hypersurface $\mathcal{S}_0$ as

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases} \mathbf{u}_0^-(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^-, \\ \mathbf{u}_0^+(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^+. \end{cases} \tag{2.16}$$

It is assumed that the initial jump in (2.16) satisfies the Rankine–Hugoniot condition, i.e., there is a smooth scalar function $\sigma(\alpha)$ so that

$$-\sigma(\alpha)\big(\mathbf{u}_0^+(\alpha) - \mathbf{u}_0^-(\alpha)\big) + \mathbf{n}(\alpha) \cdot \big(\mathbf{f}\big(\mathbf{u}_0^+(\alpha)\big) - \mathbf{f}\big(\mathbf{u}_0^-(\alpha)\big)\big) = 0, \tag{2.17}$$

and that $\sigma(\alpha)$ does not define a characteristic direction, i.e.,

$$\sigma(\alpha) \neq \lambda_i\big(\mathbf{u}_0^{\pm}\big), \quad \alpha \in \overline{\mathcal{S}_0}, 1 \leqslant i \leqslant n, \tag{2.18}$$

where λ_i, $i = 1, \dots, n$, are the eigenvalues of (1.1). It is natural to require that $\mathcal{S}(0) = \mathcal{S}_0$.

Consider the three-dimensional full Euler equations in (1.4), away from vacuum, which can be rewritten in the form

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0, \\ \partial_t E + \nabla_{\mathbf{x}} \big(\mathbf{v}(E + p)\big) = 0, \end{cases} \tag{2.19}$$

with piecewise smooth initial data

$$(\rho, \mathbf{v}, E)|_{t=0} = \begin{cases} \big(\rho_0^-, \mathbf{v}_0^-, E^+\big)(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D}_0^-, \\ \big(\rho_0^+, \mathbf{v}_0^+, E^+\big)(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D}_0^+. \end{cases} \tag{2.20}$$

THEOREM 2.5. *Assume that $\mathcal{S}_0$ is a smooth hypersurface in $\mathbb{R}^3$ and that $(\rho_0^+, \mathbf{v}_0^+, E_0^+)(\mathbf{x})$ belongs to the uniform local Sobolev space $H^s_{\mathrm{ul}}(\mathcal{D}_0^+)$, while $(\rho_0^-, \mathbf{v}_0^-, E_0^-)(\mathbf{x})$ belongs to the Sobolev space $H^s(\mathcal{D}_0^-)$, for some fixed $s \geqslant 10$. Assume also that there is a function $\sigma(\alpha) \in H^s(\mathcal{S}_0)$ so that* (2.17) *and* (2.18) *hold, and the compatibility conditions up to order $s-1$ are satisfied on $\mathcal{S}_0$ by the initial data, together with the entropy condition*

$$\mathbf{v}_0^+ \cdot \mathbf{n}(\alpha) + \sqrt{p_\rho\big(\rho_0^+, S_0^+\big)} < \sigma(\alpha) < \mathbf{v}_0^- \cdot \mathbf{n}(\alpha) + \sqrt{p_\rho\big(\rho_0^-, S_0^-\big)}, \tag{2.21}$$

and the Majda stability condition

$$\begin{aligned} &1 + \big(p\big(\rho_0^+\big) - p\big(\rho_0^-\big)\big) \frac{(\rho_0^-)^2 p_\rho(\rho_0^-, S_0^-) p_S(\rho_0^-, S_0^-)}{\theta_0^-} \\ &\quad - \big(\rho_0^-\big)^3 \big(p\big(\rho_0^+\big) - p\big(\rho_0^-\big)\big) p_\rho\big(\rho_0^-, S_0^-\big) > 0. \end{aligned} \tag{2.22}$$

Then there is a C^2 hypersurface $\mathcal{S}(t)$ together with C^1 functions $(\rho^\pm, \mathbf{v}^\pm, E^\pm)(t, \mathbf{x})$ defined for $t \in [0, T]$, with T sufficiently small, so that

$$(\rho, \mathbf{v}, E)(t, \mathbf{x}) = \begin{cases} \big(\rho^-, \mathbf{v}^-, E^-\big)(t, \mathbf{x}), \quad (t, \mathbf{x}) \in \mathcal{D}^-, \\ \big(\rho^+, \mathbf{v}^+, E^+\big)(t, \mathbf{x}), \quad (t, \mathbf{x}) \in \mathcal{D}^+, \end{cases} \tag{2.23}$$

is the discontinuous shock front solution of the Cauchy problem (2.19) *and* (2.20) *satisfying* (2.14) *and* (2.15). *In particular, the condition in* (2.22) *is always satisfied for shocks of any strength for polytropic gas with $\gamma > 1$ and for sufficiently weak shocks for general equations of state.*

In Theorem 2.5, the uniform local Sobolev space $H^s_{\mathrm{ul}}(\mathcal{D}_0^+)$ is defined as follows: A vector function $\mathbf{u}$ is in H^s_{ul}, provided that there exists some $r > 0$ so that

$$\max_{\mathbf{y} \in \mathbb{R}^d} \|w_{r,\mathbf{y}} \mathbf{u}\|_{H^s} < \infty$$

with

$$w_{r,\mathbf{y}}(\mathbf{x}) = w\left(\frac{\mathbf{x}-\mathbf{y}}{r}\right),$$

where $w \in C_0^\infty(\mathbb{R}^d)$ is a function so that $w(\mathbf{x}) \geqslant 0$, $w(\mathbf{x}) = 1$ when $|\mathbf{x}| \leqslant 1/2$ and $w(\mathbf{x}) = 0$ when $|\mathbf{x}| > 1$.

REMARK 2.7. Theorem 2.5 is taken from [240]. The compatibility conditions in Theorem 2.5 are defined in [240] and needed in order to avoid the formation of discontinuities in higher derivatives along other characteristic surfaces emanating from $\mathcal{S}_0$: Once the main condition in (2.17) is satisfied, the compatibility conditions are automatically guaranteed for a wide class of initial data functions. Further studies on the local existence and stability of shock front solutions can be found in [239–241]. The uniform time of existence of shock front solutions in the shock strength was obtained in [249]. Also see [21] for further discussions.

The idea of the proof is similar to that for Theorem 2.2 by using the existence of a strictly convex entropy and the symmetrization of (1.1), but the technical details are quite different due to the unusual features of the problem considered in Theorem 2.5 (see [240]). The shock front solutions are defined as the limit of a convergent classical iteration scheme based on a linearization by using the theory of linearized stability for shock fronts developed in [239]. The technical condition $s \geqslant 10$, instead of $s > 1 + d/2$, is required because pseudo-differential operators are needed in the proof of the main estimates. Some improved technical estimates regarding the dependence of operator norms of pseudo-differential operators on their coefficients would lower the value of s. For more details, see [240].

2.2. *Hyperbolicity*

There are many examples of $n \times n$ hyperbolic systems of conservation laws for $\mathbf{x} \in \mathbb{R}^2$ which are strictly hyperbolic; that is, they have simple characteristics. However, for $d = 3$, there are no strictly hyperbolic systems if $n \equiv 2 \pmod 4$ or, even more generally, $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$ as a corollary of Theorem 2.6. Such multiple characteristics influence the propagation of singularities.

THEOREM 2.6. *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be three matrices such that*

$$\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

has real eigenvalues for any real α, β *and* γ. *If* $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$, *then there exist* α_0, β_0 *and* γ_0 *with* $\alpha_0^2 + \beta_0^2 + \gamma_0^2 \neq 0$ *such that*

$$\alpha_0\mathbf{A} + \beta_0\mathbf{B} + \gamma_0\mathbf{C} \tag{2.24}$$

is degenerate, that is, there are at least two eigenvalues of matrix (2.24) *which coincide.*

PROOF. We prove only the case $n \equiv 2 \pmod 4$.

1. Denote $\mathcal{M}$ the set of all real $n \times n$ matrices with real eigenvalues, and $\mathcal{N}$ the set of nondegenerate matrices (that have n distinct real eigenvalues) in $\mathcal{M}$. The normalized eigenvectors $\mathbf{r}_j$ of $\mathbf{N}$ in $\mathcal{N}$, i.e.,

$$\mathbf{N}\mathbf{r}_j = \lambda_j \mathbf{r}_j, \qquad |\mathbf{r}_j| = 1, \quad j = 1, 2, \ldots, n,$$

are determined up to a factor ± 1.

2. Let $\mathbf{N}(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, be a closed curve in $\mathcal{N}$. If we fix $\mathbf{r}_j(0)$, then $\mathbf{r}_j(\theta)$ can be determined uniquely by requiring continuous dependence on θ.

Since

$$\mathbf{N}(2\pi) = \mathbf{N}(0),$$

then

$$\mathbf{r}_j(2\pi) = \tau_j \mathbf{r}_j(0), \quad \tau_j = \pm 1.$$

Clearly,

(i) each τ_j is a homotopy invariant of the closed curve,

(ii) each $\tau_j = 1$ when $\mathbf{N}(\theta)$ is constant.

3. Suppose now that the theorem is false. Then

$$\mathbf{N}(\theta) = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta$$

is a closed curve in $\mathcal{N}$ and

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_n(\theta).$$

Since $\mathbf{N}(\pi) = -\mathbf{N}(0)$, we have

$$\lambda_j(\pi) = -\lambda_{n-j+1}(0),$$
$$\mathbf{r}_j(\pi) = \rho_j \mathbf{r}_{n-j+1}(0), \quad \rho_j \pm 1.$$

4. Since the ordered basis

$$\{\mathbf{r}_1(\theta), \mathbf{r}_2(\theta), \ldots, \mathbf{r}_n(\theta)\}$$

is defined continuously, it retains its orientation. Then the ordered bases

$$\{\mathbf{r}_1(0), \mathbf{r}_2(0), \ldots, \mathbf{r}_n(0)\} \quad \text{and} \quad \{\rho_1\mathbf{r}_n(0), \rho_2\mathbf{r}_{n-1}(0), \ldots, \rho_n\mathbf{r}_1(0)\}$$

have the same orientation.

For the case $n \equiv 2 \pmod 4$, reversing the order reverses the orientation of an ordered basis, which implies

$$\prod_{j=1}^{n} \rho_j = -1.$$

Then there exists k such that

$$\rho_k \rho_{n-k+1} = -1. \tag{2.25}$$

5. Since $\mathbf{N}(\theta + \pi) = -\mathbf{N}(\theta)$, then

$$\lambda_j(\theta + \pi) = -\lambda_{n-j+1}(\theta),$$

which implies

$$\mathbf{r}_j(2\pi) = \rho_j \mathbf{r}_{n-j+1}(\pi) = \rho_j \rho_{n-j+1} \mathbf{r}_{n-j+1}(0).$$

Therefore, we have

$$\tau_j = \rho_j \rho_{n-j+1}.$$

Then (2.25) implies

$$\tau_k = 1,$$

which yields that the curve

$$\mathbf{N}(\theta) = \mathbf{A} \cos\theta + \mathbf{B} \sin\theta$$

is not homotopic to a point.

6. Suppose that all matrices of form $\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$, belong to $\mathcal{N}$. Then, since the sphere is simply connected, the curve $\mathbf{N}(\theta)$ could be contracted to a point, contracting $\tau_k = -1$. This completes the proof. □

REMARK 2.8. The proof is taken from [201] for the case $n \equiv 2 \pmod 4$. The proof for the more general case $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$ can be found in [138].

Consider the isentropic Euler equations (1.11).

When $d = 2, n = 3$, the system is strictly hyperbolic with three real eigenvalues $\lambda_- < \lambda_0 < \lambda_+$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)}, \quad \rho > 0.$$

The strict hyperbolicity fails at the vacuum states when $\rho = 0$.

However, when $d = 3, n = 4$, the system is no longer strictly hyperbolic even when $\rho > 0$ since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has double multiplicity, although the other eigenvalues

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{p'(\rho)}$$

are simple when $\rho > 0$.

Consider the adiabatic Euler equations (1.4).

When $d = 2, n = 4$, the system is nonstrictly hyperbolic since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2$$

has double multiplicity; however,

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\frac{\gamma p}{\rho}}$$

are simple when $\rho > 0$.

When $d = 3, n = 5$, the system is again nonstrictly hyperbolic since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has triple multiplicity; however,

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{\frac{\gamma p}{\rho}}$$

are simple when $\rho > 0$.

2.3. *Genuine nonlinearity*

The jth-characteristic field of system (1.1) in $\mathcal{D}$ is called genuinely nonlinear if, for each fixed $\omega \in S^{d-1}$, the jth eigenvalue $\lambda_j(\mathbf{u}; \omega)$ and the corresponding eigenvector $\mathbf{r}_j(\mathbf{u}; \omega)$ determined by

$$\big(\nabla \mathbf{f}(\mathbf{u}) \cdot \omega\big)\mathbf{r}_j(\mathbf{u}; \omega) = \lambda_j(\mathbf{u}; \omega)\mathbf{r}_j(\mathbf{u}; \omega)$$

satisfy

$$\nabla_{\mathbf{u}} \lambda_j(\mathbf{u}; \omega) \cdot \mathbf{r}_j(\mathbf{u}; \omega) \neq 0 \quad \text{for any } \mathbf{u} \in \mathcal{D}, \omega \in \mathcal{S}^{d-1}. \tag{2.26}$$

The jth-characteristic field of system (1.1) in $\mathcal{D}$ is called linearly degenerate if

$$\nabla_{\mathbf{u}}\lambda_j(\mathbf{u};\omega)\cdot\mathbf{r}_j(\mathbf{u};\omega)\equiv 0 \quad \text{for any } \mathbf{u}\in\mathcal{D}. \tag{2.27}$$

Then we immediately have the following theorem.

THEOREM 2.7. *Any scalar quasilinear conservation law in* $\mathbb{R}^d$, $d\geqslant 2$, *is never genuinely nonlinear in all directions.*

It is because, in this case,

$$\lambda(u;\omega)=\mathbf{f}'(u)\cdot\omega, \qquad r(u;\omega)=1$$

and

$$\lambda'(u;\omega)r(u;\omega)\equiv\mathbf{f}'(u)\cdot\omega$$

which is impossible to make this never equal to zero in all directions.

A multidimensional version of genuine nonlinearity for scalar conservation laws is

$$\left|\left\{u:\ \tau+\mathbf{f}'(u)\cdot\omega=0\right\}\right|=0 \quad \text{for any } (\tau,\omega)\in\mathcal{S}^d,$$

which is a generalization of (2.26).

Under this generalized nonlinearity, the following have been established:

(i) solution operators are compact as $t>0$ in [224] (also see [64,314]),
(ii) decay of periodic solutions [65,128],
(iii) initial and boundary traces of entropy solutions [82,329],
(iv) BV structure of L^∞ entropy solutions [112].

For systems with $n=2m$, $m\geqslant 1$ odd, and $d=2$, using a topological argument, we have the following theorem.

THEOREM 2.8. *Every real, strictly hyperbolic quasilinear system for* $n=2m$, $m\geqslant 1$ *odd, and* $d=2$ *is linearly degenerate in some direction.*

PROOF. We prove only for the case $m=1$.

1. For fixed $\mathbf{u}\in\mathbb{R}^n$, define

$$\mathbf{N}(\theta;\mathbf{u})=\nabla\mathbf{f}_1(\mathbf{u})\cos\theta+\nabla\mathbf{f}_2(\mathbf{u})\sin\theta.$$

Denote the eigenvalues of $\mathbf{N}(\theta;\mathbf{u})$ by $\lambda_\pm(\theta;\mathbf{u})$,

$$\lambda_-(\theta;\mathbf{u})<\lambda_+(\theta;\mathbf{u})$$

with

$$\mathbf{N}(\theta;\mathbf{u})\mathbf{r}_\pm(\theta;\mathbf{u})=\lambda_\pm(\theta;\mathbf{u})\mathbf{r}_\pm(\theta;\mathbf{u}), \qquad \left|\mathbf{r}_\pm(\theta;\mathbf{u})\right|=1. \tag{2.28}$$

This still leaves an arbitrary factor ± 1, which we fix arbitrarily at $\theta = 0$. For all other $\theta \in [0, 2\pi]$, we require $\mathbf{r}_\pm(\theta; \mathbf{u})$ to vary continuously with θ.

2. Since $\mathbf{N}(\theta + \pi; \mathbf{u}) = -\mathbf{N}(\theta; \mathbf{u})$,

$$\lambda_+(\theta + \pi; \mathbf{u}) = -\lambda_-(\theta; \mathbf{u}), \qquad \lambda_-(\theta + \pi; \mathbf{u}) = -\lambda_+(\theta; \mathbf{u}).$$

It follows from this and $|\mathbf{r}_\pm| = 1$ that

$$\mathbf{r}_+(\theta + \pi; \mathbf{u}) = \sigma_+ \mathbf{r}_-(\theta; \mathbf{u}), \qquad \mathbf{r}_-(\theta + \pi; \mathbf{u}) = \sigma_- \mathbf{r}_+(\theta; \mathbf{u}), \tag{2.29}$$

where $\sigma_\pm = 1$ or -1.

3. Since $\mathbf{r}_\pm(\theta; \mathbf{u})$ were chosen to be continuous functions of θ, we find that

(i) $\sigma_\pm$ are also continuous functions of θ and, thus, they must be constant since $\sigma_\pm = \pm 1$;

(ii) the orientation of the ordered basis $\{\mathbf{r}_-(\theta; \mathbf{u}), \mathbf{r}_+(\theta; \mathbf{u})\}$ does not change and, hence, the bases

$$\big\{\mathbf{r}_-(0; \mathbf{u}), \mathbf{r}_+(0; \mathbf{u})\big\} \quad \text{and} \quad \big\{\mathbf{r}_-(\pi; \mathbf{u}), \mathbf{r}_+(\pi; \mathbf{u})\big\}$$

have the same orientation.

Therefore, by (2.29),

$$\big\{\mathbf{r}_-(0; \mathbf{u}), \mathbf{r}_+(0; \mathbf{u})\big\} \quad \text{and} \quad \big\{\sigma_- \mathbf{r}_+(0; \mathbf{u}), \sigma_+ \mathbf{r}_-(0; \mathbf{u})\big\}$$

have the same orientation. Then

$$\sigma_+ \sigma_- = -1$$

and

$$\mathbf{r}_+(2\pi; \mathbf{u}) = \sigma_+ \mathbf{r}_-(\pi; \mathbf{u}) = \sigma_+ \sigma_- \mathbf{r}_+(0, \mathbf{u}) = -\mathbf{r}_+(0, \mathbf{u}). \tag{2.30}$$

Similarly, we have

$$\mathbf{r}_-(2\pi; \mathbf{u}) = -\mathbf{r}_-(0; \mathbf{u}). \tag{2.31}$$

4. Since the eigenvalues $\lambda_\pm(\theta; \mathbf{u})$ are periodic functions of θ with period 2π for fixed $\mathbf{u} \in \mathbb{R}^2$, so are their gradients. Then

$$\nabla_{\mathbf{u}} \lambda_\pm(2\pi; \mathbf{u}) \mathbf{r}_\pm(2\pi; \mathbf{u}) = -\nabla_{\mathbf{u}} \lambda(0; \mathbf{u}) \mathbf{r}_\pm(0; \mathbf{u}).$$

Noticing that

$$\nabla \lambda_\pm(\theta; \mathbf{u}) \mathbf{r}_\pm(\theta; \mathbf{u})$$

vary continuously with θ for any fixed $\mathbf{u} \in \mathbb{R}^2$, we conclude that there exist $\theta_\pm \in (0, 2\pi)$ such that

$$\nabla\lambda_\pm(\theta_\pm; \mathbf{u})\mathbf{r}_\pm(\theta_\pm; \mathbf{u}) = 0.$$

This completes the proof. □

REMARK 2.9. The proof of Theorem 2.8 is from [202].

REMARK 2.10. Quite often, linear degeneracy results from the loss of strict hyperbolicity. For example, even in the one-dimensional case, if there exists $j \neq k$ such that

$$\lambda_j(\mathbf{u}) = \lambda_k(\mathbf{u}) \quad \text{for all } u \in K,$$

then Boillat [23] proved that the jth- and kth-characteristic families are linearly degenerate.

For the isentropic Euler equations (1.11) with $d = 2, n = 3$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)},$$

and

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0)^\top, \qquad \mathbf{r}_\pm = \left(\pm\omega_1, \pm\omega_2, \frac{\rho}{\sqrt{p'(\rho)}}\right)^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0$$

and

$$\nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\rho p''(\rho) + 2p'(\rho)}{2p'(\rho)} = \pm\frac{\gamma + 1}{2} \neq 0.$$

For the adiabatic Euler equations (1.4) with $d = 2, n = 4$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\frac{\gamma p}{\rho}}$$

and

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0, 1)^\top, \qquad \mathbf{r}_\pm = \left(\pm\omega_1, \pm\omega_2, \sqrt{\gamma p \rho}, \rho\sqrt{\frac{\rho}{\gamma p}}\right)^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0$$

and

$$\nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\gamma+1}{2} \neq 0.$$

2.4. *Singularities*

For the one-dimensional case, singularities include the formation of shock waves and the development of vacuum states, at least for gas dynamics. For the multidimensional case, the situation is much more complicated: besides shock waves and vacuum states, singularities may include vorticity waves, focusing waves, concentration waves, complicated wave interactions, among others.

Consider the Cauchy problem of the Euler equations in (1.4) for polytropic gases in $\mathbb{R}^3$ with smooth initial data

$$(\rho, \mathbf{v}, S)|_{t=0} = (\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}) \quad \text{with } \rho_0(\mathbf{x}) > 0 \text{ for } \mathbf{x} \in \mathbb{R}^3 \tag{2.32}$$

satisfying

$$(\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}) = \left(\bar\rho, 0, \overline{S}\right) \quad \text{for } |\mathbf{x}| \geqslant R,$$

where $\bar\rho > 0$, $\overline{S}$ and R are constants. The equations in (1.4) possess a unique local C^1-solution $(\rho, \mathbf{v}, S)(t, \mathbf{x})$ with $\rho(t, \mathbf{x}) > 0$ provided that the initial data (2.32) is sufficiently regular (Theorem 2.2). The support of the smooth disturbance $(\rho_0(\mathbf{x}) - \bar\rho, \mathbf{v}_0(\mathbf{x}), S_0(\mathbf{x}) - \overline{S})$ propagates with speed at most $\sigma = \sqrt{p_\rho(\bar\rho, \overline{S})}$ (the sound speed), that is,

$$(\rho, \mathbf{v}, S)(t, \mathbf{x}) = \left(\bar\rho, 0, \overline{S}\right) \quad \text{if } |\mathbf{x}| \geqslant R + \sigma t. \tag{2.33}$$

The proof of this essential fact of finite speed of propagation for the three-dimensional case can be found in [181], as well as in [299], established through local energy estimates.

Take $\bar p = p(\bar\rho, \overline{S})$. Define

$$\begin{aligned} P(t) &= \int_{\mathbb{R}^3} \left(p(t, \mathbf{x})^{1/\gamma} - \bar p^{1/\gamma}\right) d\mathbf{x} \\ &= \kappa^{1/\gamma} \int_{\mathbb{R}^3} \left(\rho(t, \mathbf{x}) \exp\left(\frac{S(t, \mathbf{x})}{\gamma c_{\mathrm{v}}}\right) - \bar\rho \exp\left(\frac{\overline{S}}{\gamma c_{\mathrm{v}}}\right)\right) d\mathbf{x}, \\ F(t) &= \int_{\mathbb{R}^3} \rho\mathbf{v}(t, \mathbf{x}) \cdot \mathbf{x}\, d\mathbf{x}, \end{aligned}$$

which, roughly speaking, measure the entropy and the radial component of momentum. The following theorem on the formation of singularities in solutions of (1.4) and (2.32) is due to Sideris [300].

THEOREM 2.9. *Suppose that* $(\rho, \mathbf{v}, S)(t, \mathbf{x})$ *is a* C^1*-solution of* (1.4) *and* (2.32) *for* $0 < t < T$ *and*

$$P(0) \geqslant 0, \tag{2.34}$$

$$F(0) > \alpha\sigma R^4 \max_{\mathbf{x}} \rho_0(\mathbf{x}), \qquad \alpha = \frac{16\pi}{3}. \tag{2.35}$$

Then the lifespan T *of the* C^1*-solution is finite.*

PROOF. Set

$$M(t) = \int_{\mathbb{R}^3} \big(\rho(t, \mathbf{x}) - \bar{\rho}\big)\, \mathrm{d}\mathbf{x}.$$

Combining the equations in (1.4) with (2.33) and using the integration by parts, one has

$$M'(t) = -\int_{\mathbb{R}^3} \nabla \cdot (\rho\mathbf{v})\, \mathrm{d}\mathbf{x} = 0,$$

$$P'(t) = -\kappa^{1/\gamma} \int_{\mathbb{R}^3} \nabla \cdot \left(\rho\mathbf{v} \exp\left(\frac{S}{\gamma c_{\mathrm{v}}}\right)\right) \mathrm{d}\mathbf{x} = 0,$$

which implies

$$M(t) = M(0), \qquad P(t) = P(0) \tag{2.36}$$

and

$$\begin{aligned} F'(t) &= \int_{\mathbb{R}^3} \mathbf{x} \cdot (\rho\mathbf{v})_t\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \big(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\big)\, \mathrm{d}\mathbf{x} \\ &= \int_{B(t)} \big(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\big)\, \mathrm{d}\mathbf{x}, \end{aligned} \tag{2.37}$$

where $B(t) = \{\mathbf{x} \in \mathbb{R}^3\colon |\mathbf{x}| \leqslant R + \sigma t\}$. From Hölder's inequality, (2.34) and (2.36), one has

$$\begin{aligned} \int_{B(t)} p\, \mathrm{d}\mathbf{x} &\geqslant \frac{1}{|B(t)|^{\gamma-1}} \left(\int_{B(t)} p^{1/\gamma}\, \mathrm{d}\mathbf{x}\right)^{\gamma} \\ &= \frac{1}{|B(t)|^{\gamma-1}} \left(P(0) + \int_{B(t)} \bar{p}^{1/\gamma}\, \mathrm{d}\mathbf{x}\right)^{\gamma} \\ &\geqslant \int_{B(t)} \bar{p}\, \mathrm{d}\mathbf{x}, \end{aligned}$$

where $|B(t)|$ denotes the volume of ball $B(t)$. Therefore, by (2.37),

$$F'(t) \geqslant \int_{\mathbb{R}^3} \rho|\mathbf{v}|^2 \, \mathrm{d}\mathbf{x}. \tag{2.38}$$

By the Cauchy–Schwarz inequality and (2.36),

$$\begin{aligned}
F(t)^2 &= \left(\int_{B(t)} \rho\mathbf{v}\cdot\mathbf{x}\,\mathrm{d}\mathbf{x}\right)^2 \\
&\leqslant \int_{B(t)} \rho|\mathbf{v}|^2\,\mathrm{d}\mathbf{x} \int_{B(t)} \rho|\mathbf{x}|^2\,\mathrm{d}\mathbf{x} \\
&\leqslant (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\left(M(t) + \int_{B(t)} \bar{\rho}\,\mathrm{d}\mathbf{x}\right) \\
&\leqslant (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\left(\int_{B(t)} \big(\rho_0(\mathbf{x}) - \bar{\rho}\big)\,\mathrm{d}\mathbf{x} + \int_{B(t)} \bar{\rho}\,\mathrm{d}\mathbf{x}\right) \\
&\leqslant \frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}} \rho_0(\mathbf{x}) \int_{B(t)} \rho|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Then (2.38) implies that

$$F'(t) \leqslant \left(\frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}} \rho_0(\mathbf{x})\right)^{-1} F(t)^2. \tag{2.39}$$

Since $F(0) > 0$ by (2.35), $F(t)$ remains positive for $0 < t < T$, as a consequence of (2.38). Dividing by $F(t)^2$ and integrating from 0 to T in (2.39) yields

$$F(0)^{-1} > F(0)^{-1} - F(T)^{-1} \geqslant (\alpha\sigma \max \rho_0)^{-1}\big(R^{-4} - (R+\sigma T)^{-4}\big).$$

Thus,

$$(R+\sigma T)^4 < \frac{R^4 F(0)}{F(0) - \alpha\sigma R^4 \max \rho_0}.$$

This completes the proof. $\square$

REMARK 2.11. The proof is taken from [86], which is a refinement of Sideris [299]. The method of the proof above applies equally well in one and two space dimensions. In the isentropic case, the condition $P(0) \geqslant 0$ reduces to $M(0) \geqslant 0$.

REMARK 2.12. To illustrate a way in which conditions (2.34) and (2.35) may be satisfied, we consider the initial data

$$\rho_0 = \bar{\rho}, \qquad S_0 = \overline{S}.$$

Then $P(0) = 0$, and (2.35) holds if

$$\int_{|\mathbf{x}|<R} \mathbf{v}_0(x) \cdot \mathbf{x} \, \mathrm{d}\mathbf{x} > \alpha\sigma R^4.$$

Comparing both sides, one finds that the initial velocity must be supersonic in some region relative to the sound speed at infinity. The formation of a singularity (presumably a shock wave) is detected as the disturbance overtakes the wave front forcing the front to propagate with supersonic speed.

The formation of singularities occurs even without condition of largeness such as (2.35). For example, if $S_0(x) \geqslant \overline{S}$ and, for some $0 < R_0 < R$,

$$\begin{aligned} &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-1}\big(|\mathbf{x}| - r\big)^2\big(\rho_0(\mathbf{x}) - \bar{\rho}\big) \, \mathrm{d}\mathbf{x} > 0, \\ &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-3}\big(|\mathbf{x}|^2 - r^2\big)\rho_0(\mathbf{x})\mathbf{v}_0(\mathbf{x}) \cdot \mathbf{x} \, \mathrm{d}\mathbf{x} \geqslant 0 \quad \text{for } R_0 < r < R, \end{aligned} \tag{2.40}$$

then the lifespan T of the C^1-solution of (1.4) and (2.32) is finite. The assumptions in (2.40) mean that, in an average sense, the gas must be slightly compressed and outgoing directly behind the wave front. For the proof in [300], some important technical points were adopted from [298] on the nonlinear wave equations in three dimensions.

REMARK 2.13. For the multidimensional Euler equations for compressible fluids with smooth initial data that is a small perturbation of amplitude ε from a constant state, the lifespan of smooth solutions is at least $\mathrm{O}(\varepsilon^{-1})$ from the theory of symmetric hyperbolic systems [139,183]. Results on the formation of singularities show that the lifespan of a smooth solution is no better than $\mathrm{O}(\varepsilon^{-2})$ in the two-dimensional case [276] and $\mathrm{O}(\mathrm{e}^{\varepsilon^{-2}})$ [300] in the three-dimensional case. See [2,301,302] for additional discussions in this direction. Also see [246] and [279] for a compressible fluid body surrounded by the vacuum.

2.5. *BV bound*

For one-dimensional strictly hyperbolic systems, Glimm's theorem [145] indicates that, as long as $\|\mathbf{u}_0\|_{BV}$ is sufficiently small, the solution $\mathbf{u}(t,x)$ satisfies the following stability estimate

$$\big\|\mathbf{u}(t,\cdot)\big\|_{BV} \leqslant C\|\mathbf{u}_0\|_{BV}. \tag{2.41}$$

Even more strongly, for two solutions $\mathbf{u}(t,x)$ and $\mathbf{v}(t,x)$ obtained by either the Glimm scheme, wave-front tracking method or vanishing viscosity method with small total variation,

$$\big\|\mathbf{u}(t,\cdot) - \mathbf{v}(t,\cdot)\big\|_{L^1(\mathbb{R})} \leqslant C\big\|\mathbf{u}(0,\cdot) - \mathbf{v}(0,\cdot)\big\|_{L^1(\mathbb{R})}.$$

See [20,33,111,167,204] and the references cited therein.

The recent great progress for entropy solutions for one-dimensional hyperbolic systems of conservation laws based on *BV* estimates and trace theorems of *BV* fields naturally arises the expectation that a similar approach may also be effective for multidimensional hyperbolic systems of conservation laws, that is, whether entropy solutions satisfy the relatively modest stability estimate

$$\big\|\mathbf{u}(t,\cdot)\big\|_{BV} \leqslant C\|\mathbf{u}_0\|_{BV}. \tag{2.42}$$

Unfortunately, this is not the case.

Rauch [278] showed that the necessary condition for (2.42) to be held is

$$\nabla\mathbf{f}_k(\mathbf{u})\nabla\mathbf{f}_l(\mathbf{u}) = \nabla\mathbf{f}_l(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}) \quad \text{for all } k, l = 1, 2, \dots, d. \tag{2.43}$$

The analysis above suggests that only systems in which the commutativity relation (2.43) holds offer any hope for treatment in the *BV* framework. This special case includes the scalar case $n = 1$ and the one-dimensional case $d = 1$. Beyond that, it contains very few systems of physical interest.

An example is the system with fluxes

$$\mathbf{f}_k(\mathbf{u}) = \phi_k\big(|\mathbf{u}|^2\big)\mathbf{u}, \quad k = 1, 2, \dots, d,$$

which governs the flow of a fluid in an anisotropic porous medium. However, the recent study in [34] and [7] shows that, even in this case, the space *BV* is not a well-posed space, and (2.42) fails.

Even for the one-dimensional systems whose strict hyperbolicity fails or initial data is allowed to be of large oscillation, the solution is no longer in *BV* in general. However, some bounds in L^∞ or L^p may be achieved. One of such examples is the isentropic Euler equations (1.16), for which we have

$$\big\|\mathbf{u}(t,\cdot)\big\|_{L^\infty} \leqslant C\|\mathbf{u}_0\|_{L^\infty}.$$

See [75] and the references cited therein. However, for the multidimensional case, entropy solutions generally do not have even the relatively modest stability

$$\big\|\mathbf{u}(t,\cdot) - \bar{\mathbf{u}}\big\|_{L^p} \leqslant C_p\|\mathbf{u}_0 - \bar{\mathbf{u}}_0\|_{L^p}, \quad p \neq 2, \tag{2.44}$$

based on the linear theory by Brenner [31].

In this regard, it is important to identify good analytical frameworks for studying entropy solutions of multidimensional conservation laws (1.1), which are not in *BV*, or even in L^p. The most general framework is the space of divergence-measure fields, formulated recently in [67,69,83,84], which is based on the following class of entropy solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ or $L^p(\mathbb{R}^{d+1}_+)$, $1 \leqslant p \leqslant \infty$;

(ii) for any convex entropy–entropy flux pair $(\eta, \mathbf{q})$,

$$\partial_t\eta(\mathbf{u}) + \nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{u}) \leqslant 0$$

in the sense of distributions, as long as $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t, \mathbf{x})$ is a distributional field.

According to the Schwartz lemma, we have

$$\operatorname{div}_{(t,\mathbf{x})}\big(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})\big) \in \mathcal{M},$$

which implies that the vector field

$$\big(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})\big)(t, \mathbf{x})$$

is a divergence measure field. We will discuss a theory of such fields in Section 8.

3. One-dimensional Euler equations

In this section, we present some aspects of a well-posedness theory and related results for the one-dimensional Euler equations.

3.1. *Isentropic Euler equations*

Consider the Cauchy problem for the isentropic Euler equations (1.16) with initial data

$$(\rho, m)|_{t=0} = (\rho_0, m_0)(x), \tag{3.1}$$

where (ρ_0, m_0) is in the physical region $\{(\rho, m)\colon\ \rho \geqslant 0, |m| \leqslant C_0\rho\}$ for some $C_0 > 0$. The pressure function $p(\rho)$ is a smooth function in $\rho > 0$ (nonvacuum states) satisfying

$$p'(\rho) > 0, \qquad \rho p''(\rho) + 2p'(\rho) > 0 \quad \text{when } \rho > 0, \tag{3.2}$$

and

$$p(0) = p'(0) = 0, \qquad \lim_{\rho\to 0} \frac{\rho p^{(j+1)}(\rho)}{p^{(j)}(\rho)} = c_j > 0, \quad j = 0, 1. \tag{3.3}$$

More precisely, we consider a general situation of the pressure law that there exist a sequence of exponents

$$1 < \gamma := \gamma_1 < \gamma_2 < \cdots < \gamma_J \leqslant \frac{3\gamma - 1}{2} < \gamma_{J+1}$$

and a function $P(\rho)$ such that

$$\begin{aligned} &p(\rho) = \sum_{j=1}^{J} \kappa_j \rho^{\gamma_j} + \rho^{\gamma_{J+1}} P(\rho), \\ &\limsup_{\rho\to 0}\big(\big|P(\rho)\big| + \big|\rho^3 P'''(\rho)\big|\big) < \infty, \end{aligned} \tag{3.4}$$

for some κ_j, $j = 1, \ldots, J$, with $\kappa_1 > 0$. For a polytropic gas obeying the γ-law (1.12), or a mixed ideal polytropic fluid,

$$p(\rho) = \kappa_1 \rho^{\gamma_1} + \kappa_2 \rho^{\gamma_2}, \quad \kappa_2 > 0,$$

the pressure function clearly satisfies (3.2)–(3.4).

System (1.16) is strictly hyperbolic at the nonvacuum states $\{(\rho, v)\colon\ \rho > 0, |v| < \infty\}$, and strict hyperbolicity fails at the vacuum states $\{(\rho, v)\colon\ \rho = 0, |v| < \infty\}$.

Let $(\eta, q)\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ be an entropy–entropy flux pair of system (1.16). An entropy $\eta(\rho, m)$ is called a weak entropy if $\eta = 0$ at the vacuum states.

In the coordinates (ρ, v), any weak entropy function $\eta(\rho, v)$ is governed by the second-order linear wave equation

$$\begin{cases} \eta_{\rho\rho} - k'(\rho)^2 \eta_{vv} = 0, & \rho > 0, \\ \eta|_{\rho=0} = 0, \end{cases} \tag{3.5}$$

with $k(\rho) = \int_0^\rho p'(s)/s \, \mathrm{d}s$.

In the Riemann invariant coordinates $\mathbf{w} = (w_1, w_2)$ defined as

$$w_1 = v + \int_0^\rho \frac{\sqrt{p'(y)}}{y} \, \mathrm{d}y, \qquad w_2 = v - \int_0^\rho \frac{\sqrt{p'(y)}}{y} \, \mathrm{d}y, \tag{3.6}$$

any entropy function $\eta(\mathbf{w})$ is governed by

$$\eta_{w_1 w_2} + \frac{\Lambda(w_1 - w_2)}{w_1 - w_2} (\eta_{w_1} - \eta_{w_2}) = 0, \tag{3.7}$$

where

$$\Lambda(w_1 - w_2) = -\frac{k(\rho) k''(\rho)}{k'(\rho)^2} \quad \text{with } \rho = k^{-1}\left(\frac{w_1 - w_2}{2}\right). \tag{3.8}$$

The corresponding entropy flux function $q(\mathbf{w})$ is

$$q_{w_j}(\mathbf{w}) = \lambda_i(\mathbf{w}) \eta_{w_j}(\mathbf{w}), \quad i \neq j. \tag{3.9}$$

In general, any weak entropy–entropy flux pair (η, q) can be represented by

$$\eta(\rho, v) = \int_{\mathbb{R}} \chi(\rho, v; s) a(s) \, \mathrm{d}s, \qquad q(\rho, v) = \int_{\mathbb{R}} \sigma(\rho, v; s) b(s) \, \mathrm{d}s, \tag{3.10}$$

for any continuous function $a(s)$ and related function $b(s)$, where the weak entropy kernel and entropy flux kernel are determined by

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2 \chi_{vv} = 0, \\ \chi(0, v; s) = 0, \qquad \chi_\rho(0, v; s) = \delta_{v=s} \end{cases} \tag{3.11}$$

and

$$\begin{cases} \sigma_{\rho\rho} - k'(\rho)^2 \sigma_{vv} = \frac{p''(\rho)}{\rho} \chi_v, \\ \sigma(0, v; s) = 0, \qquad \sigma_\rho(0, v; s) = v\delta_{v=s}, \end{cases} \tag{3.12}$$

respectively, with $\delta_{v=s}$ the delta function concentrated at the point $v = s$.

The equations in (3.5)–(3.9) and (3.11)–(3.12) belong to the class of Euler–Poisson–Darboux-type equations. The main difficulty comes from the singular behavior of $\Lambda(w_1 - w_2)$ near the vacuum. In view of (3.8), the derivative of $\Lambda(w_1 - w_2)$ in the coefficients of (3.7) may blow up like $(w_1 - w_2)^{-(\gamma-1)/2}$ when $w_1 - w_2 \to 0$ in general, and its higher derivatives may be more singular, for which the classical theory of Euler–Poisson–Darboux equations does not apply (cf. [19,341,342]). However, for a gas obeying the γ-law,

$$\Lambda(w_1 - w_2) = \lambda := \frac{3-\gamma}{2(\gamma-1)},$$

the simplest case, which excludes such a difficulty. In particular, for this case, the weak entropy kernel is

$$\chi(\rho, v; s) = \big[\big(w_1(\rho, v) - s\big)\big(s - w_2(\rho, v)\big)\big]_+^\lambda.$$

A mathematical theory for dealing with such a difficulty for the singularities of weak entropy kernel and entropy flux kernel can be found in [74,75].

A bounded measurable function $\mathbf{u}(t, x) = (\rho, m)(t, x)$ is called an entropy solution of (1.16) and (3.1)–(3.3) in $\mathbb{R}^2_+$ if $\mathbf{u}(t, x)$ satisfies the following:

(i) there exists $C > 0$ such that

$$0 \leqslant \rho(t, x) \leqslant C, \qquad \big|m(t, x)\big| \leqslant C\rho(t, x); \tag{3.13}$$

(ii) the entropy inequality

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leqslant 0 \tag{3.14}$$

holds in the sense of distributions in $\mathbb{R}^2_+$ for any convex weak entropy–entropy flux pair $(\eta, q)(\rho, m)$.

Notice that $\eta(\rho, m) = \pm\rho, \pm m$ are trivial convex entropy functions so that (3.14) automatically implies that $(\rho, m)(t, x)$ is a weak solution in the sense of distributions.

THEOREM 3.1. *Consider the Euler equations* (1.16) *with* (3.2)–(3.4). *Let* $(\rho^h, m^h)(t, x)$, $h > 0$, *be a sequence of functions satisfying the following conditions*:

(i) *there exists* $C > 0$ *such that*

$$0 \leqslant \rho^h(t, x) \leqslant C, \qquad \big|m^h(t, x)\big| \leqslant C\rho^h(t, x) \quad \textit{for a.e. } (t, x); \tag{3.15}$$

(ii) *for any weak entropy–entropy flux pair* (η, q),

$$\partial_t \eta\big(\rho^h, m^h\big) + \partial_x q\big(\rho^h, m^h\big) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big). \tag{3.16}$$

Then the sequence $(\rho^h, m^h)(t,x)$ *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, *that is, there exist* $(\rho, m) \in L^\infty$ *and a subsequence* (*still denoted*) $(\rho^h, m^h)(t,x)$ *such that*

$$(\rho^h, m^h) \to (\rho, m) \quad \textit{in } L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) \textit{ as } h \to 0,$$

and

$$0 \leqslant \rho(t,x) \leqslant C, \qquad \big|m(t,x)\big| \leqslant C\rho(t,x).$$

REMARK 3.1. The compactness framework in Theorem 3.1 was established to replace the *BV* compactness framework (the Helly theorem), for which the detailed proof can be found in [75]. For a gas obeying the γ-law, the case $\gamma = (N+2)/N$, $N \geqslant 5$ odd, was first treated by DiPerna [123], and the general case $1 < \gamma \leqslant 5/3$ for usual gases was first completed by Chen [50] and Ding, Chen and Luo [115]. The cases $\gamma \geqslant 3$ and $5/3 < \gamma < 3$ were treated via kinetic formulation by Lions, Perthame and Tadmor [223] and Lions, Perthame and Souganidis [222], respectively.

In order to establish Theorem 3.1, it requires to establish the corresponding reduction theorem: A Young measure satisfying the Tartar commutation relations

$$\begin{aligned}
&\langle \nu_{(t,x)}, \eta_1 q_2 - \eta_2 q_1\rangle \\
&\quad = \langle \nu_{(t,x)}, \eta_1\rangle\langle \nu_{(t,x)}, q_2\rangle - \langle \nu_{(t,x)}, \eta_2\rangle\langle \nu_{(t,x)}, q_1\rangle \quad \text{for a.e. } (t,x),
\end{aligned} \tag{3.17}$$

for all weak entropy–entropy flux pairs is a Dirac mass. These conditions are derived by the method of compensated compactness, especially the div–curl lemma (see [318,319] and [258,260]). The proof was based on new properties of *cancellation of singularities* of the entropy kernel χ and the entropy flux kernel σ in the following combination

$$E(\rho, v; s_1, s_2) := \chi(\rho, v; s_1)\sigma(\rho, v; s_2) - \chi(\rho, v; s_2)\sigma(\rho, v; s_1),$$

the fractional derivative technique first introduced in [50,115], and an important observation that the following identity is an elementary consequence of the symmetric form of (3.17)

$$\begin{aligned}
&\big\langle \nu_{(t,x)}, \chi(\rho, v; s_1)\big\rangle\big\langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1}\partial_{s_3}^{\lambda+1} E(\rho, v; s_2, s_3)\big\rangle \\
&\quad + \big\langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1}\chi(\rho, v; s_2)\big\rangle\big\langle \nu_{(t,x)}, \partial_{s_3}^{\lambda+1} E(\rho, v; s_3, s_1)\big\rangle \\
&\quad + \big\langle \nu_{(t,x)}, \partial_{s_3}^{\lambda+1}\chi(\rho, v; s_3)\big\rangle\big\langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1} E(\rho, v; s_1, s_2)\big\rangle = 0
\end{aligned} \tag{3.18}$$

for all s_1, s_2 and s_3, where the derivatives are understood in the sense of distributions (also see [222,223]). It was proved that, when $s_2, s_3 \to s_1$, the second and the third terms converge in the weak-star sense of measures to the *same* term but with opposite sign. The first term is more *singular* and contains the products of functions of bounded variation by bounded measures, which are known to depend upon regularization. The first term in (3.18) converges to a nontrivial limit which is determined explicitly. Finally, the genuine nonlinearity on $p(\rho)$ is required to conclude that the Young measure ν either reduces to a Dirac mass or is supported on the vacuum line.

This compactness framework has successfully been applied for proving the convergence of the Lax–Friedrichs scheme [50,115], the Godunov scheme [116], the vanishing viscosity method [68,222] and for establishing the compactness of solution operators and the decay of periodic solutions [65,75]. Also see the references cited in [86]. In particular, we have the following theorem.

THEOREM 3.2 (Existence, compactness and decay). *Assume that there exists* $C_0 > 0$ *such that the initial data* $(\rho_0, m_0)(x)$ *satisfies*

$$0 \leqslant \rho_0(x) \leqslant C_0, \qquad \left|m_0(x)\right| \leqslant C_0 \rho_0(x).$$

Then

(i) *there exists a global solution* $(\rho, m)(t, x)$ *of the Cauchy problem for* (1.16) *satisfying* (3.13), *for some* C *depending only on* C_0 *and* γ, *and* (3.14) *in the sense of distributions for any convex weak entropy–entropy flux pairs* (η, q);

(ii) *the solution operator* $(\rho, m)(t, \cdot) = S_t(\rho_0, m_0)(\cdot)$, *determined by* (i), *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ *for* $t > 0$;

(iii) *furthermore, if the initial data* $(\rho_0, m_0)(x)$ *is periodic with period* P, *then there exists a global periodic solution* $(\rho, m)(t, x)$ *with* (3.13) *such that* $(\rho, m)(t, x)$ *asymptotically decays in* L^1 *to*

$$\frac{1}{|P|} \int_P (\rho_0, m_0)(x)\,\mathrm{d}x.$$

REMARK 3.2. All the results for entropy solutions to (1.16) in Eulerian coordinates can be presented equivalently as the corresponding results for entropy solutions to (1.17) in Lagrangian coordinates (see [52] and [332]).

REMARK 3.3. If the initial data is in *BV* and has small oscillation, or $(\gamma - 1)TV(\rho_0, m_0)$ is sufficiently small, away from vacuum, the solution is in *BV*; see [118,145,263]. In the direction of relaxing the requirement of small total variation for (1.16), see [117,287,322, 323,349]. For extensions to initial–boundary value problems, see [68,229,264,315].

REMARK 3.4. Consider the isentropic Euler equations (1.16) in nonlinear elasticity with $p = -\sigma(\tau) \in C^2(\mathbb{R})$, $\sigma'(\tau) > 0$, satisfying that

$$\operatorname{sign}\bigl((\tau - \hat{\tau})\sigma''(\tau)\bigr) \geqslant 0, \tag{3.19}$$

$$\text{there is no interval in which } \sigma \text{ is affine,} \tag{3.20}$$

and there exists a positive integer $m \in \mathcal{Z}_+$ such that, in an interval $(\hat{\tau}, \hat{\tau} + \delta)$ or $(\hat{\tau} - \delta, \hat{\tau})$ for some $\delta > 0$,

$$\sum_{k=1}^{m} \left|\sigma^{(2k)}(\tau)\right| \neq 0. \tag{3.21}$$

Then the existence, compactness and decay of entropy solutions in L^∞ has been established in [78], and the results have been extended to the equations of motion of viscoelastic media with memory in [59,78]. In the simplest model for common rubber, condition (3.19) implies that the stress σ as a function of the strain τ switches from concave in the compressive mode $\tau < \hat{\tau}$ to convex in the expansive mode $\tau > \hat{\tau}$.

3.2. *Isothermal Euler equations*

For the isothermal Euler equation (1.16) with $\gamma = 1$, we have entropy–entropy flux pairs in the form

$$\begin{aligned} \eta &= \rho^{1/(1-\xi^2)} \exp\left\{\frac{\xi}{1-\xi^2}\frac{m}{\rho}\right\}, \\ q &= \left(\frac{m}{\rho} + \xi\right)\rho^{1/(1-\xi^2)} \exp\left\{\frac{\xi}{1-\xi^2}\frac{m}{\rho}\right\}, \end{aligned} \tag{3.22}$$

which satisfy

$$\eta_{\rho\rho}\eta_{mm} - \eta_{\rho m}^2 = \frac{\xi^4}{(1-\xi^2)^3}\rho^{2\xi^2/(1-\xi^2)-2} \exp\left\{\frac{2\xi}{1-\xi^2}\frac{m}{\rho}\right\} \quad \text{for } \xi \in \mathbb{R}. \tag{3.23}$$

Then η is a weak and convex entropy for any $\xi \in (-1, 1)$.

We have the following similar compensated compactness framework for this case.

THEOREM 3.3. *Assume that* $(\rho^h, m^h)(t, x), h > 0,$ *is a sequence of functions satisfying the following conditions*:

(i) *there exists some constant* $C > 0$ *such that*

$$0 \leqslant \rho^h(t,x) \leqslant C, \qquad \left|m^h(t,x)\right| \leqslant \rho^h(t,x)\big(C + \left|\ln \rho^h(t,x)\right|\big) \quad \textit{a.e.};$$

(ii) *the sequence of entropy dissipation measures*

$$\partial_t \eta\big(\rho^h, m^h\big) + \partial_x q\big(\rho^h, m^h\big) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$$

for any entropy–entropy flux pair (η, q) *in* (3.22) *with* $\xi \in (-1, 1)$.

Then the sequence $(\rho^h, m^h)(t, x)$ *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, *that is, there exist* $(\rho, m) \in L^\infty$ *and a subsequence* (*still denoted*) (ρ^h, m^h) *such that*

$$\big(\rho^h, m^h\big) \to (\rho, m) \quad \text{in } L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) \text{ as } h \to 0,$$

and

$$0 \leqslant \rho(t, x) \leqslant C, \qquad \big|m(t, x)\big| \leqslant \rho(t, x)\big(C + \big|\ln \rho(t, x)\big|\big).$$

REMARK 3.5. This compactness framework was first established in [172]. Another proof was also provided recently in [205] by employing the approach in [75].

For strictly hyperbolic systems with smooth fluxes, the H^{-1}-compactness condition is easy to be obtained, due to the uniform boundedness of approximate solutions and Murat's lemma [259], provided that the system has a strictly convex entropy. Similar to the isentropic case, it is not clear for the case $\gamma = 1$ whether the strong entropy–entropy flux pairs satisfy the H^{-1}-compactness condition. Furthermore, for the isothermal case, the propagation speed may not be finite due to the presence of vacuum and the entropy equation is not of EPD type, which is different from the isentropic case.

The key point in the proof of [172] is to establish the commutation relations for not only the *weak* entropy–entropy flux pairs but also the *strong* ones by using the *analytic extension theorem* even though it is not known whether strong entropy–entropy flux pairs satisfy the H^{-1}-compactness condition. To achieve this, formula (3.22) of entropies parameterized by a complex variable ξ was used, which includes both weak and strong entropies determined by the value of ξ. It was shown that, for any $\xi \in (-1, 1)$, the weak entropy–entropy flux pair satisfies the H^{-1}-compactness condition. Therefore, the commutation relations are satisfied for these weak entropy–entropy flux pairs. It was observed that the two sides of the commutation relations are regular in ξ and are analytic functions with respect to ξ, which implies that the commutation relations exactly hold for the whole complex variable except two points $(-1, 0)$ and $(1, 0)$ by using the analytic extension theorem. Noting that the entropies are strong if $|\xi| > 1$ (see (3.22)), therefore, the commutation relations hold for these weak and strong entropy–entropy flux pairs so that the H^{-1}-compactness condition for strong entropy–entropy flux pairs can be bypassed. Since both weak and strong entropy–entropy flux pairs are applied to the commutation relations, the reduction theorem for the corresponding Young measure was obtained as that in the strictly hyperbolic case in [124,290], which implies the compensated compactness framework. The proof of [205] employed the approach described in Section 3.1 for the isentropic case by using only the weak entropy–entropy flux pairs.

As an application of Theorem 3.3, we have the following theorem.

THEOREM 3.4 (Existence). *Assume that the initial data satisfies*

$$0 \leqslant \rho_0(x) \leqslant C_0, \qquad \big|m_0(x)\big| \leqslant \rho_0(x)\big(C_0 + \big|\log \rho_0(x)\big|\big) \quad \text{a.e.} \tag{3.24}$$

for some constant $C_0 > 0$. *Then there exists a global entropy solution of* (1.16) *and* (3.1) (*with* $\gamma = 1$) *satisfying*

$$0 \leqslant \rho(t,x) \leqslant C, \qquad \left|m(t,x)\right| \leqslant \rho(t,x)\left(C + \left|\log \rho(t,x)\right|\right) \quad a.e., \tag{3.25}$$

where $C > 0$ *depends only on* C_0.

REMARK 3.6. The convergence of the viscosity method was established in [172]. Unlike the isentropic case, the eigenvalues of the system are no longer bounded near vacuum (which may grow with the speed $|\ln \rho|$), the construction of shock capturing scheme is more delicate since the Courant–Friedrichs–Lewy stability condition may fail for standard shock capturing schemes. In [77], such a shock capturing scheme was successfully developed and its strong convergence was established by introducing a cut-off technique to modify the approximate density functions and adjust the ratio of the time and space mesh sizes to construct the shock capturing scheme.

REMARK 3.7. Away from vacuum, the first result on the existence of *BV* solutions with large initial data was obtained in Nishida [262] by using the Glimm scheme [145] for $\gamma = 1$. Poupaud, Rascle and Vila [274] made further simplification and improved the results of [262] to the isothermal Euler–Poisson system. The existence result in Theorem 3.4 allows the initial data (ρ_0, m_0) only in L^∞, which may even contain vacuum.

3.3. *Adiabatic Euler equations*

For the full Euler equations in gas dynamics (1.15) with the following Cauchy problem

$$(\tau, v, S)|_{t=0} = (\tau_0, v_0, S_0)(x), \tag{3.26}$$

the following existence theorem holds which is due to Liu [232] (also see [85] and [321]).

THEOREM 3.5. *Let* $K \subset \{(\tau, v, S)\colon\ \tau > 0\}$ *be a compact set in* $\mathbb{R}_+ \times \mathbb{R}^2$, *and let* $N \geqslant 1$ *be any positive constant. Then there exists a constant* $C_0 = C_0(K, N)$, *independent of* $\gamma \in (1, 5/3]$, *such that, for every initial data* $(\tau_0, v_0, S_0)(x) \in K$ *with* $TV_{\mathbb{R}}(\tau_0, v_0, S_0) \leqslant N$, *when*

$$(\gamma - 1) TV_{\mathbb{R}}(\tau_0, v_0, S_0) \leqslant C_0$$

for any $\gamma \in (1, 5/3]$, *the Cauchy problem* (1.15) *and* (3.26) *has a global entropy solution* $(\tau, v, S)(t,x)$ *which is bounded and satisfies*

$$TV_{\mathbb{R}}(\tau, v, S)(t, \cdot) \leqslant C\, TV_{\mathbb{R}}(\tau_0, v_0, S_0)$$

for some constant $C > 0$ *independent of* γ.

In the direction of relaxing the requirement of small total variation for (1.15), see [268,287,322,323]. For extensions to initial–boundary value problems, see [68,229, 264,315].

For the decay of entropy solutions in BV_{loc} with periodic data or compact support, see [111,119,121,149,225,226]; also see [65] for entropy solutions only in L^∞. For additional further discussions and references to the Glimm scheme, see [111,235,294]; also see [85].

Furthermore, we have the following theorem.

THEOREM 3.6. *If the initial data functions* $\mathbf{u}_0(x)$ *and* $\mathbf{v}_0(x)$ *have sufficiently small total variation and* $\mathbf{u}_0 - \mathbf{v}_0 \in L^1(\mathbb{R})$, *then, for the corresponding exact Glimm, or wave-front tracking, or vanishing viscosity solutions* $\mathbf{u}(t,x)$ *and* $\mathbf{v}(t,x)$ *of the Cauchy problem* (1.1) *and* (2.10) ($d=1$), *there exists a constant* $C>0$ *such that*

$$\left\|\mathbf{u}(t,\cdot)-\mathbf{v}(t,\cdot)\right\|_{L^1(\mathbb{R})} \leqslant C\|\mathbf{u}_0-\mathbf{v}_0\|_{L^1(\mathbb{R})} \quad \textit{for all } t>0. \tag{3.27}$$

An immediate consequence of this theorem is that the whole sequence of approximate solutions constructed by the Glimm scheme, as well as the wave-front tracking method and the vanishing viscosity method, converges to a unique entropy solution of (1.1) and (2.10) ($d=1$) as the mesh size or the viscosity coefficient tends to zero. See also [32] for the uniqueness of limits of Glimm's random choice method. The details of the proof of Theorem 3.6 can be found in [20,33,236,238]. In the direction relaxing the requirement of small total variation for (1.1), see [207,208].

For other discussions and extensive references about the L^1-stability of BV entropy solutions and related problems, we refer to [33,111,167,204].

Furthermore, the uniqueness and stability of Riemann solutions in the class of entropy solutions with large variation satisfying only one entropy inequality for the strictly convex physical entropy S has been established in [70] as follows.

THEOREM 3.7. *Let* $\mathbf{u}(t,x)=(\tau,v,e+v^2/2)(t,x)$ *be an entropy solution of* (1.15) *and* (3.26) *in* $\Pi_T := \{(t,x)\colon\ 0\leqslant t\leqslant T\}$ *for some* $T\in(0,\infty)$, *which belongs to* $BV_{\mathrm{loc}}(\Pi_T;\mathcal{D})$ *with* $\mathcal{D}\subset\{(\tau,v,e+v^2/2)\colon\ \tau>0\}\subset\mathbb{R}^3$ *bounded. Let* $\mathbf{R}(x/t)$ *be the classical Riemann solution with Riemann data* $\mathbf{R}_0(x)$.

(i) *If* $\mathbf{u}_0=\mathbf{R}_0$, *then*

$$\mathbf{u}(t,x)=\mathbf{R}\left(\frac{x}{t}\right) \quad \textit{for a.e. } (t,x)\in\Pi_T.$$

(ii) *If* $\mathbf{u}_0-\mathbf{R}_0\in L^1\cap L^\infty\cap BV_{\mathrm{loc}}(\mathbb{R})$, *then*

$$\operatorname{ess}\lim_{t\to\infty}\int_{-L}^{L}\left|\mathbf{u}(t,\xi t)-\mathbf{R}(\xi)\right|\mathrm{d}\xi=0 \quad \textit{for any } L>0; \tag{3.28}$$

that is, the Riemann solution $\mathbf{R}(x/t)$ *is asymptotically stable in the sense* (3.28) *with respect to the corresponding initial perturbation in* $L^1\cap L^\infty\cap BV_{\mathrm{loc}}(\mathbb{R})$.

We now consider the 3×3 system of Euler equations (1.15) in Lagrangian coordinates in thermoelasticity with the following class of constitutive relations for the new state vector (τ, S) with the form

$$\begin{aligned} e &= \int_0^{\tau+\alpha S} \sigma(w)\,\mathrm{d}w + \beta S, \\ p &= -\sigma(\tau + \alpha S), \\ \theta &= \alpha\sigma(\tau + \alpha S) + \beta, \end{aligned} \tag{3.29}$$

where $\sigma(w)$ is a function with $\sigma'(w) > 0$, and α and β are positive constants. The model (3.29) is quite special. Even so, when we are dealing with solutions in which (τ, S) do not deviate far from some constant values $(\bar{\tau}, \overline{S})$, we may obtain a reasonable approximation for general constitutive relations (see [58])

$$e = \hat{e}(\tau, S), \qquad p = -\hat{\sigma}(\tau, S), \qquad \theta = \hat{\theta}(\tau, S) \tag{3.30}$$

satisfying the conditions

$$\hat{\sigma} = \hat{e}_\tau, \qquad \hat{\theta} = \hat{e}_S. \tag{3.31}$$

We also assume that, for some $\bar{w}$,

$$\sigma''(w) + 4\frac{\alpha\sigma'(w)^2}{\alpha\sigma(w) + \beta} \begin{cases} \leqslant 0 & \text{if } w < \bar{w}, \\ \geqslant 0 & \text{if } w > \bar{w}, \end{cases} \tag{3.32}$$

and

$$\sigma''(w) \neq 0 \quad \text{for } w > \bar{w}, \tag{3.33}$$

or there exists $\hat{w} > \bar{w}$ such that $\sigma(w)$ satisfies conditions (3.19)–(3.21) with $\hat{w}$ replacing $\hat{v}$.

Consider the Cauchy problem for (1.15) with initial data

$$(w, v, S)|_{t=0} = (w_0, v_0, S_0)(x) \tag{3.34}$$

for $w = \tau + \alpha S$.

THEOREM 3.8. *Assume*

$$(w_0, v_0)(x) \in \left\{(w, v)\colon \left| v \pm \int_{\hat{w}}^{w} \sqrt{\sigma'(\omega)}\,\mathrm{d}\omega \right| \leqslant C_0 \right\}$$

and $S_0(x) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$. *Then*

(i) *there exists a distributional solution*

$$(w, v, S)(t, x) \in L^{\infty}\big(\mathbb{R}^2_+; \mathbb{R}^2\big) \times \mathcal{M}_{\mathrm{loc}}\big(\mathbb{R}^2_+; \mathbb{R}\big)$$

of (1.15) *and* (3.34) *satisfying*

$$\begin{aligned} &S_t(t, x) \in \mathcal{M}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big), \qquad \theta\big(w(t, x)\big) \geqslant 0, \\ &|S|\big([0, T_0] \times \big\{|x| \leqslant cT_0\big\}\big) \leqslant CT_0^2 \end{aligned} \tag{3.35}$$

for any $c, T_0 > 0$, *with* $C > 0$ *independent of* T_0. *Moreover,* $(w, v, S)(t, x)$ *satisfies the entropy condition*

$$\partial_t \eta(w, v) + \partial_x q(w, v) \leqslant 0, \qquad S_t \geqslant 0 \tag{3.36}$$

in the sense of distributions for any C^2 *entropy–entropy flux pair* $(\eta, q)(w, v)$ *of the system*

$$\partial_t w - \partial_x v = 0, \qquad \partial_t v - \partial_x \sigma(w) = 0,$$

for which the following strong convexity condition holds:

$$\begin{aligned} &\theta \eta_{ww} - \alpha \sigma'(w) \eta_w \geqslant 0, \\ &\theta \eta_{vv} - \alpha \eta_w \geqslant 0, \\ &\big(\theta \eta_{ww} - \alpha \sigma'(w) \eta_w\big)(\theta \eta_{vv} - \alpha \eta_w) - \eta_{ww}^2 \geqslant 0; \end{aligned}$$

(ii) *any sequence* $(w^h, v^h)(t, x)$ *that is uniformly bounded in* $h > 0$ *and satisfies* (3.36) *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ *when* $t > 0$;

(iii) *furthermore, if the initial data* $(w_0, v_0, S_0)(x)$ *is periodic with period* P, *then there exists a periodic entropy solution* $(\tau, v, S)(t, x)$ *of* (1.15) *and* (3.34) *with period* P *satisfying*

$$(v, \tau + \alpha S) \in L^{\infty}\big(\mathbb{R}^2_+\big),$$

(3.35) *and* (3.36). *Moreover, the velocity* $v(t, x)$, *the pressure* $p(w(t, x))$ *and the temperature* $\theta(w(t, x))$ *asymptotically decay in* L^1 *to*

$$\bar{v} = \frac{1}{|P|} \int_P v_0(x) \, \mathrm{d}x$$

and

$$\begin{aligned} \tilde{p} &= p\bigg(\Theta^{-1}\bigg(\frac{1}{|P|} \int_P \Theta\big(w_0(x)\big) \, \mathrm{d}x\bigg)\bigg), \\ \tilde{\theta} &= \theta\bigg(\Theta^{-1}\bigg(\frac{1}{|P|} \int_P \Theta\big(w_0(x)\big) \, \mathrm{d}x\bigg)\bigg), \end{aligned}$$

respectively, where $\Theta(w) = \beta w + \alpha \int_0^w \sigma(\omega)\,\mathrm{d}\omega$.

REMARK 3.8. The first existence theorem for global entropy solutions for (1.15) and (3.29)–(3.33) was established in [58]. The existence result was extended in [65] and [78] to the existence, compactness and decay of entropy solutions of (1.15) and (3.29)–(3.33) under the weaker conditions (3.19)–(3.21) with $\hat{w}$ replacing $\hat{v}$.

REMARK 3.9. An interesting feature here is that, because of linear degeneracy of the second characteristic field of (1.15) and (3.29)–(3.33), one cannot expect the decay of all components of the solutions. However, some important quantities such as the velocity, the pressure, and the temperature do decay as $t \to \infty$.

4. Multidimensional Euler equations and related models

Multidimensional problems for the Euler equations are extremely rich and complicated. Some great developments and progress have been made in the recent decades through strong and close interdisciplinary interactions and diverse approaches including

(i) experimental data,
(ii) large and small scale computing by a search for effective numerical methods,
(iii) asymptotic and qualitative modeling,
(iv) rigorous proofs for prototype problems and an understanding of the solutions.

In some sense, the developments and progress made by using approach (iv) are behind those by using the other approaches (i)–(iii) (see [150]); however, most scientific problems are considered to be solved satisfactorily only after approach (iv) is achieved.

In this section, together with Sections 5–7, we give some samples of multidimensional models and problems for the Euler equations with emphasis on those prototype models and problems that have been solved or expected to be solved rigorously at least for some cases.

Since the multidimensional problems are so complicated in general, a natural strategy to attack these problems as a first step is to study

(i) simpler nonlinear models with strong physical motivations,
(ii) special, concrete nonlinear physical problems.

Meanwhile, extend the results and ideas from the first step to study

(i) the Euler equations in gas dynamics and elasticity,
(ii) more general problems,
(iii) nonlinear systems that the Euler equations are the main subsystem or describe the dynamics of macroscopic variables such as Navier–Stokes equations, MHD equations, combustion equations, Euler–Poisson equations, kinetic equations especially including the Boltzmann equation, among others.

In this section we first focus on some samples of multidimensional models for the Euler equations and related multidimensional hyperbolic conservation laws.

4.1. *The potential flow equation*

This approximation is well known in transonic aerodynamics, beyond the isentropic approximation (1.11) from (1.4). Denote

$$\mathrm{D}_t = \partial_t + \sum_{k=1}^{d} v_k\, \partial_{x_k},$$

the convective derivative along fluid particle trajectories. From (1.4), we have

$$\mathrm{D}_t S = 0 \tag{4.1}$$

and, by taking the curl of the momentum equations and using vector identities,

$$\mathrm{D}_t\left(\frac{\omega}{\rho}\right) = \frac{\omega}{\rho}\cdot\nabla\mathbf{v} + \frac{p_S(\rho, S)}{\rho^3}\nabla\rho\times\nabla S. \tag{4.2}$$

The identities in (4.1) and (4.2) imply that a smooth solution of (1.4) which is both isentropic and irrotational at time $t = 0$ remains isentropic and irrotational for all later time, as long as this solution stays smooth. Then the conditions $S = S_0 = \text{const}$ and $\operatorname{curl}\mathbf{v} = 0$ are reasonable for smooth solutions.

For a smooth irrotational solution of (1.4), we integrate the d-momentum equations in (1.11) through Bernoulli's law

$$\partial_t\mathbf{v} + \frac{1}{2}\nabla\big(|\mathbf{v}|^2\big) + \nabla i(\rho) = 0,$$

where $i'(\rho) = p_\rho(\rho, S_0)/\rho$.

On a simply connected space region, the condition $\operatorname{curl}\mathbf{v} = 0$ implies that there exists Φ such that

$$\mathbf{v} = \nabla\Phi.$$

Then we have

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\nabla\Phi) = 0, \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + i(\rho) = K, \end{cases} \tag{4.3}$$

where K is the Bernoulli constant, which is usually determined by the boundary conditions if such conditions are prescribed. From the second equation in (4.3), we have

$$\rho(\mathrm{D}\Phi) = i^{-1}\bigg(K - \bigg(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2\bigg)\bigg).$$

Then system (4.3) can be rewritten as the following time-dependent potential flow equation of second order

$$\partial_t \rho(\mathrm{D}\Phi) + \nabla \cdot \big(\rho(\mathrm{D}\Phi)\nabla\Phi\big) = 0. \tag{4.4}$$

For a steady solution $\Phi = \varphi(\mathbf{x})$, i.e., $\partial_t \Phi = 0$, we obtain the celebrated steady potential flow equation of aerodynamics

$$\nabla \cdot \big(\rho(\nabla\varphi)\nabla\varphi\big) = 0. \tag{4.5}$$

In applications in aerodynamics, (4.3) or (4.4) is used for discontinuous solutions, and the empirical evidence is that entropy solutions of (4.3) or (4.4) are fairly good approximations to entropy solutions for (1.4) provided that

(i) the shock strengths are small,
(ii) the curvature of shock fronts is not too large,
(iii) there is a small amount of vorticity in the region of interest.

The advantages of equation (4.4), or equivalently (4.3), as the simplest multidimensional prototype conservation laws include (cf. [242])

(i) unidirectional plane wave solutions of (4.4) reduce to solutions of a 2×2 system of conservation laws with the structure of a wave equation,

(ii) the linear structure of (4.4) is strictly hyperbolic with characteristics defined by a single light cone in several space variables,

(iii) under reasonable thermodynamic assumptions such as an ideal gas law (1.12), the system for (4.4) is genuinely nonlinear in all wave directions simultaneously and the corresponding multidimensional shock fronts are uniformly stable,

(iv) this system has the vorticity waves removed unlike (1.4) and (1.11). Such vorticity waves are linearly degenerate wave fields but represent an enormous source of instability in multidimension through Kelvin–Helmhotz instability.

The model (4.4) or (4.3) is an excellent model to capture multidimensional shock waves by ignoring vorticity waves, while the model (the incompressible Euler equations) in Section 4.2 is an excellent model to capture multidimensional vorticity waves by ignoring shock waves in fluid flow.

4.2. *Incompressible Euler equations*

In the homogeneous case, the incompressible Euler equations take the form

$$\begin{cases} \partial_t \mathbf{v} + \mathrm{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \mathrm{div}\, \mathbf{v} = 0. \end{cases} \tag{4.6}$$

This can formally be obtained from (1.11) by setting $\rho = 1$ as the equation of state and regarding p as an unknown function. As indicated above, the model (4.6) excludes the appearance of shock waves in fluid flow to capture multidimensional vorticity waves.

In the inhomogeneous case, the incompressible Euler equations are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{4.7}$$

These models can be obtained by formal asymptotics for low Mach number expansions from the compressible Euler equations. For more details, see [95,98,166,220,221,243] and the references cited therein.

4.3. *The transonic small disturbance equation*

A further simpler model than the potential flow equation in transonic aerodynamics is the unsteady transonic small disturbance equation or so-called the two-dimensional inviscid Burgers equation (see [97]),

$$\begin{cases} \partial_t u + \partial_x\left(\frac{1}{2}u^2\right) + \partial_y v = 0, \\ \partial_y u - \partial_x v = 0, \end{cases} \tag{4.8}$$

or in the form of Zabolotskaya–Khokhlov equation [346],

$$\partial_t(\partial_t u + u\,\partial_x u) + \partial_{yy} u = 0. \tag{4.9}$$

The equations in (4.8) describe the potential flow field near the reflection point in weak shock reflection, which determines the leading-order approximation of geometric optical expansions; and it can also be used to formulate asymptotic equations for the transition from regular to Mach reflection for weak shocks. See [173–175,252] and the references cited therein.

Equation (4.9) arises in many different situations. It was first derived by Timman in the context of transonic flows [325]. In nonlinear acoustics, it was derived by Zabolotskaya and Khokhlov [346] and is used to describe the diffraction of nonlinear acoustic beams [164]. Motivated by the experiments of Sturtevant and Kulkarny [310] on the focusing of shocks, Cramer and Seebass [102] used (4.9) to study caustics in nearly planar sound waves. The same equation arises as a weakly nonlinear equation for cusped caustics [174]. Hunter [173] also showed that (4.8) describes high-frequency waves near singular rays.

4.4. *Pressure-gradient equations*

The inviscid fluid motions are driven mainly by the pressure gradient and the fluid convection (i.e., transport). As for modeling, it is natural to study first the effect of the two driving factors separately. Such an idea has also been used by Argarwal and Halt [1] to formulate a flux-splitting scheme in numerical computations for airfoil flows.

Separating the pressure gradient from the Euler equations, we first have the pressure-gradient system

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t(\rho u) + \partial_x p = 0, \\ \partial_t(\rho v) + \partial_y p = 0, \\ \partial_t(\rho E) + \partial_x(up) + \partial_y(vp) = 0. \end{cases} \tag{4.10}$$

We may choose $\rho = 1$. Setting

$$p = (\gamma - 1)P, \qquad t = \frac{s}{\gamma - 1},$$

then we have the following pressure-gradient equations

$$\begin{cases} \partial_s u + \partial_x P = 0, \\ \partial_s v + \partial_y P = 0, \\ \partial_s(\ln P) + \partial_x u + \partial_y v = 0. \end{cases} \tag{4.11}$$

Eliminating the velocity (u, v), we obtain the following nonlinear wave equation for P:

$$\partial_{ss}(\ln P) - \Delta P = 0. \tag{4.12}$$

Although system (4.11) is obtained from the splitting idea, system (4.11) is a good approximation to the full Euler equations, especially when the velocity (u, v) is small and the adiabatic gas exponent $\gamma > 1$ is large (see [357]). This can be achieved by the formal expansion in terms of $\varepsilon = 1/(\gamma - 1)$

$$\begin{cases} \rho = \rho_1 + \varepsilon\rho_2 + \mathrm{O}(\varepsilon^2), \\ (u, v) = \varepsilon(u_1, v_1) + \mathrm{O}(\varepsilon^2), \\ p = \varepsilon p_1 + \mathrm{O}(\varepsilon^2). \end{cases}$$

Plugging the expansion into the Euler equations (1.4), we first compare the order of ε^2 and have

$$\partial_t \rho_1 = 0,$$

and so we may choose $\rho_1 = 1$. We then compare the order of ε and have

$$\begin{cases} \partial_t u_1 + \partial_x p_1 = 0, \\ \partial_t v_1 + \partial_y p_1 = 0, \\ \partial_t\left(\frac{p_1}{\gamma-1}\right) + p_1\, \partial_x u_1 + p_1\, \partial_y v_1 = 0. \end{cases} \tag{4.13}$$

Set

$$p_1 = (\gamma - 1)P, \qquad t = \frac{1}{\gamma - 1}\tau.$$

Then we have

$$\begin{cases} \partial_s u_1 + \partial_x P = 0, \\ \partial_s v_1 + \partial_y P = 0, \\ \partial_s (\ln P) + \partial_x u_1 + \partial_y v_1 = 0, \end{cases}$$

which is the same as (4.11) that leads to (4.12).

4.5. *Pressureless Euler equations*

With the pressure-gradient equations (4.11), the convection (i.e., transport) part of fluid flow forms the pressureless Euler equations

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) + \partial_y (\rho v) = 0, \\ \partial_t (\rho u) + \partial_x \left(\rho u^2\right) + \partial_y (\rho u v) = 0, \\ \partial_t (\rho v) + \partial_x (\rho u v) + \partial_y \left(\rho v^2\right) = 0, \\ \partial_t (\rho E) + \partial_x (\rho u E) + \partial_y (\rho v E) = 0. \end{cases} \tag{4.14}$$

This system also models the motion of free particles which stick under collision; see [30,127,348]. In general, solutions of (4.14) become measure solutions.

System (4.14) has been analyzed extensively; for example, see [26,27,30,127,161,172, 210–212,273,296,335] and the references cited therein. In particular, the existence of measure solutions of the Riemann problem was first presented in [26] for the one-dimensional case, and a connection of (4.14) with adhesion particle dynamics and the behavior of global weak solutions with random initial data were discussed in [127]. It has also been shown that δ-shocks and vacuum states do occur in the Riemann solutions even in the one-dimensional case. Since the two eigenvalues of the transport equations coincide, the occurrence of δ-shocks and vacuum states as $t > 0$ can be regarded as a result of resonance between the two characteristic fields. Such phenomena can also be regarded as the phenomena of concentration and cavitation in solutions to the Euler equations for compressible fluids as the pressure vanishes. It has shown in [79] for $\gamma > 1$ and [209] for $\gamma = 1$ that, as the pressure vanishes, any two-shock Riemann solution to the Euler equations tends to a δ-shock solution to (4.14) and the intermediate densities between the two shocks tend to a weighted δ-measure that forms the δ-shock. By contrast, any two-rarefaction-wave Riemann solution of the Euler equations has been shown in [79] to tend to a two-contact-discontinuity solution to (4.14), whose intermediate state between the two contact discontinuities is a vacuum state, even when the initial data stays away from the vacuum. Some numerical results exhibiting the formation process of δ-shocks and vacuum states have also been presented in [79].

4.6. *Euler equations in nonlinear elastodynamics*

The equations of nonlinear elastodynamics provide another excellent example of the rich special structure one encounters when dealing with hyperbolic systems of conservation laws. In three space dimensions, the state vector is $(\mathbf{v}, \mathbf{F})$, where $\mathbf{v} \in \mathbb{R}^3$ is the velocity vector and $\mathbf{F}$ is the 3×3 matrix-valued deformation gradient constrained by the requirement $\det \mathbf{F} > 0$. The system of conservation laws, which express the integrability conditions between $\mathbf{v}$ and $\mathbf{F}$ and the balance of linear momentum, reads

$$\begin{cases} \partial_t F_{i\alpha} - \partial_{x_\alpha} v_i = 0, & i, \alpha = 1, 2, 3, \\ \partial_t v_j - \sum_{\beta=1}^3 \partial_{x_\beta} S_{j\beta}(F) = 0, & j = 1, 2, 3. \end{cases} \tag{4.15}$$

The symbol $\mathbf{S}$ stands for the *Piola–Kirchhoff stress tensor*, which is determined by the (scalar-valued) *strain energy function* $\sigma(\mathbf{F})$,

$$S_{j\beta}(\mathbf{F}) = \frac{\partial \sigma(\mathbf{F})}{\partial F_{j\beta}}.$$

System (4.15) is hyperbolic if and only if

$$\sum_{i,j=1}^3 \sum_{\alpha,\beta=1}^3 \frac{\partial^2 \sigma(\mathbf{F})}{\partial F_{i\alpha}\, \partial F_{j\beta}} \xi_i \xi_j n_\alpha n_\beta > 0 \tag{4.16}$$

for any vectors $\boldsymbol{\xi}, \mathbf{n} \in \mathbf{S}^3$.

System (4.15) is endowed with an entropy–entropy flux pair

$$\eta = \sigma(\mathbf{F}) + \frac{1}{2}|\mathbf{v}|^2, \qquad q_\alpha = -\sum_{j=1}^3 v_j S_{j\alpha}(\mathbf{F}).$$

However, the laws of physics do not allow $\sigma(\mathbf{F})$, and thereby η, to be convex functions. Indeed, convexity of σ would violate the principle of *material frame indifference*

$$\sigma(\mathbf{OF}) = \sigma(\mathbf{F}) \quad \text{for all } \mathbf{O} \in SO(3),$$

and would also be incompatible with the natural requirement that $\sigma(\mathbf{F}) \to \infty$ as $\det \mathbf{F} \downarrow 0$ or $\det \mathbf{F} \uparrow \infty$ (see [106]). Consequently, the useful results on the local existence of classical solutions to the Cauchy problem and the uniqueness of classical solutions in the context of weak solutions that are available for hyperbolic systems of conservation laws endowed with a convex entropy in Section 2.1 are not directly applicable to system (4.15).

The failure of σ to be convex is also the main source of complication in elastostatics, where one is seeking to determine equilibrium configurations of the body by minimizing the total strain energy $\int \sigma(\mathbf{F})$. The following alternative conditions, weaker than convexity and physically reasonable, are relevant in that context [13]:

(i) *polyconvexity*,

$$\sigma(\mathbf{F}) = g\big(\mathbf{F}, \mathbf{F}^*, \det \mathbf{F}\big),$$

where $\mathbf{F}^*$ is the adjugate of $\mathbf{F}$ (the matrix of cofactors of $\mathbf{F}$), $\mathbf{F}^* = (\det \mathbf{F})\mathbf{F}^{-1}$, and $g(\mathbf{F}, \mathbf{G}, w)$ is a convex function of 19 variables,

(ii) *quasiconvexity* in the sense of Morrey [254],

(iii) *rank-one convexity*, expressed by (4.16).

It is known that convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-one convexity, however, none of the converse statements is generally valid. It is important to investigate the relevance of the above conditions in elastodynamics. A first start was made in [106] where it was shown that rank-one convexity suffices for the local existence of classical solutions, quasiconvexity yields the uniqueness of classical solutions in the context of the class of entropy-admissible weak solutions, and polyconvexity renders the system symmetrizable (also see [275]).

To achieve this for polyconvexity, one of the main ideas is to enlarge system (4.15) with the state vector $(\mathbf{v}, \mathbf{F})$ into a large, albeit equivalent, system for the new state vector $(\mathbf{v}, \mathbf{F}, \mathbf{F}^*, w)$ with $w = \det \mathbf{F}$

$$\partial_t w = \sum_{\alpha=1}^{3} \sum_{i=1}^{3} \partial_{x_\alpha}\big(F^*_{\alpha i} v_i\big), \tag{4.17}$$

$$\partial_t F^*_{\gamma k} = \sum_{\alpha,\beta=1}^{3} \sum_{i,j=1}^{3} \partial_{x_\alpha}(\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk} F_{j\beta} v_i), \quad \gamma, k = 1, 2, 3, \tag{4.18}$$

where $\varepsilon_{\alpha\beta\gamma}$ and ε_{ijk} denote the standard permutation symbols. Then the enlarged system with 21 equations, which consists of (4.15) augmented by (4.17) and (4.18), is endowed a uniformly convex entropy

$$\eta = \sigma\big(\mathbf{F}, \mathbf{F}^*, w\big) + \frac{1}{2}|\mathbf{v}|^2$$

so that the local existence of classical solutions and the stability of Lipschitz solutions may be inferred directly from Theorem 2.3. See [111,113,275] for more details.

4.7. *The Born–Infeld system in electromagnetism*

The Born–Infeld system is a nonlinear version of Maxwell equations,

$$\begin{cases} \partial_t B + \operatorname{curl} \frac{\partial W}{\partial D} = 0, \\ \partial_t D - \operatorname{curl} \frac{\partial W}{\partial B} = 0, \end{cases} \tag{4.19}$$

where $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is the given energy density. The Born–Infeld model corresponds to the special case

$$W_{\rm BI}(B, D) = \sqrt{1 + |B|^2 + |D|^2 + |P|^2}.$$

When W is strongly convex (i.e., $D^2 W > 0$), system (4.19) is endowed with a strictly convex entropy, which implies that the system is symmetric and hyperbolic and, therefore, the Cauchy problem is locally well posed in H^s for $s > 5/2$. However, $W_{\rm BI}$ is not convex for a large enough field.

As in Section 4.6, the Born–Infeld model is enlarged from 6 to 10 equations in [29], by adjunction of the conservation laws satisfied by $P := B \times D$ and W so that the augmented system turns out to be a set of conservation laws in the unknowns

$$(h, B, D, P) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3,$$

endowed with a strongly convex entropy, which is symmetric and hyperbolic,

$$\begin{cases} \partial_t h + \operatorname{div} P = 0, \\ \partial_t B + \operatorname{curl}\left(\frac{P \times B + D}{h}\right) = 0, \\ \partial_t D + \operatorname{curl}\left(\frac{P \times D - B}{h}\right) = 0, \\ \partial_t P + \operatorname{Div}\left(\frac{P \otimes P - B \otimes B - D \otimes D - I}{h}\right) = 0, \end{cases}$$

where I is the 3×3 identity matrix. The physical region is

$$\left\{(h, B, D, P)\colon\ P = D \times B, h = \sqrt{1 + |B|^2 + |D|^2 + |P|^2} > 0\right\}.$$

Also see [295] for another enlarged system consisting of 9 scalar evolution equations in 9 unknowns (B, D, P), where P stands for the relaxation of the expression $D \times B$.

4.8. *Lax systems*

Let $f(\mathbf{u})$ be an analytic function of a single complex variable $\mathbf{u} = u + v\mathrm{i}$. We impose on the complex valued function $\mathbf{u} = \mathbf{u}(t, z)$, $z = x + y\mathrm{i}$, and the real variable t the following nonlinear partial differential equation

$$\partial_t \bar{\mathbf{u}} + \partial_z f(\mathbf{u}) = 0, \tag{4.20}$$

where the bar denotes the complex conjugate and $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\,\partial_y)$. Then we can express this equation in terms of the real and imaginary parts of $\mathbf{u}$ and $\frac{1}{2} f(\mathbf{u}) = a(u, v) + b(u, v)\mathrm{i}$. Then (4.20) gives

$$\begin{cases} \partial_t u + \partial_x a(u, v) + \partial_y b(u, v) = 0, \\ \partial_t v - \partial_x b(u, v) + \partial_y a(u, v) = 0. \end{cases} \tag{4.21}$$

In particular, when $f(\mathbf{u}) = \mathbf{u}^2 = u^2 + v^2 + 2uv\mathrm{i}$, system (4.20) is called the complex Burger equation, which becomes

$$\begin{cases} \partial_t u + \frac{1}{2}\,\partial_x\big(u^2 + v^2\big) + \partial_y(uv) = 0, \\ \partial_t v - \partial_x(uv) + \frac{1}{2}\,\partial_y\big(u^2 + v^2\big) = 0. \end{cases} \tag{4.22}$$

System (4.21) is a symmetric hyperbolic system of conservation laws with a strictly convex entropy

$$\eta(u, v) = u^2 + v^2,$$

so that local well posedness of classical solutions can be inferred directly from Theorem 2.3; see [202] for more details. For the one-dimensional case, this system is an archetype of hyperbolic systems of conservation laws with umbilic degeneracy, which has been analyzed in [72,286] and the references cited therein.

5. Multidimensional steady supersonic problems

Multidimensional steady problems for the Euler equations are fundamental in fluid dynamics. In particular, understanding of these problems will help us to understand the asymptotic behavior of evolution solutions for large time, especially global attractors. One of the excellent sources of steady problems is Courant–Friedrichs' book [100].

In this section we first discuss some of recent developments in the study of two-dimensional steady supersonic problems.

The two-dimensional steady Euler flows are governed by

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x\big(\rho u^2 + p\big) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y\big(\rho v^2 + p\big) = 0, \\ \partial_x\big(u(E + p)\big) + \partial_y\big(v(E + p)\big) = 0, \end{cases} \tag{5.1}$$

where (u, v) is the velocity and E is the total energy, and the constitutive relations among the thermodynamical variables ρ, p, e, θ and S are determined by (1.5)–(1.9). For the barotropic (isentropic or isothermal) case

$$p = p(\rho) = \frac{\kappa\rho^\gamma}{\gamma}, \quad \gamma \geqslant 1,$$

and then the first three equations in (5.1) form a self-contained system, the Euler system for steady barotropic fluids. The quantity

$$c = \sqrt{p_\rho(\rho, S)}$$

is defined as the sonic speed and, for polytropic gases, $c = \sqrt{\gamma p/\rho}$.

System (5.1) governing a supersonic flow (i.e., $u^2 + v^2 > c^2$) has all real eigenvalues and is hyperbolic, while system (5.1) governing a subsonic flow (i.e., $u^2 + v^2 < c^2$) has complex eigenvalues and is both elliptic–hyperbolic mixed and composite.

5.1. *Wedge problems involving supersonic shocks*

The mathematical study of two-dimensional steady supersonic flows past wedges whose vertex angles are less than the critical angle can date back to the 1940s since the stability of such flows is fundamental in applications (cf. [100] and [336]). Local solutions around the wedge vertex were first constructed in [162,219,285] and the references cited therein. Global potential solutions have been constructed in [89–91] when the wedge has some convexity or the wedge is a small perturbation of the straight wedge with fast decay in the flow direction and in [353,354] for piecewise smooth curved wedges that are a small perturbation of the straight wedge.

As indicated in Section 4.1, the potential flow equation is an excellent model for the flow containing only weak shocks since it approximates to the isentropic Euler equations up to third order in shock strength. For the flow containing shocks of large strength, the full Euler equations (5.1) are required to govern the physical flow. For the wedge problem, when the vertex angle is large, the flow contains a large shock front emanating from the wedge vertex and, for this case, the Euler equations should take the position to describe the physical flow. Thus it is important to study the two-dimensional steady supersonic flows governed by the Euler equations for the wedge problem with a large vertex angle. When a wedge is straight and the wedge vertex angle is less than the critical angle ω_{crit}, there exists a supersonic shock front emanating from the wedge vertex so that the constant states on both sides of the shock are supersonic; the critical angle condition is necessary and sufficient for the existence of the supersonic shock. This can be seen through the shock polar (see Figures 1 and 2; also see [88,100]).

Consider two-dimensional steady supersonic Euler flows past two-dimensional Lipschitz curved wedges whose vertex angles are less than the critical angle ω_{crit}, along which the total variation of the tangent angle functions is suitably small. More specifically,

(i) there exists a Lipschitz function $g \in \mathrm{Lip}(\mathbb{R}_+)$ with $g' \in BV(\mathbb{R}_+)$ and $g(0) = 0$ such that $\omega_0 := \arctan(g'(0+)) < \omega_{\mathrm{crit}}$,

$$\begin{aligned} &TV\big\{g'(\cdot); \mathbb{R}_+\big\} \leqslant \varepsilon \quad \text{for some constant } \varepsilon > 0, \\ &\Omega := \big\{(x, y)\colon\ y > g(x), x \geqslant 0\big\}, \qquad \Gamma := \big\{(x, y)\colon\ y = g(x), x \geqslant 0\big\} \end{aligned} \tag{5.2}$$

and $\mathbf{n}(x\pm) = (-g'(x\pm), 1)/\sqrt{(g'(x\pm))^2 + 1}$ are the outer normal vectors to Γ at points $x\pm$, respectively (see Figure 3);

(ii) the uniform upstream flow $U_- = (\rho_-, u_-, 0, p_-)$ satisfies

$$u_- > c_- := \sqrt{\frac{\gamma p_-}{\rho_-}}$$

so that a strong supersonic shock emanates from the wedge vertex.

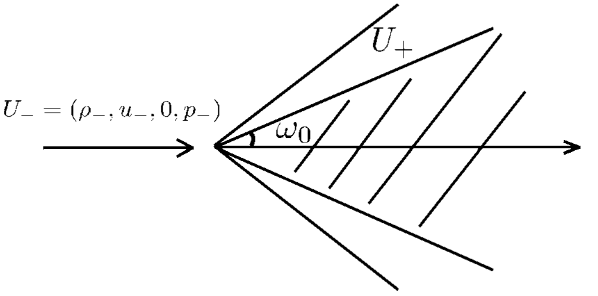

Fig. 1. Supersonic shock emanating from the wedge vertex.

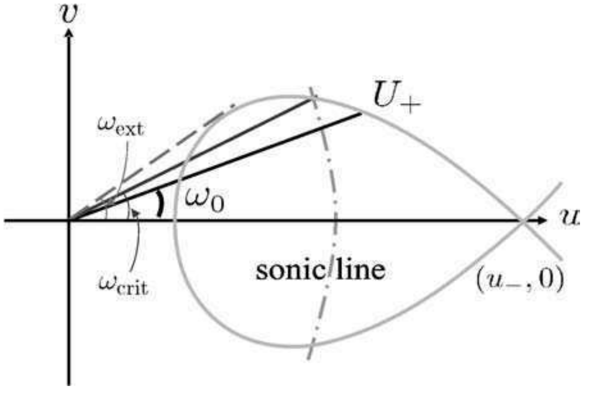

Fig. 2. Shock polar in the (u, v)-plane.

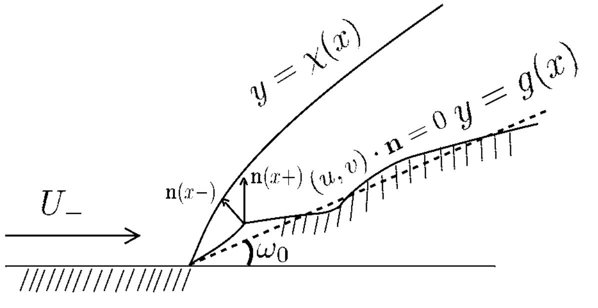

Fig. 3. Supersonic flow past a curved wedge.

With this setup, the wedge problem can be formulated into the following problem of initial–boundary value type for system (5.1)

$$\textit{Cauchy condition}: \quad U|_{x=0} = U_-; \tag{5.3}$$

$$\textit{boundary condition}: \quad (u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{5.4}$$

DEFINITION 1 (Entropy solutions). A function $U = U(x, y) \in BV(\Omega)$ is called an entropy solution of problem (5.1) and (5.3)–(5.4) provided that

(i) U is a weak solution of (5.1): U satisfies the equations in the sense of distributions and the Cauchy and boundary conditions (5.3) and (5.4) in the trace sense,

(ii) U satisfies the entropy inequality in the sense of distributions,

$$\partial_x(\rho u S) + \partial_y(\rho v S) \geqslant 0, \tag{5.5}$$

that is, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi \geqslant 0$,

$$\int_\Omega (\rho u S\varphi_x + \rho v S\varphi_y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_0^\infty \rho_- u_- S_- \varphi(0, y)\,\mathrm{d}y. \tag{5.6}$$

Then we have the following theorem.

THEOREM 5.1 (Existence and stability). *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, if* (5.2) *holds for* $\varepsilon \leqslant \varepsilon_0$, *there exists a pair of functions*

$$U \in BV\big(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}_+\big), \qquad \sigma \in BV(\mathbb{R}_+; \mathbb{R})$$

with $\chi = \int_0^x \sigma(s)\,\mathrm{d}s \in \mathrm{Lip}(\mathbb{R}_+; \mathbb{R}_+)$ *such that*

(i) U *is a global entropy solution of problem* (5.1) *and* (5.3)–(5.4) *in* Ω *with*

$$TV\big\{U(x, \cdot)\colon \big[g(x), -\infty\big)\big\} \leqslant C\,TV\big(g'(\cdot)\big) \quad \textit{for every } x \in \mathbb{R}_+,$$
$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0 \qquad \textit{in the trace sense};$$

(ii) *the curve* $y = \chi(x)$ *is a strong shock front with* $\chi(x) > g(x)$ *for any* $x > 0$ *and*

$$U|_{\{y>\chi(x)\}} = U_-, \qquad \sqrt{u^2 + v^2}\big|_{\{g(x)<y<\chi(x)\}} < u_-;$$

(iii) *there exist constants* p_∞ *and* σ_∞ *such that*

$$\lim_{x\to\infty} \sup\big\{\big|p(x, y) - p_\infty\big|\colon\ g(x) < y < \chi(x)\big\} = 0,$$
$$\lim_{x\to\infty} \big|\sigma(x) - \sigma_\infty\big| = 0$$

and

$$\lim_{x\to\infty} \sup\left\{\left|\arctan\left(\frac{v(x, y)}{u(x, y)}\right) - \omega_\infty\right|\colon\ g(x) < y < \chi(x)\right\} = 0,$$

where $\omega_\infty = \lim_{x\to\infty} \arctan(g'(x+))$.

This theorem has been established in [88]. It indicates that, under the *BV* perturbation of the wedge boundary as long as the wedge vertex angle is less than the critical angle, the strong shock front emanating from the wedge vertex is nonlinearly stable in structure globally, although there may be many weak shocks and vortex sheets between the wedge boundary and the strong shock front. This asserts that any supersonic shock for the wedge problem is nonlinearly stable.

In order to establish this theorem, we first developed a modified Glimm scheme whose mesh grids are designed to follow the slope of the Lipschitz wedge boundary, which are not standard rectangle mesh grids, so that the lateral Riemann building blocks contain only one shock or rarefaction wave emanating from the mesh points on the boundary. Such a design makes the *BV* estimates more convenient for the Glimm approximate solutions. Then careful interaction estimates were made. One of the essential estimates is the estimate of the strength δ_1 of the reflected 1-waves in the interaction between the 4-strong shock front and weak waves $(\alpha_1, \beta_2, \beta_3, \beta_4)$, that is,

$$\delta_1 = \alpha_1 + K_{s1}\beta_4 + \mathrm{O}(1)|\alpha_1|\big(|\beta_2| + |\beta_3|\big) \quad \text{with } |K_{s1}| < 1.$$

The second essential estimate is the interaction estimate between the wedge boundary and weak waves.

Based on the construction of the modified Glimm scheme and interaction estimates, we successfully identified a Glimm-type functional to incorporate the curved wedge boundary and the strong shock front naturally and to trace the interactions not only between the wedge boundary and weak waves but also between the strong shock front and weak waves. In particular, the Glimm-type functional on the mesh curve J is defined by

$$F(J) = C_*\big|\sigma^J - \sigma_0\big| + L(J) + KQ(J).$$

Here the linear part measuring the total variation is

$$L(J) = K_0L_0(J) + L_1(J) + K_2L_2(J) + K_3L_3(J) + K_4L_4(J)$$

with

$$L_0(J) = \sum\big\{\big|\omega(C_l)\big|: C_l \in \Omega_J\big\},$$

$$L_j(J) = \sum\big\{|\alpha_j|: \alpha_j \text{ crosses } J\big\}, \quad 1 \leqslant j \leqslant 4,$$

and the quadratic part measuring the potential wave interaction is

$$\mathcal{Q}(J) := \sum\big\{|\alpha||\beta|: \alpha, \beta \text{ interacting waves crossing } J\big\},$$

where Ω_J is the set of the mesh corner points lying in J and the boundary, σ^J stands for the speed of the strong shock crossing J, the constants K, C_*, K_0, K_2, K_3 and K_4 can be appropriately chosen with the aid of the important fact that $|K_{s1}| < 1$ so that the identified

Glimm functional monotonically decreases in the flow direction. Another essential estimate is to trace the approximate strong shocks in order to establish the nonlinear stability and asymptotic behavior of the strong shock emanating from the wedge vertex under the *BV* wedge perturbation.

Condition (5.2) can be relaxed by combining the analysis in [88] with the argument in [322,323]. The existence and stability of transonic flows past a curved wedge is under investigation with the aid of free boundary approaches (see Section 6.3).

For *the cone problem*, the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with small vertex angle was established upon the perturbation of the obstacle in [203]. It would be interesting to combine the analysis in [203] with the argument in [88] to study the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with arbitrary vertex angle. Other related results and analysis for this problem can be seen in [92,93] and the references cited therein.

5.2. *Stability of supersonic vortex sheets*

Another natural problem is the stability of supersonic vortex sheets above Lipschitz walls along which the total variation of the tangent angle functions is suitably small. More precisely,

(i) there exists a Lipschitz function $g \in \mathrm{Lip}(\mathbb{R}_+; \mathbb{R})$ with $g(0) = 0$, $g'(0+) = 0$, $\lim_{x\to\infty} \arctan(g'(x+)) = 0$, and $g' \in BV(\mathbb{R}_+; \mathbb{R})$ such that

$$
\begin{aligned}
&TV\big(g'(\cdot)\big) \leqslant \varepsilon \quad \text{for some constant } \varepsilon > 0,\\
&\Omega = \big\{(x, y)\colon\ y > g(x), x \geqslant 0\big\}, \qquad \Gamma = \big\{(x, y)\colon\ y = g(x), x \geqslant 0\big\},
\end{aligned}
\tag{5.7}
$$

and $\mathbf{n}(x\pm) = (-g'(x\pm), 1)/\sqrt{(g'(x\pm))^2 + 1}$ are the outer normal vectors to Γ at points $x\pm$, respectively (see Figure 4);

(ii) the upstream flow consists of one supersonic straight vortex sheet $y = y_0 > 0$ and two constant vectors $U_0 = (\rho_0, u_0, 0, p_0)$ when $y > y_0 > 0$ and $U_1 = (\rho_1, u_1, 0, p_0)$ when $0 < y < y_0$ satisfying

$$
u_1 > u_0 > 0, \qquad u_i > c_i, \quad i = 0, 1,
$$

where $c_i = \sqrt{\gamma p_i/\rho_i}$ is the sonic speed of states U_i, $i = 0, 1$.

With this setup, the vortex sheet problem can be formulated into the following problem of initial–boundary value type for system (5.1):

$$
\textit{Cauchy condition}: \qquad U|_{x=0} = \begin{cases} U_0, & 0 < y < y_0,\\ U_1, & y > y_0; \end{cases} \tag{5.8}
$$

$$
\textit{boundary condition}: \quad (u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{5.9}
$$

The stability of supersonic vortex sheets has been studied by classical linearized stability analysis, large-scale numerical simulations, and asymptotic analysis. In particular,

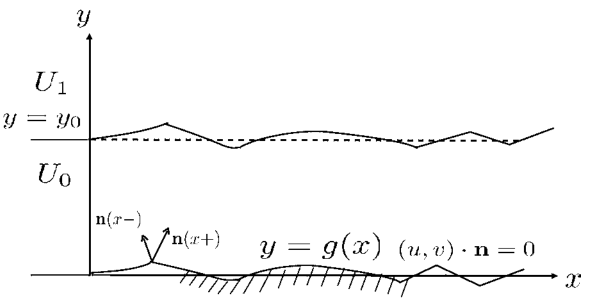

Fig. 4. Stability of the supersonic vortex sheet.

the nonlinear development of instabilities of supersonic vortex sheets has been predicted at high Mach number as time evolves; see [11,339] and the references cited therein. Motivated by the phenomenon of evolution instabilities, we are interested in whether steady supersonic vortex sheets, as time-asymptotics, are stable under a *BV* perturbation of the Lipschitz walls. In contrast with the prediction of instability in time, it has been proved that steady supersonic vortex sheets, as time-asymptotics, are stable in structure globally, even under the *BV* perturbation of the Lipschitz walls in [87].

THEOREM 5.2 (Existence and stability). *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, if* (5.7) *holds for* $\varepsilon \leqslant \varepsilon_0$, *there exists a pair of functions*

$$U \in BV(\mathbb{R}_+; \mathbb{R}), \qquad \chi \in \mathrm{Lip}(\mathbb{R}_+; \mathbb{R}_+)$$

with $\chi(0) = y_0$ *such that*

(i) U *is a global entropy solution of problem* (5.1) *and* (5.8)–(5.9) *in* Ω *with*

$$TV\{U(x, \cdot)\colon [g(x), \infty)\} \leqslant C\,TV(g'(\cdot)) \quad \textit{for every } x \in [0, \infty),$$
$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0 \qquad \textit{in the trace sense};$$

(ii) *the curve* $\{y = \chi(x)\}$ *is a strong supersonic vortex sheet with* $\chi(x) > g(x)$ *for any* $x > 0$ *and*

$$\big|U|_{\{g(x)<y<\chi(x)\}} - U_0\big| \leqslant C\varepsilon, \qquad \big|U|_{\{y>\chi(x)\}} - U_1\big| \leqslant C\varepsilon;$$

(iii) *there exist constants* p_∞ *and* χ_∞ *such that*

$$\lim_{x\to\infty} \sup\big\{\big|p(x, y) - p_\infty\big|\colon g(x) < y < \chi(x)\big\} = 0,$$
$$\lim_{x\to\infty} \big|\chi(x) - \chi_\infty\big| = 0$$

and

$$\lim_{x\to\infty} \sup\left\{\left|\arctan\left(\frac{v(x, y)}{u(x, y)}\right)\right|\colon y > g(x)\right\} = 0.$$

This theorem indicates that the strong supersonic vortex sheets are nonlinearly stable in structure globally under the *BV* perturbation of the Lipschitz wall, although there may be many weak shocks and supersonic vortex sheets away from the strong vortex sheet.

In order to establish this theorem, as in Section 5.1, we first developed a modified Glimm scheme whose mesh grids are designed to follow the slope of the Lipschitz boundary, which are not standard rectangle mesh grids, so that the lateral Riemann building blocks contain only one wave emanating from the mesh points on the boundary. For this case, one of the essential estimates is the estimate of the strength δ_1 of the reflected 1-wave in the interaction between the 4-weak wave α_4 and the strong vortex sheet from below, that is,

$$\delta_1 = K_{01}\alpha_4, \quad |K_{01}| < 1.$$

Another essential estimate is the estimate of the strength δ_4 of the reflected 4-wave in the interaction between the 1-weak wave β_1 and the strong vortex sheet from above is also less than one, that is,

$$\delta_4 = K_{11}\beta_1, \quad |K_{11}| < 1.$$

The third essential estimate is the interaction estimate between the boundary and weak waves.

Based on the construction of the modified Glimm scheme and the new interaction estimates, we successfully identified a Glimm-type functional by both incorporating the Lipschitz wall and the strong vortex sheet naturally and tracing the interactions not only between the boundary and weak waves but also between the strong vortex sheet and weak waves so that the Glimm-type functional monotonically decreases in the flow direction. Another essential estimate is to trace the approximate supersonic vortex sheets in order to establish the nonlinear stability and asymptotic behavior of the strong vortex sheet under the *BV* boundary perturbation. For more details, see [87].

6. Multidimensional steady transonic problems

In this section we discuss another important class of multidimensional steady problems: transonic problems. In the last decade, a program has been initiated on the existence and stability of multidimensional transonic shocks, and some new analytical approaches including techniques, methods and ideas have been developed. We focus here on the potential flow equation for the velocity potential $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, which is a second-order nonlinear equation of mixed elliptic–hyperbolic type,

$$\operatorname{div}\big(\rho\big(|\nabla\varphi|^2\big)\nabla\varphi\big) = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \tag{6.1}$$

where the density $\rho(q^2)$ is

$$\rho\big(q^2\big) = \big(1 - \theta q^2\big)^{1/(\gamma-1)}$$

with adiabatic exponent $\gamma > 1$. Equation (6.1) is elliptic at $\nabla\varphi$ with $|\nabla\varphi| = q$ if

$$\rho\big(q^2\big) + 2q^2\rho'\big(q^2\big) > 0$$

and hyperbolic if

$$\rho\big(q^2\big) + 2q^2\rho'\big(q^2\big) < 0.$$

We are interested in compressible potential flows with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega}, \qquad \mathcal{S} = \partial\Omega^+ \cap \Omega.$$

Let $\varphi \in C^{0,1}(\Omega)$ be a weak solution of (6.1) and in $C^1(\overline{\Omega^\pm})$ so that $\nabla\varphi$ experiences a jump across $\mathcal{S}$ that is a $(d-1)$-dimensional smooth surface. Then φ satisfies the following Rankine–Hugoniot conditions on $\mathcal{S}$

$$[\varphi]_{\mathcal{S}} = 0, \qquad \big[\rho\big(|\nabla\varphi|^2\big)\nabla\varphi \cdot \mathbf{n}\big]_{\mathcal{S}} = 0, \tag{6.2}$$

where $\mathbf{n}$ is the unit normal to $\mathcal{S}$ from Ω^- to Ω^+, and the bracket denotes the difference between the values of the function along $\mathcal{S}$ on the $\Omega^\pm$ sides. Moreover, a function $\varphi \in C^1(\overline{\Omega^\pm})$, which satisfies $|\nabla\varphi| \leqslant \sqrt{2/(\gamma-1)}$, (6.2), and equation (6.1) in $\Omega^\pm$, respectively, is a weak solution of (6.1) in the whole domain Ω. Set $\varphi^\pm = \varphi|_{\Omega^\pm}$. Then we can also write (6.2) as

$$\varphi^+ = \varphi^- \quad \text{on } \mathcal{S} \tag{6.3}$$

and

$$\rho\big(\big|\nabla\varphi^+\big|^2\big)\nabla\varphi^+ \cdot \mathbf{n} = \rho\big(\big|\nabla\varphi^-\big|^2\big)\nabla\varphi^- \cdot \mathbf{n} \quad \text{on } \mathcal{S}. \tag{6.4}$$

Note that the function

$$\Phi(p) := \left(1 - \frac{\gamma-1}{2}p^2\right)^{1/(\gamma-1)} p \tag{6.5}$$

is continuous on $[0, \sqrt{2/(\gamma-1)}\,]$ and satisfies

$$\Phi(p) > 0 \quad \text{for } p \in \left(0, \sqrt{\frac{2}{\gamma-1}}\right), \qquad \Phi(0) = \Phi\left(\sqrt{\frac{2}{\gamma-1}}\right) = 0, \tag{6.6}$$

$$0 < \Phi'(p) < 1 \quad \text{on } (0, c_*), \qquad \Phi'(p) < 0 \quad \text{on } \left(c_*, \sqrt{\frac{2}{\gamma-1}}\right), \tag{6.7}$$

$$\Phi''(p) < 0 \qquad \text{on } (0, c_*], \tag{6.8}$$

where $c_* = \sqrt{2/(\gamma+1)}$ is the sonic speed, for which a flow is called supersonic if $|\nabla\varphi| > c_*$ and subsonic if $|\nabla\varphi| < c_*$.

Suppose that $\varphi \in C^1(\overline{\Omega^\pm})$ is a weak solution satisfying

$$|\nabla\varphi| < c_* \quad \text{in } \Omega^+, \qquad |\nabla\varphi| > c_* \quad \text{in } \Omega^-, \qquad \nabla\varphi^\pm \cdot \mathbf{n}|_{\mathcal{S}} > 0. \tag{6.9}$$

Then φ is a *transonic shock solution* with *transonic shock* $\mathcal{S}$ dividing Ω into the *subsonic region* Ω^+ and the *supersonic region* Ω^- and satisfying the physical entropy condition (see [100])

$$\rho\left(\left|\nabla\varphi^-\right|^2\right) < \rho\left(\left|\nabla\varphi^+\right|^2\right) \quad \text{along } \mathcal{S}. \tag{6.10}$$

Note that (6.1) is elliptic in the subsonic region and hyperbolic in the supersonic region.

Let $(x_1, \mathbf{x}')$ be the coordinates in $\mathbb{R}^d$, where $x_1 \in \mathbb{R}$ and $\mathbf{x}' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. Fix $\mathbf{V}_0 \in \mathbb{R}^d$, and let

$$\varphi_0(\mathbf{x}) := \mathbf{V}_0 \cdot \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^d.$$

If $|\mathbf{V}_0| \in (0, c_*)$ (resp. $|\mathbf{V}_0| \in (c_*, \sqrt{2/(\gamma-1)}))$, then $\varphi_0(\mathbf{x})$ is a subsonic (resp. supersonic) solution in $\mathbb{R}^d$, and $\mathbf{V}_0 = \nabla\varphi_0$ is its velocity.

Let $q_0^- > 0$ and $\mathbf{V}_0' \in \mathbb{R}^{d-1}$ be such that the vector $\mathbf{V}_0^- := (q_0^-, \mathbf{V}_0')$ satisfies $|\mathbf{V}_0^-| > c_*$. Then, using the properties of function (6.5), we conclude from (6.6)–(6.8) that there exists a unique $q_0^+ > 0$ such that

$$\begin{aligned}&\left(1 - \frac{\gamma-1}{2}\left(\left|q_0^+\right|^2 + \left|\mathbf{V}_0'\right|^2\right)\right)^{1/(\gamma-1)} q_0^+ \\ &\quad = \left(1 - \frac{\gamma-1}{2}\left(\left|q_0^-\right|^2 + \left|\mathbf{V}_0'\right|^2\right)\right)^{1/(\gamma-1)} q_0^-.\end{aligned} \tag{6.11}$$

The entropy condition (6.10) implies $q_0^+ < q_0^-$. By denoting $\mathbf{V}_0^+ := (q_0^+, \mathbf{V}_0')$ and defining functions

$$\varphi_0^\pm(\mathbf{x}) := V_0^\pm \cdot \mathbf{x} \quad \text{on } \mathbb{R}^d,$$

then φ_0^+ (resp. φ_0^-) is a subsonic (resp. supersonic) solution. Furthermore, from (6.4) and (6.11), the function

$$\begin{aligned}\varphi_0(\mathbf{x}) &:= \min\left(\varphi_0^-(\mathbf{x}), \varphi_0^+(\mathbf{x})\right) \\ &= \begin{cases} \mathbf{V}_0^+ \cdot \mathbf{x}, & \mathbf{x} \in \Omega_0^- := \left\{\mathbf{x} \in \mathbb{R}^d\colon\ x_1 < 0\right\}, \\ \mathbf{V}_0^- \cdot \mathbf{x}, & \mathbf{x} \in \Omega_0^+ := \left\{\mathbf{x} \in \mathbb{R}^d\colon\ x_1 > 0\right\}, \end{cases}\end{aligned} \tag{6.12}$$

is a plane transonic shock solution in $\mathbb{R}^d$, Ω_0^- and Ω_0^+ are respectively its subsonic and supersonic regions, and $\mathcal{S} = \{x_1 = 0\}$ is a transonic shock. Note that, if $\mathbf{V}_0' = 0$, the velocities $V_0^\pm$ are orthogonal to the shock $\mathcal{S}$ and, if $\mathbf{V}_0' \neq 0$, the velocities are not orthogonal to $\mathcal{S}$.

In order to deal with multidimensional transonic shocks in an unbounded domain Ω, we define the following weighted Hölder seminorms and norms in a domain $\mathcal{D} \subset \mathbb{R}^d$.

Let $\mathbf{x} \to \delta_{\mathbf{x}}$ be a given nonnegative function defined on $\mathcal{D}$, which will be specified in each case we consider below. Let $\delta_{\mathbf{x},\mathbf{y}} := \min(\delta_{\mathbf{x}}, \delta_{\mathbf{y}})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{D}$. For $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $m \in \mathcal{Z}_+$, we define

$$
\begin{aligned}
&[u]^{(k)}_{m,0,\mathcal{D}} = \sum_{|\beta|=m} \sup_{\mathbf{x}\in\mathcal{D}} \big(\delta_{\mathbf{x}}^{m+k} \big|\mathrm{D}^\beta u(\mathbf{x})\big|\big),\\
&[u]^{(k)}_{m,\alpha,\mathcal{D}} = \sum_{|\beta|=m} \sup_{\mathbf{x},\mathbf{y}\in\mathcal{D},\mathbf{x}\neq\mathbf{y}} \left(\delta_{\mathbf{x},\mathbf{y}}^{m+\alpha+k} \frac{|\mathrm{D}^\beta u(\mathbf{x}) - \mathrm{D}^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}\right),\\
&\|u\|^{(k)}_{m,0,\mathcal{D}} = \sum_{j=0}^{m} [u]^{(k)}_{j,0,\mathcal{D}}, \qquad \|u\|^{(k)}_{m,\alpha,\mathcal{D}} = \|u\|^{(k)}_{m,0,\mathcal{D}} + [u]^{(k)}_{m,\alpha,\mathcal{D}},
\end{aligned}
\tag{6.13}
$$

where $\mathrm{D}^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$, $\beta = (\beta_1, \ldots, \beta_d)$ is a multiindex with $\beta_j \geqslant 0, \beta_j \in \mathcal{Z}$ and $|\beta| = \beta_1 + \cdots + \beta_d$. We denote by $\|u\|_{m,\alpha,\mathcal{D}}$ the (nonweighted) Hölder norms in a domain $\mathcal{D}$, i.e., the norms defined as above with $\delta_{\mathbf{x}} = \delta_{\mathbf{x},\mathbf{y}} = 1$.

6.1. *Transonic shock problems in* $\mathbb{R}^d$

We now consider multidimensional perturbations of the uniform transonic shock solution (6.12) in the whole space $\mathbb{R}^d$ with $d \geqslant 3$.

Since it suffices to specify the supersonic perturbation φ^- only in a neighborhood of the unperturbed shock surface $\{x_1 = 0\}$, we introduce domains

$$
\Omega := (-1, \infty) \times \mathbb{R}^{d-1}, \qquad \Omega_1 := (-1, 1) \times \mathbb{R}^{d-1}.
$$

Note that we expect the subsonic region Ω^+ to be close to the half-space $\Omega_0^+ = \{x_1 > 0\}$. We use the norms in (6.13) with the weight function

$$
\delta_{\mathbf{x}} = 1 + |\mathbf{x}|
$$

and consider the following problem.

PROBLEM 6.1. Given a supersonic solution $\varphi^-(\mathbf{x})$ of (6.1) in Ω_1 satisfying that, for some $\alpha > 0$,

$$
\big\|\varphi^- - \varphi_0^-\big\|^{(d-1)}_{2,\alpha,\Omega_1} \leqslant \sigma \tag{6.14}
$$

with $\sigma > 0$ small, find a transonic shock solution $\varphi(\mathbf{x})$ in Ω such that

$$
\Omega^- \subset \Omega_1, \qquad \varphi(\mathbf{x}) = \varphi^-(\mathbf{x}) \quad \text{in } \Omega^-,
$$

where $\Omega^- := \Omega \setminus \Omega^+$ and $\Omega^+ := \{\mathbf{x} \in \Omega \colon |\nabla\varphi(\mathbf{x})| < c_*\}$, and

$$\varphi = \varphi^-, \qquad \partial_{x_1}\varphi = \partial_{x_1}\varphi^- \quad \text{on } \{x_1 = -1\}, \tag{6.15}$$

$$\lim_{R\to\infty} \|\varphi - \varphi_0^+\|_{C^1(\Omega^+\setminus B_R(0))} = 0. \tag{6.16}$$

Condition (6.15) determines that the solution has supersonic upstream, while condition (6.16) determines, in particular, that the uniform velocity state at infinity in the downstream direction is equal to the unperturbed downstream velocity state. The additional requirement in (6.16) that $\varphi \to \varphi_0^+$ at infinity within Ω^+ fixes the position of shock at infinity. This allows us to determine the solution of Problem 6.1 uniquely.

Then we have the following theorem (see [62]).

THEOREM 6.1. *Let* $|(q_0^-, \mathbf{V}_0')| \in (c_*, \sqrt{2/(\gamma-1)})$ *and* $q_0^+ \in (0, c_*)$ *satisfy* (6.11), *and let* $\varphi_0(\mathbf{x})$ *be the transonic shock solution* (6.12). *Then there exist positive constants* σ_0, C_1 *and* C_2 *depending only on* d, γ, α, $|\mathbf{V}_0'|$ *and* q_0^- *such that, for every* $\sigma \leqslant \sigma_0$ *and any supersonic solution* $\varphi^-(\mathbf{x})$ *of* (6.1) *satisfying the conditions stated in Problem* 6.1, *there exists a unique solution* $\varphi(\mathbf{x})$ *of Problem* 6.1 *satisfying*

$$\|\varphi - \varphi_0^+\|^{(d-2)}_{2,\alpha,\Omega^+} \leqslant C_1\sigma \tag{6.17}$$

with Ω^+ *defined in Problem* 6.1. *In addition,*

$$\Omega^+ = \{x_1 > f(\mathbf{x}')\}, \tag{6.18}$$

where $f : \mathbb{R}^{d-1} \to \mathbb{R}$ *satisfies*

$$\|f\|^{(d-2)}_{2,\alpha,\mathbb{R}^{d-1}} \leqslant C_2\sigma, \tag{6.19}$$

that is, the shock surface

$$\mathcal{S} = \{(x_1, \mathbf{x}') \colon x_1 = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\}$$

is in $C^{2,\alpha}$ *and converges at infinity, with an appropriate algebraic rate, to the hyperplane*

$$\mathcal{S}_0 = \{x_1 = 0\}.$$

Moreover, there exist a nonnegative nondecreasing function $\Psi \in C([0,\infty))$ *satisfying* $\Psi(0) = 0$ *and a constant* σ_0 *depending only on* d, γ, α, $|\mathbf{V}_0'|$ *and* q_0^- *such that, if* $\sigma < \sigma_0$ *and smooth supersonic solutions* $\varphi^-(\mathbf{x})$ *and* $\hat{\varphi}^-(\mathbf{x})$ *of* (6.1) *satisfy* (6.14), *the unique solutions* $\varphi(\mathbf{x})$ *and* $\hat{\varphi}(\mathbf{x})$ *of Problem* 6.1 *for* $\varphi^-(\mathbf{x})$ *and* $\hat{\varphi}^-(\mathbf{x})$, *respectively, satisfy*

$$\|f_\varphi - f_{\hat{\varphi}}\|^{(d-2)}_{2,\alpha,\mathbb{R}^{d-1}} \leqslant \Psi\left(\|\varphi^- - \hat{\varphi}^-\|^{(d-1)}_{2,\alpha,\Omega_1}\right), \tag{6.20}$$

where $f_\varphi(\mathbf{x}')$ *and* $f_{\hat{\varphi}}(\mathbf{x}')$ *are the free boundary functions of* $\varphi(\mathbf{x})$ *and* $\hat{\varphi}(\mathbf{x})$ *in* (6.18), *respectively.*

This existence result can be extended to the case that the regularity of the steady perturbation φ^- is only $C^{1,1}$. That is, (6.14) can be replaced by

$$\|\varphi^- - \varphi_0^-\|_{1,1,\Omega_1}^{(d-1)} \leqslant \sigma. \tag{6.21}$$

Another related problem is the stability of transonic shocks near a spherical transonic shock, which can be established by following similar arguments (see [60,62]).

6.2. *Nozzle problems involving transonic shocks*

We now consider multidimensional transonic shocks in the following infinite nozzle Ω with arbitrary smooth cross-sections

$$\Omega = \Psi(\Lambda \times \mathbb{R}) \cap \{x_1 > -1\}, \tag{6.22}$$

where $\Lambda \subset \mathbb{R}^{d-1}$ is an open bounded connected set with a smooth boundary, and $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth map, which is close to the identity map. For simplicity, we assume that

$$\partial\Lambda \text{ is in } C^{[d/2]+3,\alpha}, \qquad \|\Psi - I\|_{[d/2]+3,\alpha,\mathbb{R}^d} \leqslant \sigma \tag{6.23}$$

for some $\alpha \in (0,1)$ and small $\sigma > 0$, where $[s]$ is the integer part of s, $I : \mathbb{R}^d \to \mathbb{R}^d$ is the identity map, and $\partial_l\Omega := \Psi(\mathbb{R} \times \partial\Lambda) \cap \{x_1 > -1\}$. Such nozzles especially include the slowly varying de Laval nozzles [100,336]. For concreteness, we also assume that there exists $L > 1$ such that

$$\Psi(\mathbf{x}) = \mathbf{x} \quad \text{for any } \mathbf{x} = (x_1, \mathbf{x}') \text{ with } x_1 > L, \tag{6.24}$$

that is, the nozzle slowly varies in a bounded domain as the de Laval nozzles.

In the two-dimension case, the domain Ω defined above has the following simple form

$$\Omega = \{(x_1, x_2):\ x_1 > -1, b^-(x_2) < x_2 < b^+(x_2)\},$$

where $\|b^\pm - b^\pm_\infty\|_{4,\alpha,\mathbb{R}} \leqslant \sigma$ and $b^\pm \equiv b^\pm_\infty$ on $[L,\infty)$ for some constants $b^\pm_\infty$ satisfying $b^+_\infty > b^-_\infty$.

For the multidimensional case, the geometry of the nozzles is much richer.

Note that our setup implies that $\partial\Omega = \overline{\partial_o\Omega} \cup \partial_l\Omega$ with

$$\partial_l\Omega := \Psi\big[(-\infty,\infty) \times \partial\Lambda\big] \cap \big\{(x_1, \mathbf{x}'):\ x_1 > -1\big\},$$
$$\partial_o\Omega := \Psi\big((-\infty,\infty) \times \Lambda\big) \cap \big\{(x_1, \mathbf{x}'):\ x_1 = -1\big\}.$$

Then our transonic nozzle problem can be formulated into the following form.

PROBLEM 6.2 (Transonic nozzle problem). Given the supersonic upstream flow at the entrance $\partial_o \Omega$,

$$\varphi = \varphi_e^-, \qquad \varphi_{x_1} = \psi_e^- \quad \text{on } \partial_o \Omega, \tag{6.25}$$

the slip boundary condition on the nozzle boundary $\partial_l \Omega$,

$$\nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial_l \Omega, \tag{6.26}$$

and the uniform subsonic flow condition at the infinite exit $x_1 = \infty$,

$$\left\|\varphi(\cdot) - \omega x_1\right\|_{C^1(\Omega\cap\{x_1>R\})} \to 0 \quad \text{as } R \to \infty \text{ for some } \omega \in (0, c_*), \tag{6.27}$$

find a multidimensional transonic flow φ of problem (6.1) and (6.25)–(6.27) in Ω.

The standard local existence theory of smooth solutions for the initial–boundary value problem (6.25) and (6.26) for second-order quasilinear hyperbolic equations implies that, as σ is sufficiently small in (6.23) and (6.30), there exists a supersonic solution φ^- of (6.1) in

$$\Omega_2 := \{-1 \leqslant x_1 \leqslant 1\},$$

which is a C^{l+1} perturbation of $\varphi_0^- = q_0^- x_1$: For any $\alpha \in (0, 1]$,

$$\left\|\varphi^- - \varphi_0^-\right\|_{l,\alpha,\Omega_2} \leqslant C_0\sigma, \quad l = 1, 2, \tag{6.28}$$

for some constant $C_0 > 0$, and satisfies

$$\nabla\varphi^- \cdot \mathbf{n} = 0 \quad \text{on } \partial_l \Omega_2, \tag{6.29}$$

provided that (φ_e^-, ψ_e^-) on $\partial_o \Omega$ satisfies

$$\left\|\varphi_e^- - q_0^- x_1\right\|_{H^{s+l}} + \left\|\psi_e^- - q_0^-\right\|_{H^{s+l-1}} \leqslant \sigma, \quad l = 1, 2, \tag{6.30}$$

for some integer $s > d/2 + 1$ and the compatibility conditions up to order $s + 1$, where the norm $\|\cdot\|_{H^s}$ is the Sobolev norm with $H^s = W^{s,2}$.

Then we have the following theorem (see [61]).

THEOREM 6.2. *Let $q_0^- \in (c_*, \sqrt{2/(\gamma-1)})$ and $q_0^+ \in (0, c_*)$ satisfy* (6.11), *and let φ_0 be the transonic shock solution* (6.12) *with $\mathbf{V}' = 0$. Then there exist $\sigma_0 > 0$, C_1 and C_2, depending only on d, α, γ, q_0^-, Λ and L, such that, for every $\sigma \in (0, \sigma_0)$, any map Ψ satisfying* (6.23) *and* (6.24), *and any supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o\Omega$ satisfying* (6.30) *with $l = 1$, there exists a solution $\varphi \in C^{0,1}(\Omega)$ of Problem* 6.2 *satisfying*

$$\Omega^+(\varphi) = \{x_1 > f(\mathbf{x}')\}, \qquad \Omega^-(\varphi) = \{x_1 < f(\mathbf{x}')\}, \tag{6.31}$$

$$\left\|\varphi - \varphi_0^-\right\|_{1,\alpha,\Omega^-} \leqslant C_1\sigma, \qquad \left\|\nabla\varphi - q_0^+ e_1\right\|_{0,0,\Omega^+} \leqslant C_2\sigma.$$

Moreover, this solution satisfies $\varphi \in C^{0,1}(\Omega) \cap C^{\infty}(\Omega^+)$ *and the following properties.*

(i) *The constant* ω *in* (6.27) *must be* q^+,

$$\omega = q^+, \tag{6.32}$$

where q^+ *is the unique solution in the interval* $(0, c_*)$ *of the equation*

$$\rho\big((q^+)^2\big)q^+ = Q^+ \tag{6.33}$$

with

$$Q^+ = \frac{1}{|\Lambda|}\int_{\partial_o\Omega} \rho\big(\big|\nabla_{\mathbf{x}'}\varphi_e^-\big|^2 + \big(\psi_e^-\big)^2\big)\psi_e^- \,\mathrm{d}\mathcal{H}^{d-1}.$$

Thus, φ *and* q^+ *satisfy*

$$\big\|\varphi - q^+x_1\big\|_{C^1(\Omega\cap\{x_1>R\})} \to 0 \quad \text{as } R \to \infty \tag{6.34}$$

and

$$\big|q^+ - q_0^+\big| \leqslant C_2\sigma. \tag{6.35}$$

(ii) *The function* $f(\mathbf{x}')$ *in* (6.31) *satisfies*

$$\|f\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant C_2\sigma, \tag{6.36}$$

and the surface $S = \{(f(\mathbf{x}'), \mathbf{x}') \colon \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$ *is orthogonal to* $\partial_l\Omega$ *at every intersection point.*

(iii) *Furthermore,* $\varphi \in C^{1,\alpha}(\overline{\Omega^+})$ *with*

$$\big\|\varphi - q^+x_1\big\|_{1,\alpha,\Omega^+} \leqslant C_2\sigma. \tag{6.37}$$

In addition, if the supersonic uniform flow (φ_e^-, ψ_e^-) *on* $\partial_o\Omega$ *satisfies* (6.30) *with* $l = 2$, *then the solution* $\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ *with*

$$\big\|\varphi - q^+x_1\big\|_{2,\alpha,\Omega^+} \leqslant C_2\sigma,$$

and the solution with a transonic shock is unique and stable with respect to the nozzle boundary and the smooth supersonic upstream flow at the entrance.

When the initial data $(\varphi_e^-, \psi_e^-) \equiv (-\psi_e^-, \psi_e^-)$ is constant and the nozzle

$$\Omega \cap \{-1 \leqslant x_1 \leqslant -1+\varepsilon\} = [-1, -1+\varepsilon] \times \Lambda \quad \text{for some } \varepsilon > 0$$

as the de Laval nozzles, then the compatibility conditions are automatically satisfied. In fact, in this case, $\varphi^-(\mathbf{x}) = \psi_e^- x_1$ is a solution near $x_1 = -1$ in the nozzle.

When $d = 2$, condition (6.30) for the supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o\Omega$ in Theorem 6.2 can be replaced by the C^3-condition,

$$\|\varphi_e^- + q_0^-\|_{C^3} + \|\psi_e^- - q_0^-\|_{C^2} \leqslant \sigma, \tag{6.38}$$

which can be achieved by following the arguments in [216]. For the isothermal gas $\gamma = 1$, the same results can be obtained by following similar arguments.

The techniques have been extended and applied to the nozzle problem for the full Euler equations in [57].

Other transonic problems include the stability of transonic flows past infinite nonsmooth wedges or cones which are under investigation with the aid of the approaches which will be discussed in Section 6.3.

A further problem is subsonic flow past an airfoil or an obstacle. Shiffman [297], Bers [18] and Finn and Gilbarg [136] studied subsonic (elliptic) solutions of (6.1) outside an obstacle when the upstream flows are sufficiently subsonic; also see [125]. Morawetz in [250] first showed that the flows of (6.1) past an obstacle may contain transonic shocks in general. An important problem is to construct global entropy solutions of the airfoil problem (see [251,253] and [141]).

6.3. *Free boundary approaches*

We now describe two of the free boundary approaches for Problems 6.1 and 6.2, developed recently in [60–62].

6.3.1. *Free boundary problems.* The transonic shock problems can be formulated into a one-phase free boundary problem for a nonlinear elliptic equation: Given $\varphi^- \in C^{1,\alpha}(\overline{\Omega})$, find a function φ that is continuous in Ω and satisfies

$$\varphi \leqslant \varphi^- \quad \text{in } \overline{\Omega}, \tag{6.39}$$

equation (6.1), the ellipticity condition in the noncoincidence set $\Omega^+ = \{\varphi < \varphi^-\}$, the free boundary condition (6.4) on the boundary $S = \partial\Omega^+ \cap \Omega$, as well as the prescribed conditions on the fixed boundary $\partial\Omega$ and at infinity. These conditions are different in different problems, for example, conditions (6.15) and (6.16) for Problem 6.1 and (6.25)–(6.27) for Problem 6.2.

The free boundary is the location of the shock, and the free boundary conditions (6.3) and (6.4) are the Rankine–Hugoniot conditions in (6.2). Note that condition (6.39) is motivated by the similar property (6.12) of unperturbed shocks; and (6.39), locally on the shock, is equivalent to the entropy condition (6.10). Condition (6.39) transforms the transonic shock problem, in which the subsonic region Ω^+ is determined by the gradient condition $|\nabla\varphi(\mathbf{x})| < c_*$, into a free boundary problem in which Ω^+ is the noncoincidence set.

In order to solve this free boundary problem, equation (6.1) is modified to be uniformly elliptic and then the free boundary condition (6.4) is correspondingly modified. Then this modified free boundary problem is solved. Since φ^- is a small $C^{1,\alpha}$ perturbation of φ_0^-, the solution φ of the free boundary problem is shown to be a small $C^{1,\alpha}$ perturbation of the given subsonic shock solution φ_0^+ in Ω^+. In particular, the gradient estimate implies that φ in fact satisfies the original free boundary problem, hence the transonic shock problem, Problem 6.1 (Problem 6.2, respectively).

The modified free boundary problem does not directly fit into the variational framework of Alt and Caffarelli [4] and Alt, Caffarelli and Friedman [5], as well as the regularization framework of Berestycki, Caffarelli and Nirenberg [16]. Also, the nonlinearity of the free boundary problem makes it difficult to apply the Harnack inequality approach of Caffarelli [38]. In particular, a boundary comparison principle for positive solutions of nonlinear elliptic equations in Lipschitz domains is not available yet for the equations that are not homogeneous with respect to $\nabla^2 u$, ∇u and u, which, however, is our case.

6.3.2. *Iteration approach.* The first approach we developed in [60,61] is an iteration scheme based on the nondegeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. Our iteration process is as follows: Suppose the domain Ω_k^+ is given so that $S_k := \partial\Omega_k^+ \setminus \partial\Omega$ is $C^{1,\alpha}$. Consider the oblique derivative problem in Ω_k^+ obtained by rewriting the (modified) equation (6.1) and free boundary condition (6.4) in terms of the function $u := \varphi - \varphi_0^+$. Then the problem has the following form:

$$\begin{aligned} &\operatorname{div}\mathbf{A}(\mathbf{x}, \nabla u) = F(\mathbf{x}) \qquad \text{in } \Omega_k^+ := \{u > 0\},\\ &\mathbf{A}(\mathbf{x}, \nabla u)\cdot\mathbf{n} = G(\mathbf{x}, \mathbf{n}) \quad \text{on } \mathcal{S} := \partial\Omega_k^+ \big\backslash \partial\Omega, \end{aligned} \tag{6.40}$$

plus the fixed boundary conditions on $\partial\Omega_k^+ \cap \partial\Omega$ and the conditions at infinity. The equation is quasilinear, uniformly elliptic, $\mathbf{A}(\mathbf{x}, 0) \equiv 0$, while $G(\mathbf{x}, \mathbf{n})$ has a certain structure. Let $u_k \in C^{1,\alpha}(\overline{\Omega_k^+})$ be the solution of (6.40). Then $\|u_k\|_{1,\alpha,\Omega_k^+}$ is estimated to be small if the perturbation is small, where appropriate weighted Hölder norms are actually needed in the unbounded domains. The function $\varphi_k := \varphi_0^+ + u_k$ from Ω_k^+ is extended to Ω so that the $C^{1,\alpha}$ norm of $\varphi_k - \varphi_0^+$ in Ω is controlled by $\|u_k\|_{1,\alpha,\Omega_k^+}$. Define

$$\Omega_{k+1}^+ := \bigl\{\mathbf{x} \in \Omega\colon\ \varphi_k(\mathbf{x}) < \varphi^-(\mathbf{x})\bigr\}$$

for the next step. Note that, since $\|\varphi_k - \varphi_0^+\|_{1,\alpha,\Omega}$ and $\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega}$ are small, we have

$$\bigl|\nabla\varphi^-\bigr| - |\nabla\varphi_k| \geqslant \delta > 0 \quad \text{in } \Omega,$$

and this nondegeneracy implies that $S_{k+1} := \partial\Omega_{k+1}^+ \setminus \partial\Omega$ is $C^{1,\alpha}$ and its norm is estimated in terms of the data of the problem.

The fixed point Ω^+ of this process determines a solution of the free boundary problem since the corresponding solution φ satisfies $\Omega^+ = \{\varphi < \varphi^-\}$ and the Rankine–Hugoniot condition holds on $\mathcal{S} := \partial\Omega^+ \cap \Omega$.

On the other hand, the elliptic estimates alone are not sufficient to get the existence of a fixed point, because the right-hand side of the boundary condition in problem (6.40) depends on the unit normal **n** of the free boundary. One way is to require the orthogonality of the flat shocks so that

$$\rho\big(\big|\nabla\varphi_0^+\big|^2\big)\nabla\varphi_0^+ = \rho\big(\big|\nabla\varphi_0^-\big|^2\big)\nabla\varphi_0^- \quad \text{in } \Omega \tag{6.41}$$

to obtain better estimates for the iteration and to prove the existence of a fixed point. Note that (6.41) is a vector identity, and the Rankine–Hugoniot condition (6.4) is the normal part of (6.41) on the unperturbed free boundary S_0.

The uniqueness and stability of solutions for the transonic shock problems are obtained by using the regularity and nondegeneracy of solutions.

For more details, see [60,61].

6.3.3. *Partial hodograph approach.* The second approach we developed in [60,62] is a partial hodograph procedure, with which we can handle the existence and stability of multidimensional transonic shocks that are not nearly orthogonal to the flow direction. One of the main ingredients in this new approach is to employ a partial hodograph transform to reduce the free boundary problem to a co-normal boundary value problem for the corresponding nonlinear second-order elliptic equation of divergence form in unbounded domains and then develop techniques to solve the co-normal boundary value problem in the unbounded domain. To achieve this, the strategy is to construct first solutions in the intersection domains between the physical unbounded domain under consideration and a series of half-balls with radius R, then make uniform estimates in R, and finally send $R \to \infty$. It requires delicate a priori estimates to achieve this. A uniform bound in a weighted L^∞ norm can be achieved by both employing a comparison principle and identifying a global function with the same decay rate as the fundamental solution of the elliptic equation with constant coefficients which controls the solutions. Then, by scaling arguments, the uniform estimates can be obtained in a weighted Hölder norm for the solutions, which lead to the existence of a solution in the unbounded domain with some decay rate at infinity. For such decaying solutions, a comparison principle holds, which implies the uniqueness for the co-normal problem. Finally, by the gradient estimate, the limit function can be shown to be a solution of the multidimensional transonic shock problem, and then the existence result can be extended to the case that the regularity of the steady perturbation is only $C^{1,1}$. We can further prove that the multidimensional transonic shock solution is stable with respect to the $C^{2,\alpha}$ supersonic perturbation.

When the regularity of the steady perturbation is $C^{3,\alpha}$ or higher, that is,

$$\big\|\varphi^- - \varphi_0^-\big\|_{3,\alpha,\Omega_1}^{(d-1)} \leqslant \sigma, \tag{6.42}$$

we introduced another simpler approach to deal with the existence and stability problem.

We also extend the approach by using the partial hodograph transform in the radial direction in the polar coordinates to establish the existence and stability of multidimensional transonic shocks near spheres in $\mathbb{R}^d$, $d \geqslant 3$. The case $d = 2$ can also be handled with similar approaches.

Another approach can be found in [40,41,357].

7. Multidimensional unsteady problems

In this section, we introduce some sample multidimensional time-dependent problems with a simplifying feature that the data (domain and/or the initial data) coupled with the structure of the underlying equations obey certain geometric structure so that the multidimensional problems can be reduced to lower dimensional problems with more complicated couplings. Different types of geometric structure call for different techniques.

The Euler equations for compressible fluids with geometric structure describe many important fluid flows, including spherically symmetric flow and self-similar flow. Such geometric flows are motivated by many physical problems, such as shock diffraction, supernovae formation in stellar dynamics, inertial confinement fusion, and underwater explosions. For the initial data with large amplitude having geometric structure, the physical insight we seek is

(a) whether the solution has the same geometric structure globally,

(b) whether the solution blows up to infinity in a finite time, for example, the density near the origin for spherically symmetric flow.

These questions are not easily understood in physical experiments and numerical simulations, especially for the blow-up, because of the limited capacity of available instruments and computers.

7.1. *Spherically symmetric solutions*

The first problem is the study of singularity at the origin for the Euler equations for isentropic or adiabatic fluids under spherical symmetry in $\mathbb{R}^d$, $d \geqslant 2$. The singularity at the origin makes the problem truly multidimensional. The central difficulty of this problem in the unbounded domain is the singularity at the origin and the reflection of waves from infinity and their strengthening as they move radially inwards.

Consider the Cauchy problem for (1.11),

$$(\rho, \mathbf{m})|_{t=0} = \big(\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x})\big) \tag{7.1}$$

with the following geometric structure,

$$\big(\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x})\big) = \left(\rho_0\big(|\mathbf{x}|\big), m_0\big(|\mathbf{x}|\big)\frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{7.2}$$

where $m_0(x)$ is a scalar function of $x = |\mathbf{x}| \geqslant 0$. Such a problem describes dynamic behavior of many physical problems with spherically symmetric initial structure such as explosion waves in air and other media [100,336]. Motivated by the physical experiments, we look for the solutions with spherical symmetry

$$\rho(t, \mathbf{x}) = \rho\big(t, |\mathbf{x}|\big), \qquad \mathbf{m}(t, \mathbf{x}) = m\big(t, |\mathbf{x}|\big)\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{7.3}$$

The function $(\rho, \mathbf{m})(t, x)$, $x = |\mathbf{x}|$, is determined by the one-dimensional isentropic Euler equations with geometric source terms

$$\begin{cases} \partial_t \rho + \partial_x m = -\frac{d-1}{x} m, & x > 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho)\right) = -\frac{d-1}{x} \frac{m^2}{\rho}. \end{cases} \tag{7.4}$$

It is evident that the density ρ blows up as $|\mathbf{x}| \to 0$ in general, for instance, for the focusing case. One of the challenging open problems is to understand the order of singularity

$$\rho\big(t, |\mathbf{x}|\big) \sim |\mathbf{x}|^{-\alpha}$$

for bounded Cauchy data.

On the other hand, a criterion was observed in [54] for L^∞ Cauchy data functions of arbitrarily large amplitude to guarantee the global existence of L^∞ spherically symmetric solutions which model outgoing blast waves and large-time asymptotic solutions.

THEOREM 7.1. *Consider the Cauchy problem for the Euler equations* (1.11) *with spherically symmetric initial data* (7.1) *and* (7.2). *Assume that the initial data satisfies*

$$0 \leqslant \int_0^{\rho_0(\mathbf{x})} \frac{\sqrt{p'(s)}}{s} \,\mathrm{d}s \leqslant \frac{|\mathbf{m}_0(\mathbf{x})|}{\rho_0(\mathbf{x})} \leqslant C_0 < \infty \tag{7.5}$$

for some constant $C_0 > 0$. Then there exists a global entropy solution $(\rho, \mathbf{m})(t, \mathbf{x}) \in L^\infty$ of the Cauchy problem (1.11) *and* (7.1)–(7.3) *satisfying*

$$0 \leqslant \rho(t, \mathbf{x}) \leqslant C, \qquad 0 \leqslant \big|\mathbf{m}(t, \mathbf{x})\big| \leqslant C\rho(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{7.6}$$

for some constant $C > 0$ and

$$\frac{1}{T} \int_0^T (\rho, \mathbf{m})(t, \mathbf{x}) \,\mathrm{d}t \to 0 \quad \textit{for a.e. } \mathbf{x} \in \mathbb{R}^d \textit{ when } T \to \infty. \tag{7.7}$$

PROOF. This theorem was established in [54] by developing the fractional Godunov scheme through system (7.4). The proof is divided into five steps, and we now briefly describe them for the case of polytropic gases with $p = \kappa\rho^\gamma$, $1 < \gamma \leqslant 2$.

Step 1. *Construction of approximate solutions via the fractional-step Godunov scheme.* Partition $\mathbb{R}_+$ by the sequence $t_k = kh$, $k \in \mathcal{Z}_+$, with mesh size h and partition $\mathbb{R}_+$ into cells with the jth cell centered at $x_j = jl$, $j \in \mathcal{Z}_+$, with mesh size l. Denote $\mathbf{u}^h = (\rho^h, m^h)$ as the approximate solutions satisfying the inequality

$$\Lambda \equiv \max_{i=1,2} \big(\sup \big|\lambda_i\big(\mathbf{u}^h\big)\big|\big) \leqslant \frac{l}{4h} \leqslant 2\Lambda. \tag{7.8}$$

We will prove that $\mathbf{u}^h(t,x)$ have a uniform bound with respect to h so that it is possible to construct $\mathbf{u}^h(t,x)$ satisfying (7.8).

Assume that $\mathbf{u}^h(t,x)$ have been defined for $t < kh$. Then we define

$$\begin{aligned}
&\mathbf{u}^h(kh+0,x) = \mathbf{u}_j^n \equiv \frac{1}{l}\int_{(j-1/2)l}^{(j+1/2)l} \mathbf{u}^h(kh-0,x)X^h(x)\,\mathrm{d}x,\\
&\left(j-\frac{1}{2}\right)l \leqslant x \leqslant \left(j+\frac{1}{2}\right)l,
\end{aligned} \tag{7.9}$$

where $X^h(x)$ is the characteristic function on $[Nl, \frac{1}{h}]$, where $N = N(C_0) > 0$ is some large constant depending only on $C_0 > 0$, which is solely determined by the initial data (see (7.18)).

In the strip $kh \leqslant t < (k+1)h,\ jl < x \leqslant (j+1)l,\ j,k \in \mathcal{Z}_+$, we define

$$\begin{cases}
\rho^h(t,x) = \rho_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+,\\
m^h(t,x) = m_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+,
\end{cases} \tag{7.10}$$

where $\mathbf{u}_0^h(t,x)$ are the Riemann solutions of (1.16) with initial data $(\mathbf{u}_j^k, \mathbf{u}_{j+1}^k)$ with respect to $x = (j+1/2)l$ at $t = kh$.

From this, we define the fractional step Godunov scheme

$$\mathbf{u}_j^{k+1} = \frac{1}{l}\int_{(j-1/2)l}^{(j+1/2)l} \mathbf{u}^h(kh-0,x)X^h(x)\,\mathrm{d}x. \tag{7.11}$$

In this way, for $kh \leqslant t < (k+1)h$, $k \geqslant 0$ integers, we have

$$\begin{cases}
w_1^h(t,x)\\
\quad = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} + \big(\rho_0^h(t,x)\big)^{(\gamma-1)/2}\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+^{(\gamma-1)/2},\\
w_2^h(t,x)\\
\quad = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} - \big(\rho_0^h(t,x)\big)^{(\gamma-1)/2}\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+^{(\gamma-1)/2},
\end{cases} \tag{7.12}$$

where (w_1, w_2) are the Riemann invariants introduced in (3.6).

Step 2. *L^∞ estimate for the approximate solutions.* There exists $C = C(C_0) > 0$, independent of h, such that

$$0 \leqslant \rho^h(t,x) \leqslant C, \qquad 0 \leqslant m^h(t,x) \leqslant C\rho^h(t,x), \quad (t,x) \in \mathbb{R}_+^2. \tag{7.13}$$

In order to show this estimate, we first need some properties of the Riemann solutions for the homogeneous Euler equations (1.16) with initial Riemann data

$$\mathbf{u} = \begin{cases} \mathbf{u}_- \equiv (\rho_-, m_-), & x < x_0, x_0 > 0, \\ \mathbf{u}_+ \equiv (\rho_+, m_+), & x > x_0, \end{cases} \tag{7.14}$$

and lateral Riemann data

$$\begin{cases} \mathbf{u}|_{t=0} = \mathbf{u}_+, & x > 0, \\ m|_{x=0} = 0, & t > 0, \end{cases} \tag{7.15}$$

where $\rho_\pm \geqslant 0$ and $m_\pm$ are the constants with $|m_\pm/\rho_\pm| < \infty$. The discontinuity in the weak solutions of (1.16) satisfies the Rankine–Hugoniot condition

$$\sigma(\mathbf{u} - \mathbf{u}_0) = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_0), \tag{7.16}$$

where σ is the propagation speed of the discontinuity, $\mathbf{u}_0$ and $\mathbf{u}$ are the corresponding left and right state, respectively. A discontinuity is a shock if it satisfies the entropy condition

$$\sigma\big(\eta(\mathbf{u}) - \eta(\mathbf{u}_0)\big) - \big(q(\mathbf{u}) - q(\mathbf{u}_0)\big) \geqslant 0 \tag{7.17}$$

for any convex entropy–entropy flux pair (η, q).

For the Riemann problems (7.14) and (7.15) for system (1.16), the Riemann solutions generally contain rarefaction waves and shocks satisfying the following facts.

FACT (i). There exists a unique piecewise smooth entropy solution $(\rho, m)(t, x)$ containing vacuum states on the quarter $t \geqslant 0, x \geqslant 0$ for each problem of (7.14) and (7.15), at least locally in time.

FACT (ii). The regions

$$\Sigma = \big\{(\rho, m)\colon\ w_1 \leqslant w_0, w_2 \geqslant z_0, w_1 - w_2 \geqslant 0\big\}$$

and

$$\Sigma = \big\{(\rho, m)\colon\ w_1 \leqslant w_0, w_2 \geqslant z_0, w_1 - w_2 \geqslant 0\big\}, \quad z_0 \leqslant 0 \leqslant \frac{w_0 + z_0}{2},$$

are invariant for the Riemann problems (7.14) and (7.15) for system (1.16), respectively. More precisely, if the Riemann data lies in Σ, the corresponding Riemann solution lies in Σ and its corresponding integral average in x over $[a, b]$ also lies in Σ. □

With these properties of the Riemann solutions to (1.16), we can now establish estimate (7.13).

First we have from assumption (7.5) that

$$0 \leqslant \int_0^{\rho_0^h(x)} \frac{\sqrt{p'(s)}}{s}\,\mathrm{d}s \leqslant \frac{m_0^h(x)}{\rho_0^h(x)} \leqslant C_0 < \infty.$$

This means that there exists $w_0 = w_0(C_0) > 0$ such that

$$\begin{cases} w_1\big(\rho_0^h(x), m_0^h(x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(x), m_0^h(x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(x), m_0^h(x)\big) - w_2\big(\rho_0^h(x), m_0^h(x)\big) \geqslant 0. \end{cases}$$

Fact (ii) indicates that, for $0 \leqslant t \leqslant h$, $(\rho_0^h, m_0^h)(t, x)$ satisfy

$$\begin{cases} w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) - w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \end{cases}$$

which means that there exists $\widehat{C} = \widehat{C}(C_0)$ such that

$$\rho_0^h(t,x) \geqslant 0, \qquad 0 \leqslant \frac{m_0^h(t,x)}{\rho_0^h(t,x)} \leqslant \widehat{C} < \infty.$$

Choose $N = N(C_0)$ such that

$$\frac{(d-1)\widehat{C}(C_0)}{4\Lambda(C_0)N} = 1, \quad \text{that is,} \quad N = N(C_0) \equiv \frac{(d-1)\widehat{C}(C_0)}{4\Lambda(C_0)}. \tag{7.18}$$

Then, for $t \in [0, h)$, we have

$$\begin{cases} \rho^h(t,x) = \rho_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}t\big) \geqslant 0, \\ \frac{m^h(t,x)}{\rho^h(t,x)} = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} \geqslant 0. \end{cases}$$

For $0 \leqslant t < h$, we have

$$\begin{cases} w_1\big(\rho^h(t,x), m^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho^h(t,x), m^h(t,x)\big) - w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0. \end{cases}$$

Suppose that the above inequality holds for $t < kh$. Then, at $t = kh$, we similarly have from Fact (ii) that

$$\begin{cases} w_1\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \leqslant w_0, \\ w_2\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \geqslant 0, \\ w_1\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) - w_2\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \geqslant 0. \end{cases}$$

It follows from Fact (ii) that, for $kh \leqslant t < (k+1)h$,

$$\begin{cases} w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) - w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0. \end{cases}$$

Therefore, we have

$$\begin{cases} w_1\big(\rho^h(t,x), m^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho^h(t,x), m^h(t,x)\big) - w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0 \end{cases}$$

from the fact

$$\begin{cases} \rho^h(t,x) = \rho_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+, \\ \frac{m^h(t,x)}{\rho^h(t,x)} = \frac{m_0^h(t,x)}{\rho_0^h(t,x)}. \end{cases}$$

Then we have again

$$0 \leqslant \rho^h(t,x) \leqslant \widehat{C}, \qquad 0 \leqslant m^h(t,x) \leqslant \widehat{C}\rho^h(t,x),$$

where $\widehat{C} = \widehat{C}(C_0)$ is solely determined by the initial data.

Step 3. *H^{-1}-compactness of entropy dissipation measures for the approximate solutions.* The measure sequence

$$\eta\big(\mathbf{u}^h\big)_t + q\big(\mathbf{u}^h\big)_x \quad \text{is compact in } H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) \tag{7.19}$$

for any weak entropy–entropy flux pair (η, q).

Without loss of generality, we assume that the initial data has compact support because of the finiteness of propagation speed of approximate solutions, hence one can assume

$$\int_0^\infty \rho_0(x)\,\mathrm{d}x + \int_0^\infty m_0(x)\,\mathrm{d}x + \int_0^\infty \eta_*\big(\rho_0(x), m_0(x)\big)\,\mathrm{d}x < \infty,$$

where

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \frac{\kappa}{\gamma-1}\rho^\gamma$$

with corresponding entropy flux

$$q_* = m\left(\frac{m^2}{2\rho^2} + \frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1}\right).$$

For any function $\phi \in C^\infty(\Pi_T)$ with $\Pi_T = [0, T] \times \mathbb{R}_+$, the entropy dissipation measures can be calculated in the form

$$\begin{aligned}
&\iint_{\Pi_T} \big(\eta\big(\mathbf{u}^h\big)\phi_t + q\big(\mathbf{u}^h\big)\phi_x\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = M^h(\phi) + N^h(\phi) + L^h(\phi) + \Sigma^h(\phi),
\end{aligned} \tag{7.20}$$

where

$$\begin{aligned}
M^h(\phi) &= \int_0^\infty \phi(T, x)\eta\big(\mathbf{u}_0^h(T, x)\big)\,\mathrm{d}x - \int_0^\infty \phi(0, x)\eta\big(\mathbf{u}_0^h(0, x)\big)\,\mathrm{d}x, \\
N^h(\phi) &= \iint_{\Pi_T} \big(\big(\eta\big(\mathbf{u}^h\big) - \eta\big(\mathbf{u}_0^h\big)\big)\phi_t + \big(q\big(\mathbf{u}^h\big) - q\big(\mathbf{u}_0^h\big)\big)\phi_x\big)\,\mathrm{d}x\,\mathrm{d}t, \\
L^h(\phi) &= \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \big(\eta\big(\mathbf{u}_{0-}^{hk}\big) - \eta\big(\mathbf{u}_j^k\big)\big)\phi(kh, x)\,\mathrm{d}x, \\
\Sigma^h(\phi) &= \int_0^T \sum \big\{\sigma[\eta]_0 - [q]_0\big\}\phi\big(t, x(t)\big)\,\mathrm{d}t,
\end{aligned}$$

where $\mathbf{u}_-^{hk} = \mathbf{u}^h(kh - 0, x)$, $\phi_j^k = \phi(kh, jl)$, the summation is taken over all the shocks in $\mathbf{u}_0^h$ at a fixed time t, σ is the propagating speed of the shock, and $[\eta]_0$ and $[q]_0$ denote the jumps of $\eta(\mathbf{u}_0^h(t, x))$ and $q(\mathbf{u}_0^h(t, x))$ across the shock in $\mathbf{u}_0^h(t, x)$ from the left to right, respectively.

Noting that (ρ^h, m^h) have compact support in Π_T, one can substitute

$$(\eta, q, \phi) = (\rho, m, 1) \quad \text{and} \quad \left(m, \frac{m^2}{\rho} + p(\rho), 1\right)$$

in (7.20). We conclude

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} m^h(kh - 0, x)\,\mathrm{d}x\,h \leqslant C < \infty, \tag{7.21}$$

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} \frac{(m^h(kh - 0, x))^2}{\rho^h(kh - 0, x)}\,\mathrm{d}x\,h \leqslant C < \infty, \tag{7.22}$$

using the Rankine–Hugoniot condition (7.16) and noting that

$$\begin{cases}
\sum_{k\geqslant 1} \int_0^\infty \big(\rho_{0-}^{hk} - \rho_j^k\big)\,\mathrm{d}x = \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} m^h(kh - 0, x)\,\mathrm{d}x\,h, \\
\sum_{k\geqslant 1} \int_0^\infty \big(m_{0-}^{hk} - m_j^k\big)\,\mathrm{d}x = \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} \frac{(m^h(kh-0,x))^2}{\rho^h(kh-0,x)}\,\mathrm{d}x\,h.
\end{cases}$$

Then we choose $(\eta, q) = (\eta_*, q_*)$ and $\phi = 1$ in (7.20) and use estimates (7.21) and (7.22) to obtain

$$\int_0^T \sum \bigl\{\sigma[\eta_*]_0 - [q_*]_0\bigr\}\,\mathrm{d}t \leqslant C, \tag{7.23}$$

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \int_0^1 (1-\theta)\bigl(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\bigr)^\top \times \nabla^2\eta_*\bigl(\mathbf{u}_j^k + \theta\bigl(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\bigr)\bigr)\bigl(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\bigr)\,\mathrm{d}\theta\,\mathrm{d}x \leqslant C. \tag{7.24}$$

In particular, since $\nabla^2\eta_* \geqslant c_0 > 0$, we obtain

$$\sum_{j,k,0\leqslant jl\leqslant L} \int_{(j-1/2)l}^{(j+1/2)l} \bigl|\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\bigr|^2\,\mathrm{d}x \leqslant C(L). \tag{7.25}$$

Noting that $\mathbf{u}_0^h(t,x)$ are of the form $V(\frac{x-jl}{t-kh})$, then there exists $C(L) < \infty$ such that

$$\sum_{j,k,0\leqslant jl\leqslant L} \int_{(k-1)h}^{kh} \int_{(j-1/2)l}^{(j+1/2)l} \bigl|\mathbf{u}_0^h(t,x) - \mathbf{u}_0^h(kh-0,x)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C(L)h. \tag{7.26}$$

Then, similarly to the proof of Ding, Chen and Luo [116], we use (7.21)–(7.26) to conclude (7.19).

Step 4. *Convergence and consistency.* Applying Theorem 3.1 with (7.13) and (7.19), we see that there exist a subsequence (still denoted by) $\mathbf{u}^h(t,x)$ and an L^∞ function $\mathbf{u}(t,x) \equiv (\rho, m)(t,x)$ such that

$$\mathbf{u}^h(t,x) \to \mathbf{u}(t,x) \quad \text{a.e. when } h \to 0,$$

and

$$0 \leqslant \rho(t,x) \leqslant C, \qquad \bigl|m(t,x)\bigr| \leqslant C\rho(t,x) \quad \text{a.e.} \tag{7.27}$$

It now suffices to check the consistency of the limit function $(\rho, m)(t,x)$ with (7.4). For any nonnegative function $\psi(t,x) \in C_0^\infty(\mathbb{R}_+^2)$, set $\phi(t,x) = x^{d-1}\psi(t,x)$ and $a(x) = (d-1)/x$. Then we have

$$\begin{aligned}
&\iint_{\mathbb{R}_+^2} \bigl(\eta\bigl(\mathbf{u}^h\bigr)\phi_t + q\bigl(\mathbf{u}^h\bigr)\phi_x - a(x)\nabla\eta\bigl(\mathbf{u}^h\bigr)g\bigl(\mathbf{u}^h\bigr)\phi\bigr)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\infty \eta\bigl(\mathbf{u}^h(0,x)\bigr)\phi(0,x)\,\mathrm{d}x \\
&\equiv I_1^h + I_2^h,
\end{aligned} \tag{7.28}$$

where

$$I_1^h = \iint_{\mathbb{R}_+^2} \big(\eta\big(\mathbf{u}_0^h\big)\phi_t + q\big(\mathbf{u}_0^h\big)\phi_x + a(x)\nabla\eta\big(\mathbf{u}_0^h\big)g\big(\mathbf{u}_0^h\big)\phi\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^\infty \eta\big(\mathbf{u}_0^h(0,x)\big)\phi(0,x)\,\mathrm{d}x.$$

Notice that $|\mathbf{u}^h - \mathbf{u}_0^h| \leqslant a(x)|g(\mathbf{u}_0^h)|h$ and $\mathbf{u}^h$ are uniformly bounded. We have

$$\begin{aligned}\big|I_2^h\big| \leqslant C\bigg(h + \iint_{\operatorname{supp}\phi} \big|g\big(\mathbf{u}^h\big) - g\big(\mathbf{u}_0^h\big)\big|\,\mathrm{d}x\,\mathrm{d}t \\ + \iint_{\operatorname{supp}\phi} \big|\nabla\eta\big(\mathbf{u}^h\big) - \nabla\eta\big(\mathbf{u}_0^h\big)\big|\,\mathrm{d}x\,\mathrm{d}t\bigg) \to 0\end{aligned} \tag{7.29}$$

when $h \to 0$. Furthermore,

$$\begin{aligned}I_1^h &= \sum_{k\geqslant 1}\int_0^\infty \big(\eta\big(\mathbf{u}_{0-}^{hk}\big) - \eta\big(\mathbf{u}_j^k\big)\big)\phi(kh,x)\,\mathrm{d}x \\ &\quad - \iint_{\mathbb{R}_+^2} a(x)\nabla\eta\big(\mathbf{u}_0^h\big)g\big(\mathbf{u}_0^h\big)\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\equiv I_{11}^h + I_{12}^h.\end{aligned} \tag{7.30}$$

Notice that

$$\begin{aligned}\big|I_{11}^h\big| &= \bigg|\sum_{j,k}\int_{(j-1/2)l}^{(j+1/2)l} \big(\phi - \phi_j^k\big)\big(\eta\big(\mathbf{u}_{0-}^{hk}\big) - \eta\big(\mathbf{u}_j^k\big)\big)\,\mathrm{d}x\bigg| \\ &\leqslant C\sqrt{h}\|\phi\|_{C_0^1}\bigg(\sum_{j,k,0\leqslant jl\leqslant L}\int_{(j-1/2)l}^{(j+1/2)l} \big|\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\big|^2\,\mathrm{d}x\bigg)^{1/2} \\ &\leqslant C\sqrt{h} \to 0\end{aligned} \tag{7.31}$$

when $h \to 0$ and

$$\begin{aligned}I_{12}^h &\geqslant -\bigg|\sum_{j,k}\int_{(k-1)h}^{kh}\mathrm{d}t\int_{(j-1/2)l}^{(j+1/2)l} a(x)\big(\phi\nabla\eta\big(\mathbf{u}_0^h\big)g\big(\mathbf{u}_0^h\big) \\ &\qquad\qquad - \phi_j^k\nabla\eta\big(\mathbf{u}_{0-}^{hk}\big)g\big(\mathbf{u}_{0-}^{hk}\big)\big)\,\mathrm{d}x\bigg| \\ &\geqslant -\big(J_1^h + J_2^h\big),\end{aligned} \tag{7.32}$$

where

$$J_1^h = \Big| \sum_{j,k} \int_{(k-1)h}^{kh} \mathrm{d}t \int_{(j-1/2)l}^{(j+1/2)l} a(x)\big(\phi - \phi_j^k\big)\nabla\eta\big(\mathbf{u}_{0-}^{hk}\big)g\big(\mathbf{u}_{0-}^{hk}\big)\,\mathrm{d}x \Big|$$
$$\leqslant Ch\|\phi\|_{C^1} \to 0 \tag{7.33}$$

when $h \to 0$ and

$$J_2^h \leqslant \iint_{\{\operatorname{supp}\phi\}\cap\Omega_h} a(x)\big|\nabla\eta\big(\mathbf{u}_{0-}^{hk}\big)g\big(\mathbf{u}_{0-}^{hk}\big) - \nabla\eta\big(\mathbf{u}_0^h\big)g\big(\mathbf{u}_0^h\big)\big|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\{\operatorname{supp}\phi\}\cap\Omega_h^c} a(x)\big|\nabla\eta\big(\mathbf{u}_{0-}^{hk}\big)g\big(\mathbf{u}_{0-}^{hk}\big) - \nabla\eta\big(\mathbf{u}_0^h\big)g\big(\mathbf{u}_0^h\big)\big|\,\mathrm{d}x\,\mathrm{d}t$$

with $\Omega_h = \{(t,x)\colon\ \rho_0(t,x) \geqslant h^{1/4}\}$, and thus

$$J_2^h \leqslant C\left(h^{-1/4}\left(\iint_{\operatorname{supp}\phi} \big|\mathbf{u}_0^h - \mathbf{u}_{0-}^{hk}\big|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} + h^{1/4}\right) \leqslant Ch^{1/4} \to 0 \tag{7.34}$$

when $h \to 0$, by using the Hölder inequality.

Since $(\rho^h, m^h) \to (\rho, m)$ a.e., then it is routine to use the dominated convergence theorem to show that the function $(\rho, \mathbf{m})(t, \mathbf{x})$ determined by the function $(\rho, m)(t, x)$,

$$(\rho, \mathbf{m})(t, \mathbf{x}) = \left(\rho\big(t, |\mathbf{x}|\big), m\big(t, |\mathbf{x}|\big)\frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{7.35}$$

satisfies the standard notion of entropy solutions.

Finally, notice that the pairs $\pm(\rho, m)$ and $\pm(m, m^2/\rho + p(\rho))$ are all convex entropy–entropy flux pairs. It follows that (ρ, m) satisfies (7.4) in the sense of distributions. For (7.4), we take the test function $\phi(t,x) = \alpha(t)X^k(x)$ with

$$\alpha(t) \in C_0^\infty(0,\infty), \quad \alpha(t) \geqslant 0,$$

and

$$\begin{cases} X^k(x) \in C_0^\infty(0,\infty), \quad X^k\big|_{[0,x_0/2]} \equiv 1,\ 0 \leqslant X^k(x) \leqslant 1, \\ X^k(x) \to \chi_{[0,x_0]}(x) \quad \text{as } k \to \infty \end{cases}$$

for Lebesgue points $x_0 \in (0,\infty)$, $x_0 \to 0$, of the function

$$\int_0^\infty \left(m(t,x) + \frac{m(t,x)^2}{\rho(t,x)} + p\big(\rho(t,x)\big)\right)\alpha(t)\,\mathrm{d}t.$$

Then, adding the two identities and setting $k \to \infty$, we have

$$\begin{aligned}
&\int_0^\infty \left(m(t,x_0) + \frac{m(t,x_0)^2}{\rho(t,x_0)} + p\big(\rho(t,x_0)\big)\right)\alpha(t)\,\mathrm{d}t \\
&\quad \leqslant C\left(TV\big(\alpha(\cdot)\big)x_0 + \int_0^\infty\int_0^{x_0} \frac{d-1}{x}\left(m(t,x) + \frac{m(t,x)^2}{\rho(t,x)}\right)\alpha(t)\,\mathrm{d}x\,\mathrm{d}t\right)
\end{aligned}$$

since $\int_0^\infty (\rho(t,x) + m(t,x))\,\mathrm{d}x \leqslant C < \infty$.

Therefore, we have

$$\lim_{x_0\to 0}\int_0^\infty \left(m(t,x_0) + \frac{m(t,x_0)^2}{\rho(t,x_0)} + p\big(\rho(t,x_0)\big)\right)\alpha(t)\,\mathrm{d}t = 0$$

for any $\alpha(t) \in C_0^\infty(0,\infty)$ by using

$$\int_0^\infty\int_0^\infty \frac{d-1}{x} m(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int_0^\infty \frac{d-1}{x}\frac{m(t,x)^2}{\rho(t,x)}\,\mathrm{d}x\,\mathrm{d}t \leqslant C < \infty.$$

Similarly, we take $(\eta, q) = (\rho, m)$ and $(m, \frac{m^2}{\rho} + p(\rho))$ in (5.3), respectively, and take $\phi(t,x) = \alpha^l(t)X^k(x)$ with

$$\begin{cases}
\alpha^l(t) \in C_0^\infty(0,\infty), & \alpha^l\big|_{[0,T-\varepsilon]} \equiv 1,\ 0 \leqslant \alpha^l(t) \leqslant 1, \\
\alpha^l(t) \to \chi_{[0,T]}(t) & \text{as } l \to \infty, \\
TV\big(\alpha^l(\cdot)\big) \leqslant C, & C \text{ independent of } l,
\end{cases}$$

and

$$\begin{cases}
X^k(y) \in C_0^\infty(0,\infty), & X^k\big|_{[x_0/2,x/2]} \equiv 1,\ 0 \leqslant X^k(y) \leqslant 1, \\
X^k(y) \to \chi_{[x_0,x]}(y) & \text{as } k \to \infty \ \text{ for } x \in [x_0,\infty).
\end{cases}$$

Adding the two identities and letting $k \to \infty$, we have

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left(m(t,x) + \frac{m(t,x)^2}{\rho(t,x)} + p\big(\rho(t,x)\big)\right)\alpha^l(t)\,\mathrm{d}t \\
&\quad \leqslant \frac{1}{T} TV\big(\alpha^l(\cdot)\big) \sup_{0\leqslant t\leqslant T}\int_0^\infty \big(\rho(t,y) + m(t,y)\big)\,\mathrm{d}y \\
&\qquad + \frac{1}{T}\int_0^\infty\int_{x_0}^x \frac{d-1}{x}\left(m(t,y) + \frac{m(t,y)^2}{\rho(t,y)}\right)\mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

$$+ \frac{1}{T}\int_0^T \left(m(t,x_0) + \frac{m(t,x_0)^2}{\rho(t,x_0)} + p\big(\rho(t,x_0)\big)\right) \mathrm{d}t$$
$$\leqslant C\left(\frac{1}{T} + \frac{1}{T}\int_0^T \left(m(t,x_0) + \frac{m(t,x_0)^2}{\rho(t,x_0)} + p\big(\rho(t,x_0)\big)\right) \mathrm{d}t\right).$$

Set $x_0 \to 0$ and then $l \to \infty$. We obtain

$$\frac{1}{T}\int_0^T \left(m(t,x) + \frac{m(t,x)^2}{\rho(t,x)} + p\big(\rho(t,x)\big)\right) \mathrm{d}t \leqslant \frac{C}{T} \quad \text{for a.e. } x \in \mathbb{R}_+.$$

This implies that

$$\frac{1}{T}\int_0^T (\rho, m)(t,x)\, \mathrm{d}t \to (0,0) \quad \text{for a.e. } x \in \mathbb{R}_+ \text{ as } T \to \infty,$$

which arrives at (7.7). This completes the proof. □

7.2. *Self-similar solutions*

The second type of geometric structure is self-similarity. One of the most challenging problems is to study solutions with data that give rise to self-similar solutions (such solutions especially include Riemann solutions) and to develop a unifying framework to treat hyperbolic–elliptic mixed problems with mixed boundary conditions that are derived from compressible flows.

Compressible flow equations in two space dimensions with one or more linearly degenerate modes of wave propagation have additional difficulties. In that case, the global flow is governed by a reduced (self-similar) system which is of both (hyperbolic–elliptic) mixed and composite type in the subsonic region. The linearly degenerate waves give rise to one or more families of degenerate characteristics which remain real in the subsonic region. The reduced equations typically couple a hyperbolic–elliptic mixed problem for the density and/or the pressure with a hyperbolic (transport) equation for the vorticity.

For the Euler equations (1.4) for $\mathbf{x} \in \mathbb{R}^2$, self-similar solutions

$$(\rho, u_1, u_2, p) = (\rho, u_1, u_2, p)(\xi, \eta), \quad (\xi, \eta) = \frac{\mathbf{x}}{t},$$

are determined by

$$\begin{cases} \partial_\xi(\rho U) + \partial_\eta(\rho V) = -2\rho, \\ \partial_\xi\big(\rho U^2 + p\big) + \partial_\eta(\rho U V) = -3\rho U, \\ \partial_\xi(\rho U V) + \partial_\eta\big(\rho V^2 + p\big) = -3\rho V, \\ \partial_\xi\big(U(E+p)\big) + \partial_\eta\big(V(E+p)\big) = -2(E+p), \end{cases} \tag{7.36}$$

where $(U, V) = (u_1 - \xi, u_2 - \eta)$ is the pseudovelocity and $E = \rho(e + (U^2 + V^2)/2)$.

It is straightforward to calculate and obtain four eigenvalues

$$\lambda_0 = \frac{V}{U} \quad \text{(two multiplicity)}$$

and

$$\lambda_{\pm} = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2},$$

where c is the sonic speed.

When $U^2 + V^2 > c^2$, system (7.36) is hyperbolic with four real eigenvalues and the flow is called pseudosupersonic.

When $U^2 + V^2 < c^2$, system (7.36) is hyperbolic–elliptic composite type (two repeated eigenvalues are real and the other two are complex): two equations are hyperbolic and the other two are elliptic.

The region $U^2 + V^2 = c^2$ in the (ξ, η) plane is called the pseudosonic region in the flow.

In general, system (7.36) is both hyperbolic–elliptic mixed and composite type, and the flow is pseudotransonic.

For a bounded solution (ρ, u_1, u_2, p), the flow must be pseudosupersonic when $\xi^2 + \eta^2 \to \infty$.

An important prototype problem for both practical applications and the theory of multidimensional complex wave patterns is the problem of diffraction of a shock wave which is incident along an inclined ramp. When a plane shock hits a wedge head on, a self-similar reflected shock moves outward as the original shock moves forward (e.g., [15,43,100,150, 252,291,328]). The computational and asymptotic analysis shows that various patterns of reflected shocks may occur, including regular and Mach reflections. The reflected shock is a transonic shock in the self-similar coordinates, for which the corresponding equation changes its type from hyperbolic to elliptic across the shock. There has been no rigorous mathematical result on the *global* existence and structure of shock reflections for the potential flow equation and the full Euler equations. Some results were recently obtained for simplified models. The transonic small-disturbance (TSD) equation in Section 4.3 was derived and used in [173,177,187,252] and the references cited therein for asymptotic analysis of shock reflections; and some steps of this analysis have been justified in [40]. Zheng [357] made an effort on the existence of a regular reflection solution for the pressure gradient equation when the wedge is close to a flat wall.

It is important to establish the existence and stability of shock reflection solutions and clarify the transition among regular reflection, simple Mach reflection, double Mach reflection, and complex Mach reflection.

A good starting point is the potential flow equation (4.4) for this problem. A self-similar solution is a solution of the form

$$\Psi = t\phi(\xi, \eta), \quad (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

By introducing the function

$$\varphi(\xi,\eta) = -\frac{\xi^2+\eta^2}{2} + \phi(\xi,\eta),$$

the system can be rewritten in the form of a second-order equation of mixed hyperbolic–elliptic type

$$\operatorname{div}_{(\xi,\eta)}\big(\rho\big(|\nabla\varphi|^2,\varphi\big)\nabla\varphi\big) + 2\rho\big(|\nabla\varphi|^2,\varphi\big) = 0 \tag{7.37}$$

with

$$\rho(q^2,z) = \left(1 - \frac{q^2+2z}{2}\right)^{1/(\gamma-1)}.$$

Similar to (6.1), equation (7.37) at $|\nabla\varphi| = q$ is hyperbolic (pseudosupersonic) if

$$\rho\big(q^2,z\big) + q\rho_q\big(q^2,z\big) < 0$$

and elliptic (pseudosubsonic) if

$$\rho\big(q^2,z\big) + q\rho_q\big(q^2,z\big) > 0.$$

The nature of the shock reflection pattern has been explored in [252] for weak incident shocks (strength b) and small wedge angles $2\theta_w$ by a number of different scalings, a study of mixed equations, and matching asymptotics for the different scalings, where the parameter $\beta = c_1\theta_w^2/b(\gamma+1)$ ranges from 0 to ∞ and c_1 is the sound speed behind the incident shock. It was shown that, for $\beta > 2$, regular reflection of both strong and weak kinds is possible as well as Mach reflection; for $\beta < 1/2$, Mach reflection occurs and the flow behind the reflection is subsonic and can be constructed in principle (with an open elliptic problem) and matched; and for $1/2 < \beta < 2$, the flow behind a Mach reflection may be transonic and the corresponding nonlinear boundary value problem of mixed type has been discussed. The basic pattern of reflection was shown to be an almost semicircular shock, for regular reflection, emanating from the reflection point on the wedge and, for Mach reflection, matched with a local interaction flow. It is important to establish some rigorous proofs for this problem with the aid of free boundary approaches as discussed in Section 6.3. Such a rigorous proof for the existence of shock reflection solutions has successfully been established in [63] when the wedge angle is large.

7.3. *Global solutions with special Cauchy data*

Several cases of initial data for the Cauchy problem may be solved for constructing global solutions for the compressible Euler equations (1.11) or (1.4).

CASE 1. Initial data of the form

$$(\rho, u_1, u_2)|_{t=0} = \begin{cases} (\rho_-, u_{1-}, u_{2-}) & \text{if } L(\mathbf{x}) < 0, \mathbf{x} \in \mathbb{R}^2, \\ (\rho_+, u_{1+}, u_{2+}) & \text{if } L(\mathbf{x}) > 0, \end{cases} \tag{7.38}$$

for (1.11). The initial discontinuity $L(\mathbf{x}) = 0$ is a smooth curve which separates the $\mathbf{x}$-plane into two unbounded parts, and $\nabla_{\mathbf{x}} L$ is continuous. This Cauchy problem (1.11) and (7.38) can be considered as a multidimensional generalization of the one-dimensional Riemann problem. It is also a natural problem from the viewpoint of physics. Conventional self-similarity transformations or symmetric transformations are not available to such a problem.

Certain preliminary observations have shown in the case where the global solutions are connected by two-dimensional rarefaction waves, with the discontinuity $L(\mathbf{x}) = 0$ being convex or concave, and two initial constants (ρ_-, u_{1-}, u_{2-}) and (ρ_+, u_{1+}, u_{2+}) satisfying a natural relation. A natural strategy is to develop the so-called envelope method and some particular implicit functions which may enable the construction of the two-dimensional rarefaction waves to be possible. It has also been observed that the state functions inside the rarefaction waves and the intermediate state functions between the two rarefaction waves must be smooth. It is interesting to obtain a complete global solution. For the pressureless Euler equations, some results have been obtained by Yang and Huang [343].

CASE 2. Initial data of the form

$$(\rho, u_1, u_2)|_{t=0} = \begin{cases} (\rho_-, u_{1-}, u_{2-}) & \text{if } L(\mathbf{x}) < 0, \\ (\rho_-, u_{1+}, u_{2+})(\mathbf{x}) & \text{if } L(\mathbf{x}) > 0, \end{cases}$$

where (ρ_-, u_{1-}, u_{2-}) is a constant state and $(\rho_+, u_{1+}, u_{2+})(\mathbf{x})$ is a smooth initial function. It is important to determine the class of initial functions $(\rho_+, u_{1+}, u_{2+})(\mathbf{x})$ which leads to the existence of two-dimensional global solutions that have only a single shock for such special initial data. In this regard, see [159] and [160].

CASE 3. Initial data consists of four different constant states $\mathbf{u}_i = (\rho_i, u_1^i, u_2^i)$, $i = 1, 2, 3, 4$, corresponding to four quadrants with a special relationship among the states, so that the unfolding solution at infinity consists of only one rarefaction wave along the direction of each semiaxis. Chang, Chen and Yang [44,45] and Lax and Liu [203] have similar numerical results for this case. The contour curve of the density ρ is simple: the two groups of planar rarefaction waves, R_{12} (along the η^+ axis) and R_{41} (along the ξ^+ axis), R_{34} (along the η^- axis) and R_{23} (along the ξ^- axis), are connected by a family of straight lines $\xi + \eta = \alpha$, where α is a constant parameter. Hence, ρ is symmetric about $\xi - \eta = \tilde{\alpha}$ for a particular $\tilde{\alpha}$, while the contour curve of the self-Mach number is relatively complex but follows some rules.

It is interesting to construct two-dimensional global solutions to the Euler equations (1.11) with this type of initial data. The idea is first to estimate the solution of ρ from its contour curve, then to plug ρ into equations (1.11), according to the symmetry of ρ, so as to construct u_1 and u_2.

8. Divergence-measure fields and hyperbolic conservation laws

Naturally, we want to approach the questions of existence, stability, uniqueness, and long-time behavior of entropy solutions for multidimensional compressible flows for fluids (as represented by the Euler equations of inviscid flows such as system (1.4) and system (1.11)) and solids (as represented by the equations of nonlinear elastodynamics such as system (4.15), (4.17) and (4.18)) with as much generality as possible. In this section, we discuss some recent efforts in developing a theory of divergence-measure fields to construct a global framework for studying solutions of multidimensional compressible flows and, more generally, hyperbolic systems of conservation laws.

8.1. *Connections*

Consider a system of hyperbolic conservation laws in d space dimensions in (1.1). As mentioned earlier, the main feature of nonlinear hyperbolic conservation laws (1.1), especially (1.4) or (1.11), is that, no matter how smooth the initial data is, solutions may develop singularities and form shock waves and vorticity waves, among others, in finite time. For the one-dimensional problem of (1.1), in particular, for the one-dimensional version of the Euler equations (1.15) or (1.17) in Lagrangian coordinates, one may expect solutions in *BV*; this is indeed the case by Glimm's theorem [145] which indicates that there exists a global entropy solution in *BV* when the initial data has sufficiently small total variation and stays away from vacuum. On the other hand, when the initial data is large, even away from vacuum, solutions may develop vacuum instantaneously as $t > 0$ or approach the vacuum states indefinitely. In this case, the specific volume $\tau = 1/\rho$ may become a Radon measure or an L^1 function, rather than a *BV* function (cf. [332]).

In particular, we emphasize again that, as discussed in Section 2.6, the *BV* bound generically fails for multidimensional hyperbolic conservation laws. In general, for multidimensional conservation laws, especially the Euler equations, because of complex interactions among shocks, rarefaction waves, vortex sheets, and vorticity waves, solutions of (1.1) are expected to be in the following class of entropy solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ or $L^p(\mathbb{R}^{d+1}_+)$, $1 \leqslant p \leqslant \infty$,

(ii) $\mathbf{u}(t,\mathbf{x})$ satisfies the Lax entropy inequality

$$\mu_\eta := \partial_t \eta\big(\mathbf{u}(t,\mathbf{x})\big) + \nabla_{\mathbf{x}} \cdot \mathbf{q}\big(\mathbf{u}(t,\mathbf{x})\big) \leqslant 0 \quad \text{in the sense of distributions,} \tag{8.1}$$

for any convex entropy–entropy flux pair $(\eta,\mathbf{q}):\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^d$ so that $\eta(\mathbf{u}(t,\mathbf{x}))$ and $\mathbf{q}(\mathbf{u}(t,\mathbf{x}))$ are distributions.

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information on solutions, including large-time behavior, uniqueness, stability, and existence of traces, *with neither specific reference to any particular method for constructing the solutions nor additional regularity assumptions*.

The Schwartz lemma infers from (8.1) that the distribution μ_η is in fact a Radon measure,

$$\operatorname{div}_{(t,\mathbf{x})}\big(\eta\big(\mathbf{u}(t,\mathbf{x})\big), \mathbf{q}\big(\mathbf{u}(t,\mathbf{x})\big)\big) \in \mathcal{M}\big(\mathbb{R}^{d+1}_+\big).$$

Furthermore, when $\mathbf{u} \in L^\infty$, this is also true for any C^2 entropy–entropy flux pair $(\eta, \mathbf{q})$ (η not necessarily convex) if (1.1) has a strictly convex entropy, which was first observed in [51].

More generally, we have the following definition.

DEFINITION. Let $\mathcal{D} \subset \mathbb{R}^N$ be open. For $1 \leqslant p \leqslant \infty$, $\mathbf{F}$ is called a $\mathcal{D}M^p(\mathcal{D})$ field if $\mathbf{F} \in L^p(\mathcal{D}; \mathbb{R}^N)$ and

$$\|\mathbf{F}\|_{\mathcal{D}M^p(\mathcal{D})} := \|\mathbf{F}\|_{L^p(\mathcal{D};\mathbb{R}^N)} + \|\operatorname{div}\mathbf{F}\|_{\mathcal{M}(\mathcal{D})} < \infty, \tag{8.2}$$

and the field $\mathbf{F}$ is called a $\mathcal{D}M^{\text{ext}}(\mathcal{D})$-field if $\mathbf{F} \in \mathcal{M}(\mathcal{D}; \mathbb{R}^N)$ and

$$\|\mathbf{F}\|_{\mathcal{D}M^{\text{ext}}(\mathcal{D})} := \big\|(\mathbf{F}, \operatorname{div}\mathbf{F})\big\|_{\mathcal{M}(\mathcal{D})} < \infty. \tag{8.3}$$

Furthermore, for any bounded open set $\mathcal{D} \subset \mathbb{R}^N$, $\mathbf{F}$ is called a $\mathcal{D}M^p_{\text{loc}}(\mathbb{R}^N)$ field if $\mathbf{F} \in \mathcal{D}M^p(\mathcal{D})$, and $\mathbf{F}$ is called a $\mathcal{D}M^{\text{ext}}_{\text{loc}}(\mathbb{R}^N)$ if $\mathbf{F} \in \mathcal{D}M^{\text{ext}}(\mathcal{D})$. A field $\mathbf{F}$ is simply called a $\mathcal{D}M$ field in $\mathcal{D}$ if $\mathbf{F} \in \mathcal{D}M^p(\mathcal{D})$, $1 \leqslant p \leqslant \infty$, or $\mathbf{F} \in \mathcal{D}M^{\text{ext}}(\mathcal{D})$.

It is easy to check that these spaces, under the respective norms $\|\mathbf{F}\|_{\mathcal{D}M^p(\mathcal{D})}$ and $\|\mathbf{F}\|_{\mathcal{D}M^{\text{ext}}(\mathcal{D})}$ are Banach spaces. These spaces are larger than the space of BV fields. The establishment of the Gauss–Green theorem, traces, and other properties of BV functions in the 1950s (cf. [133]; also [8,144,330]) has significantly advanced our understanding of solutions of nonlinear partial differential equations and related problems in the calculus of variations, differential geometry and other areas, especially for the one-dimensional theory of hyperbolic conservation laws. A natural question is whether the $\mathcal{D}M$ fields have similar properties, especially the normal traces and the Gauss–Green formula to deal with entropy solutions for multidimensional conservation laws. At a first glance, it seems impossible due to the Whitney paradox [338].

EXAMPLE 8.1 (Whitney paradox [338]). The field

$$\mathbf{F}(y_1, y_2) = \left(\frac{-y_2}{y_1^2 + y_2^2}, \frac{y_1}{y_1^2 + y_2^2}\right)$$

belongs to $\mathcal{D}M^1_{\text{loc}}(\mathbb{R}^2)$; however, for $\Omega = (0,1) \times (0,1)$,

$$\int_\Omega \operatorname{div}\mathbf{F} = 0 \neq \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}\mathcal{H}^1 = \frac{\pi}{2},$$

if one understands $\mathbf{F} \cdot \mathbf{n}$ in the classical sense. This implies that the classical Gauss–Green theorem fails.

EXAMPLE 8.2. For any $\mu_i \in \mathcal{M}(\mathbb{R})$, $i = 1, 2$, with finite total variation,

$$\mathbf{F}(y_1, y_2) = \big(\mu_1(y_2), \mu_2(y_1)\big) \in \mathcal{D}M^{\text{ext}}\big(\mathbb{R}^2\big).$$

A nontrivial example of such fields is provided by the Riemann solutions of the one-dimensional Euler equations in Lagrangian coordinates for which vacuum generally develops (see [69]).

On the other hand, motivated by various nonlinear problems from conservation laws, as well as for rigorous derivation of systems of balance laws with measure source terms from the physical principle of balance law and the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation, a suitable notion of normal traces and corresponding Gauss–Green formula for divergence-measure fields are required.

Some earlier efforts were made on generalizing the Gauss–Green theorem for some special situations, and relevant results can be found in [9] for an abstract formulation for $\mathbf{F} \in L^\infty$, Rodrigues [283] for $\mathbf{F} \in L^2$, and Ziemer [358] for a related problem for $\operatorname{div} \mathbf{F} \in L^1$; also see [12,36] and [359]. In [67], an explicit way to calculate the suitable normal traces was first observed for $\mathbf{F} \in \mathcal{DM}^\infty$, under which a generalized Gauss–Green theorem was shown to hold, which has motivated the establishment of a theory of divergence-measure fields in [67,69,83,84].

8.2. *Basic properties of divergence-measure fields*

Now we list some basic properties of divergence-measure fields.

PROPOSITION 8.1. (i) *Let $\{\mathbf{F}_j\}$ be a sequence in $\mathcal{DM}^p(\mathcal{D})$ such that*

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } L^p_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \text{ for } 1 \leqslant p < \infty, \tag{8.4}$$

$$\mathbf{F}_j \overset{*}{\rightharpoonup} \mathbf{F} \quad \text{in } L^\infty_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \text{ for } p = \infty. \tag{8.5}$$

Then

$$\|\mathbf{F}\|_{L^p(\mathcal{D})} \leqslant \liminf_{j\to\infty} \|\mathbf{F}_j\|_{L^p(\mathcal{D})}, \qquad |\operatorname{div}\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}).$$

(ii) *Let $\{\mathbf{F}_j\}$ be a sequence in $\mathcal{DM}^{\text{ext}}(\mathcal{D})$ such that*

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } \mathcal{M}_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big).$$

Then

$$|\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\mathbf{F}_j|(\mathcal{D}), \qquad |\operatorname{div}\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}).$$

In particular, if $\mathbf{F}$ has compact support in $\mathcal{D}$, then

$$|\operatorname{div}\mathbf{F}_j|(\mathcal{D}) \to |\operatorname{div}\mathbf{F}|(\mathcal{D}) \quad \text{as } j \to \infty.$$

This proposition immediately implies that spaces $\mathcal{D}M^p(\mathcal{D})$, $1 \leqslant p \leqslant \infty$, and $\mathcal{D}M^{\text{ext}}(\mathcal{D})$ are Banach spaces under norms (8.2) and (8.3), respectively.

PROPOSITION 8.2. *Let $\{\mathbf{F}_j\}$ be a sequence in $\mathcal{D}M(\mathcal{D})$ satisfying*

$$\lim_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}) = |\operatorname{div}\mathbf{F}|(\mathcal{D})$$

and one of the following three conditions

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \textit{in } L^p_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \textit{ for } 1 \leqslant p < \infty,$$
$$\mathbf{F}_j \overset{*}{\rightharpoonup} \mathbf{F} \quad \textit{in } L^\infty_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \textit{ for } p = \infty,$$
$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \textit{in } \mathcal{M}_{\text{loc}}\big(\mathcal{D};\mathbb{R}^N\big).$$

Then, for every open set $\Omega \subset \mathcal{D}$,

$$|\operatorname{div}\mathbf{F}|\big(\overline{\Omega}\cap\mathcal{D}\big) \geqslant \limsup_{j\to\infty} |\operatorname{div}\mathbf{F}_j|\big(\overline{\Omega}\cap\mathcal{D}\big). \tag{8.6}$$

In particular, if $|\operatorname{div}\mathbf{F}|(\partial\Omega\cap\mathcal{D}) = 0$, then

$$|\operatorname{div}\mathbf{F}|(\Omega) = \lim_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\Omega). \tag{8.7}$$

We now use the standard positive symmetric mollifiers $\omega:\mathbb{R}^N \to \mathbb{R}$ satisfying

$$\omega(\mathbf{y}) \in C_0^\infty\big(\mathbb{R}^N\big), \quad \omega(\mathbf{y}) \geqslant 0,\ \omega(\mathbf{y}) = \omega\big(|\mathbf{y}|\big),\ \int_{\mathbb{R}^N} \omega(\mathbf{y})\,\mathrm{d}\mathbf{y} = 1,$$
$$\operatorname{supp}\omega(\mathbf{y}) \subset B_1 \equiv \big\{\mathbf{y}\in\mathbb{R}^N:\ |\mathbf{y}| < 1\big\}.$$

We denote

$$\omega^\varepsilon(\mathbf{y}) = \varepsilon^{-N}\omega\left(\frac{\mathbf{y}}{\varepsilon}\right), \qquad \mathbf{F}^\varepsilon = \mathbf{F}*\omega^\varepsilon, \tag{8.8}$$

that is,

$$\mathbf{F}^\varepsilon(\mathbf{y}) = \varepsilon^{-N}\int_{\mathbb{R}^N} \mathbf{F}(\mathbf{x})\omega\left(\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}\right)\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \mathbf{F}(\mathbf{y}+\varepsilon\mathbf{x})\omega(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{8.9}$$

Then $\mathbf{F}^\varepsilon \in C^\infty(\Omega;\mathbb{R}^N)$ for any $\Omega \Subset \mathcal{D}$ when ε is sufficiently small. We recall that, for any $f, g \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f^\varepsilon g\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} f g^\varepsilon\,\mathrm{d}\mathbf{x}. \tag{8.10}$$

The following fact for $\mathcal{D}M$ fields is analogous to a well-known property of BV functions.

PROPOSITION 8.3. *Let* $\mathbf{F} \in \mathcal{DM}(\mathcal{D})$. *Let* $\Omega \Subset \mathcal{D}$ *be open and* $|\operatorname{div}\mathbf{F}|(\partial\Omega) = 0$. *Then, for any* $\varphi \in C(\mathcal{D};\mathbb{R})$,

$$\lim_{\varepsilon\to 0}\left\langle \operatorname{div}\mathbf{F}^{\varepsilon}, \varphi\chi_{\Omega}\right\rangle = \langle \operatorname{div}\mathbf{F}, \varphi\chi_{\Omega}\rangle.$$

Furthermore, if $\mathbf{F} \in \mathcal{DM}^{\mathrm{ext}}(\mathcal{D})$ *and* $|\mathbf{F}|(\partial\Omega) = 0$, *then, for any* $\varphi \in C(\mathcal{D};\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0}\left\langle \mathbf{F}^{\varepsilon}, \varphi\chi_{\Omega}\right\rangle = \langle \mathbf{F}, \varphi\chi_{\Omega}\rangle.$$

Now we discuss some product rules for divergence-measure fields.

PROPOSITION 8.4. *Let* $\mathbf{F} = (F_1,\dots,F_N) \in \mathcal{DM}(\mathcal{D})$. *Let* $g \in BV \cap L^{\infty}(\mathcal{D})$ *be such that*

(i) $\partial_{y_j} g(\mathbf{y})$ *is* $|F_j|$*-integrable, for each* $j = 1,\dots,N$,
(ii) *the set of non-Lebesgue points of* $\partial_{y_j} g(\mathbf{y})$ *has* $|F_j|$*-measure zero,*
(iii) $g(\mathbf{y})$ *is* $(|\mathbf{F}| + |\operatorname{div}\mathbf{F}|)$*-integrable,*
(iv) *the set of non-Lebesgue points of* $g(\mathbf{y})$ *has* $(|\mathbf{F}| + |\operatorname{div}\mathbf{F}|)$*-measure zero.*

Then $g\mathbf{F} \in \mathcal{DM}(\mathcal{D})$ *and*

$$\operatorname{div}(g\mathbf{F}) = g\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla g. \tag{8.11}$$

In particular, if $\mathbf{F} \in \mathcal{DM}^{\infty}(\mathcal{D})$, *then* $g\mathbf{F} \in \mathcal{DM}^{\infty}(\mathcal{D})$ *for any* $g \in BV \cap L^{\infty}(\mathcal{D})$; *moreover, if* g *is also Lipschitz over any compact set in* $\mathcal{D}$, *then*

$$\operatorname{div}(g\mathbf{F}) = g\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla g. \tag{8.12}$$

In fact, for $\mathbf{F} \in \mathcal{DM}^{\infty}(\mathcal{D})$, one may refine the above result to yield that (8.12) holds a.e. in a more general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. For $g \in BV$, let $(\nabla g)_{\mathrm{ac}}$ and $(\nabla g)_{\mathrm{sing}}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. Then we have the proposition.

PROPOSITION 8.5. *Given* $\mathbf{F} \in \mathcal{DM}^{\infty}(\mathcal{D})$ *and* $g \in BV \cap L^{\infty}(\mathcal{D})$, *the identity*

$$\operatorname{div}(g\mathbf{F}) = \bar{g}\operatorname{div}\mathbf{F} + \overline{\mathbf{F}\cdot\nabla g}$$

holds in the sense of Radon measures in $\mathcal{D}$, *where* $\bar{g}$ *is the limit of a mollified sequence for* g *through a positive symmetric mollifier, and* $\overline{\mathbf{F}\cdot\nabla g}$ *is a Radon measure, absolutely continuous with respect to* $|\nabla g|$, *whose absolutely continuous part with respect to the Lebesgue measure in* $\mathcal{D}$ *coincides with* $\mathbf{F}\cdot(\nabla g)_{\mathrm{ac}}$ *almost everywhere in* $\mathcal{D}$.

8.3. *Normal traces and the Gauss–Green formula*

We now discuss the Gauss–Green formula for $\mathcal{DM}$ fields over $\Omega \subset \mathcal{D}$ by introducing a suitable notion of normal traces over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary, established in [67,69].

DEFINITION 8.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that

(i) for any $\mathbf{x} \in \partial\Omega$, there exist $r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(\mathbf{x}, r) = \left\{\mathbf{y} \in \mathbb{R}^N:\ \gamma(y_1, \ldots, y_{N-1}) < y_N\right\} \cap Q(\mathbf{x}, r),$$

where $Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N:\ |y_i - x_i| \leqslant r,\ i = 1, \ldots, N\}$,

(ii) there exists $\Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ such that Ψ is a homeomorphism, bi-Lipschitz over its image, and $\Psi(\omega, 0) = \omega$ for any $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0, 1]$, and denote Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$.

REMARK 8.1. The domains with deformable Lipschitz boundaries clearly include bounded domains with Lipschitz boundaries, the star-shaped domains and the domains whose boundaries satisfy the cone property. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\overline{\Omega}$ with a Lipschitz deformable boundary, then Ω itself possesses a Lipschitz deformable boundary.

For $\mathcal{D}M^p$ fields with $1 < p \leqslant \infty$, we have the following theorem.

THEOREM 8.1. *Let $\mathbf{F} \in \mathcal{D}M^p(\mathcal{D})$, $1 < p \leqslant \infty$. Let $\Omega \subset \mathcal{D}$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $\mathbf{F} \cdot \mathbf{n}$ over $\mathrm{Lip}(\partial\Omega)$ such that, for any $\phi \in \mathrm{Lip}(\mathbb{R}^N)$,*

$$\langle \mathbf{F} \cdot \mathbf{n}, \phi\rangle_{\partial\Omega} = \langle \operatorname{div} \mathbf{F}, \phi\rangle_{\Omega} + \int_{\Omega} \nabla\phi \cdot \mathbf{F}\, \mathrm{d}x. \tag{8.13}$$

Moreover, let $\mathbf{n} : \Psi(\partial\Omega \times [0, 1]) \to \mathbb{R}^N$ be such that $\mathbf{n}(\mathbf{x})$ is the unit outer normal to $\partial\Omega_s$ at $\mathbf{x} \in \partial\Omega_s$, defined for a.e. $\mathbf{x} \in \Psi(\partial\Omega \times [0, 1])$. Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$, that is,

$$h(\mathbf{y}) := \begin{cases} 0 & \text{for } \mathbf{y} \in \mathbb{R}^N - \overline{\Omega}, \\ 1 & \text{for } \mathbf{y} \in \Omega - \Psi\big(\partial\Omega \times [0, 1]\big), \\ s & \text{for } \mathbf{y} \in \partial\Omega_s, 0 \leqslant s \leqslant 1. \end{cases}$$

Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \psi\rangle_{\partial\Omega} = -\lim_{s \to 0} \frac{1}{s} \int_{\Psi(\partial\Omega \times (0,s))} \mathcal{E}(\psi) \nabla h \cdot \mathbf{F}\, \mathrm{d}\mathbf{x}, \tag{8.14}$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to the whole space $\mathbb{R}^N$.

In the case $p = \infty$, *the normal trace* $\mathbf{F} \cdot \mathbf{n}$ *is a function in* $L^\infty(\partial\Omega)$ *satisfying*

$$\|\mathbf{F} \cdot \mathbf{n}\|_{L^\infty(\partial\Omega)} \leqslant C\|\mathbf{F}\|_{L^\infty(\Omega)}$$

for some constant C *independent of* $\mathbf{F}$. *Furthermore, for any field* $\mathbf{F} \in \mathcal{D}M^\infty(\Omega)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \psi\rangle_{\partial\Omega} = \operatorname{ess}\lim_{s\to 0} \int_{\partial\Omega_s} (\mathbf{F} \cdot \mathbf{n})\big(\psi \circ \Psi_s^{-1}\big)\, \mathrm{d}\mathcal{H}^{N-1} \quad \textit{for any } \psi \in L^1(\Omega). \tag{8.15}$$

Finally, for $\mathbf{F} \in \mathcal{D}M^p(\Omega)$ *with* $1 < p < \infty$, $\mathbf{F} \cdot \mathbf{n}$ *can be extended to a continuous linear functional over* $W^{1-1/p,p} \cap L^\infty(\partial\Omega)$.

EXAMPLE 8.3. The field

$$\mathbf{F}(y_1, y_2) = \left(\sin\left(\frac{1}{y_1 - y_2}\right), \sin\left(\frac{1}{y_1 - y_2}\right)\right)$$

belongs to $\mathcal{D}M^\infty(\mathbb{R}^2)$. It is impossible to define any reasonable notion of traces over the line $y_1 = y_2$ for the component $\sin(1/(y_1 - y_2))$. Nevertheless, the unit normal $\mathbf{n}_\tau$ to the line $y_1 - y_2 = \tau$ is the vector $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ so that the scalar product $\mathbf{F}(y_1, y_1 - \tau) \cdot \mathbf{n}_\tau$ is identically zero over this line. Hence, we find that

$$\mathbf{F} \cdot \mathbf{n} \equiv 0 \quad \text{over the line } y_1 = y_2$$

and the Gauss–Green formula implies that, for any $\phi \in C_0^1(\mathbb{R}^2)$,

$$0 = \langle \operatorname{div} \mathbf{F}|_{y_1 > y_2}, \phi\rangle = -\int_{y_1 > y_2} \mathbf{F} \cdot \nabla\phi \, \mathrm{d}\mathbf{y}.$$

This identity could also be directly obtained by applying the dominated convergence theorem to the analogous identity obtained from the classical Gauss–Green formula.

As indicated by Examples 8.1 and 8.2, it is more delicate for fields in $\mathcal{D}M^1$ and $\mathcal{D}M^{\mathrm{ext}}$. Then we have to define the normal traces as functionals over the spaces $\operatorname{Lip}(\gamma, \partial\Omega)$ with $\gamma > 1$ (see [309]).

For $1 < \gamma \leqslant 2$, the elements of $\operatorname{Lip}(\gamma, \partial\Omega)$ are $(N + 1)$-components vectors, where the first component is the function itself, and the other N components are its "first-order partial derivatives". In particular, as a functional over $\operatorname{Lip}(\gamma, \partial\Omega)$, the values of the normal trace of a field in $\mathcal{D}M^1$ or $\mathcal{D}M^{\mathrm{ext}}$ on $\partial\Omega$ depend on not only the values of the respective functions over $\partial\Omega$ but also the values of their first-order derivatives over $\partial\Omega$. To define the normal traces for $\mathbf{F} \in \mathcal{D}M^1(\Omega)$ or $\mathcal{D}M^{\mathrm{ext}}(\Omega)$, we resort to the properties of the Whitney extensions of functions in $\operatorname{Lip}(\gamma, \partial\Omega)$ to $\operatorname{Lip}(\gamma, \mathbb{R}^N)$.

We have the following analogue of Theorem 8.1 which covers fields in $\mathcal{D}M^1$ and $\mathcal{D}M^{\mathrm{ext}}$.

THEOREM 8.2. *Let* $\mathbf{F} \in \mathcal{D}M^1(\mathcal{D})$ *or* $\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$. *Let* $\Omega \subset \mathcal{D}$ *be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional* $\mathbf{F} \cdot \mathbf{n}$ *over* $\mathrm{Lip}(\gamma, \partial\Omega)$ *for any* $\gamma > 1$ *such that, for any* $\phi \in \mathrm{Lip}(\gamma, \mathbb{R}^N)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \phi \rangle_{\partial\Omega} = \langle \operatorname{div} \mathbf{F}, \phi \rangle_{\Omega} + \langle \mathbf{F}, \nabla\phi \rangle_{\Omega}. \tag{8.16}$$

Moreover, let $h : \mathbb{R}^N \to \mathbb{R}$ *be the level set function as defined in Theorem* 8.1; *and in the case that* $\mathbf{F} \in \mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$, *we also assume that* $\partial_{x_i} h$ *is* $|F_i|$*-measurable and its set of non-Lebesgue points has* $|F_i|$*-measure zero,* $i = 1, \ldots, N$. *Then, for any* $\psi \in \mathrm{Lip}(\gamma, \partial\Omega)$, $\gamma > 1$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle_{\partial\Omega} = -\lim_{s \to 0} \frac{1}{s} \bigl\langle \mathbf{F}, \mathcal{E}(\psi) \nabla h \bigr\rangle_{\Psi(\partial\Omega \times (0,s))}, \tag{8.17}$$

where $\mathcal{E}(\psi) \in \mathrm{Lip}(\gamma, \mathbb{R}^N)$ *is the Whitney extension of* ψ *on* $\partial\Omega$ *to* $\mathbb{R}^N$.

REMARK 8.2. In general, for $\mathbf{F} \in \mathcal{D}M^1(\mathcal{D})$ or $\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$, the normal trace $\mathbf{F} \cdot \mathbf{n}$ may be no longer a function. This can be seen in Example 8.1 for $\mathbf{F} \in \mathcal{D}M^1_{\mathrm{loc}}(\mathbb{R}^2)$ with $\Omega = \{\mathbf{y}\colon\ y_1^2 + y_2^2 < 1,\ y_2 > 0\}$, for which $\mathbf{F} \cdot \mathbf{n}$ is a measure over $\partial\Omega$.

As a corollary of the Gauss–Green formula for $\mathcal{D}M^\infty$ fields, we have the following proposition.

PROPOSITION 8.6. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set with Lipschitz boundary and*

$$\mathbf{F}_1 \in \mathcal{D}M^\infty(\Omega), \qquad \mathbf{F}_2 \in \mathcal{D}M^\infty\bigl(\mathbb{R}^N - \overline{\Omega}\bigr).$$

Then

$$\mathbf{F}(y) = \begin{cases} \mathbf{F}_1(\mathbf{y}), & \mathbf{y} \in \Omega, \\ \mathbf{F}_2(\mathbf{y}), & \mathbf{y} \in \mathbb{R}^N - \overline{\Omega}, \end{cases} \tag{8.18}$$

belongs to $\mathcal{D}M^\infty(\mathbb{R}^N)$, *and*

$$\begin{aligned} \|\mathbf{F}\|_{\mathcal{D}M^\infty(\mathbb{R}^N)} \leqslant{}& \|\mathbf{F}_1\|_{\mathcal{D}M^\infty(\Omega)} + \|\mathbf{F}_2\|_{\mathcal{D}M^\infty(\mathbb{R}^N - \overline{\Omega})} \\ &+ \|\mathbf{F}_1 \cdot \mathbf{n} - \mathbf{F}_2 \cdot \mathbf{n}\|_{L^\infty(\partial\Omega)} \mathcal{H}^{N-1}(\partial\Omega). \end{aligned}$$

The analysis above over sets with Lipschitz boundary has been extended to the analysis over sets of finite perimeter for $\mathbf{F} \in \mathcal{D}M^\infty(\mathcal{D})$.

DEFINITION 8.2. Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set $\mathcal{D} \subset \mathbb{R}^N$, we say that E is a set of finite perimeter in $\mathcal{D}$ if the characteristic function of E, χ_E, belongs to $BV(\mathcal{D})$. We refer to a set of finite perimeter in $\mathbb{R}^N$ simply as a set of finite perimeter.

REMARK 8.3. If E is a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$, then $\nabla\chi_E$ (the gradient of χ_E in the sense of distributions) is a vector-valued Radon measure in $\mathcal{D}$. We denote the total variation of $\nabla\chi_E$ as $|\nabla\chi_E|$. It can be shown (cf. [8,132]) that

$$\nabla\chi_E = \mathbf{n}_E|\nabla\chi_E|,$$

where $\mathbf{n}_E$ is the measure-theoretic inward unit normal to E.

DEFINITION 8.3. Let E be a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $\mathbf{y} \in \operatorname{supp}(|\nabla\chi_E|) \cap \mathcal{D}$ such that

(i) $\int_{B(\mathbf{y},r)} |\nabla\chi_E| > 0$ for all $r > 0$;

(ii) $\lim_{r\to 0}(\int_{B(\mathbf{y},r)} \nabla\chi_E / \int_{B(\mathbf{y},r)} |\nabla\chi_E|) = \mathbf{n}_E(\mathbf{y})$;

(iii) $|\mathbf{n}_E(\mathbf{y})| = 1$.

We recall that the space of functions of bounded variation, BV, in fact represents an equivalence class of functions so that changing the value of a function in this class on a set of $\mathcal{L}^N$-measure zero does not change the function itself. From Definition 8.2, it follows that the same is true for sets of finite perimeter. Since we are concerned with only equivalence classes of sets, *we assume here that a set of finite perimeter E is the representative given by the following proposition*, which can be found in [144].

PROPOSITION 8.7. *If $E \subset \mathbb{R}^N$ is a Borel set, then there exists a Borel set $\widetilde{E}$ equivalent to E, which differs only by a set of $\mathcal{L}^N$-measure zero, such that*

$$0 < \left|\widetilde{E} \cap B(\mathbf{y}, r)\right| < \omega_N r^N \tag{8.19}$$

for all $\mathbf{y} \in \partial\widetilde{E}$ and all small $r > 0$, where ω_N is the measure of the unit ball in $\mathbb{R}^N$.

DEFINITION 8.4. For every $\alpha \in [0, 1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, define

$$E^\alpha := \left\{\mathbf{y} \in \mathbb{R}^N \colon \lim_{r\to 0} \frac{|E \cap B(\mathbf{y}, r)|}{|B(\mathbf{y}, r)|} = \alpha\right\}, \tag{8.20}$$

the set of all points with density $\alpha \in [0, 1]$. We now define the *essential boundary* of E, $\partial^s E$, as

$$\partial^s E = \mathbb{R}^N \setminus \left(E^0 \cup E^1\right). \tag{8.21}$$

The sets E^0 and E^1 may be considered as the measure-theoretic exterior and interior of E, which motivate the definition of essential boundaries.

REMARK 8.4. If E is a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$, it has been shown (cf. [8]) that

$$\partial^* E \subset E^{1/2} \subset \partial^s E, \tag{8.22}$$

$$\mathcal{H}^{N-1}\left(\partial^s E \setminus \partial^* E\right) = 0, \tag{8.23}$$

and

$$|\nabla \chi_E| = \mathcal{H}^{N-1} \lfloor \partial^* E. \tag{8.24}$$

DEFINITION 8.5. Let $f \in L^1(\mathcal{D})$ and $\mathbf{a} \in \mathbb{R}^N$. We say that $f_{\mathbf{a}}(\mathbf{y}_0)$ is the approximate limit of f at $\mathbf{y}_0 \in \mathcal{D}$ restricted to $\Pi_{\mathbf{a}} := \{\mathbf{y} \in \mathbb{R}^N\colon \mathbf{y} \cdot \mathbf{a} \geqslant 0\}$ if, for any $\delta > 0$,

$$\lim_{r \to 0} \frac{|\{\mathbf{y} \in \mathbb{R}^N\colon |f(\mathbf{y}) - f_{\mathbf{a}}(\mathbf{y}_0)| < \delta\} \cap B(\mathbf{y}_0, r) \cap \Pi_{\mathbf{a}}|}{|B(\mathbf{y}_0, r) \cap \Pi_{\mathbf{a}}|} = 1. \tag{8.25}$$

DEFINITION 8.6. We say that $\mathbf{y}_0 \in \mathcal{D}$ is a regular point of a function $f \in BV(\mathcal{D})$ if there exists a vector $\mathbf{a} \in \mathbb{R}^N$ such that the approximate limits $f_{\mathbf{a}}(\mathbf{y}_0)$ and $f_{-\mathbf{a}}(\mathbf{y}_0)$ exist. The vector $\mathbf{a}$ is called a *defining vector*.

If $\mathbf{y}_0$ is a regular point of $f \in BV(\mathcal{D})$, then there are two possibilities

$$\text{either} \quad f_{\mathbf{a}}(\mathbf{y}_0) = f_{-\mathbf{a}}(\mathbf{y}_0) \quad \text{or} \quad f_{\mathbf{a}}(\mathbf{y}_0) \neq f_{-\mathbf{a}}(\mathbf{y}_0).$$

It can be proved (cf. [330]) that, in the first case, any $\mathbf{b} \in \mathbb{R}^N$ is a defining vector and $f_{\mathbf{b}}(\mathbf{y}_0) = f_{\mathbf{a}}(\mathbf{y}_0)$; in the second case, $\mathbf{a}$ is unique up to the sign, i.e., the only defining vectors are $\mathbf{a}$ and $-\mathbf{a}$.

REMARK 8.5. A classical result in the BV theory says that $\mathcal{H}^{N-1}$ almost every $\mathbf{y} \in \mathcal{D}$ is a regular point of $f \in BV(\mathcal{D})$; see [8,132,330].

DEFINITION 8.7. Given $f \in L^1_{\mathrm{loc}}(\mathcal{D})$, we define

$$\bar{f}(\mathbf{y}) := \lim_{\varepsilon \to 0} f^\varepsilon(\mathbf{y}), \tag{8.26}$$

where $f^\varepsilon := f * \omega^\varepsilon$ with $\omega^\varepsilon(\mathbf{y}) = \varepsilon^{-N}\omega(\mathbf{y}/\varepsilon)$ for the standard positive symmetric mollifier ω defined in (8.8).

REMARK 8.6. It can be proved that, if $f \in BV(\mathcal{D})$, then $\bar{f}$ is defined at each regular point. Moreover, if $\mathbf{y}_0$ is a regular point of f, then

$$\bar{f}(\mathbf{y}_0) = \frac{1}{2}\big(f_{\mathbf{a}}(\mathbf{y}_0) + f_{-\mathbf{a}}(\mathbf{y}_0)\big),$$

where $\mathbf{a}$ is a defining vector (cf. [330]).

If E is a set of finite perimeter in $\mathcal{D}$, we have from Remark 8.6 that $\bar{\chi}_E$ is defined $\mathcal{H}^{N-1}$-almost everywhere. In fact, we have

$$\bar{\chi}_E(\mathbf{y}) = \begin{cases} \frac{1}{2} & \text{if } \mathbf{y} \in \partial^* E, \\ 1 & \text{if } \mathbf{y} \in E^1, \\ 0 & \text{if } \mathbf{y} \in E^0. \end{cases} \tag{8.27}$$

We recall here that $\mathcal{H}^{N-1}(\partial^s E \setminus \partial^* E) = 0$.

As Proposition 8.8 indicates,

$$\operatorname{div} \mathbf{F} \ll \mathcal{H}^{N-1} \quad \text{for } \mathbf{F} \in \mathcal{D}M^\infty(\mathcal{D}).$$

Thus, the values of $\bar{\chi}_E$ on the set $\partial^s E \setminus \partial^* E$ can be ignored. This fact is essential in the proof of the Gauss–Green formula for $\mathcal{D}M^\infty$ fields over sets of finite perimeter.

PROPOSITION 8.8. *Let* $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$. *Then the Radon measure* $\operatorname{div} \mathbf{F}$ *in* $\mathcal{D}$ *is absolutely continuous with respect to the* $(N-1)$*-Hausdorff measure* $\mathcal{H}^{N-1}$. *That is, if* $A \subset \mathcal{D}$ *be a Borel measurable set such that* $\mathcal{H}^{N-1}(A) = 0$, *then* $|\operatorname{div} \mathbf{F}|(A) = 0$.

PROOF. Since there are Borel measurable sets $\mathcal{D}_+$ and $\mathcal{D}_-$, $\mathcal{D}_+ \cup \mathcal{D}_- = \mathcal{D}$, such that $\operatorname{div} \mathbf{F}$ is a nonnegative measure over $\mathcal{D}_+$ and a nonpositive measure over $\mathcal{D}_-$, one may assume $A \subset \mathcal{D}_+$ and hence $|\operatorname{div} \mathbf{F}|(A) = (\operatorname{div} \mathbf{F})_+(A) = \operatorname{div} \mathbf{F}(A)$. Also, since $(\operatorname{div} \mathbf{F})_+$ is a Radon measure, it suffices to prove the assertion for any compact A.

Now, for any $\delta > 0$, we can find a finite number J of balls of radius less than δ such that

$$A \subset \bigcup_{i=1}^{J} B(\mathbf{y}_i; r_i), \qquad \sum_{i=1}^{J} r_i^{N-1} < \delta,$$

since $\mathcal{H}^{N-1}(A) = 0$. Then we may apply the Gauss–Green formula for $\mathcal{D}M^\infty$ fields over the set

$$\Omega = \Omega_\delta \equiv \bigcup_{i=1}^{J} B(\mathbf{y}_i; r_i)$$

with Lipschitz deformable boundary and any function $\phi \in C_0^1(\mathbb{R}^N)$ that is identically equal to one over $\overline{\Omega}_\delta$. Then

$$\left| \operatorname{div} \mathbf{F}(\Omega_\delta) \right| \leqslant \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial \Omega_\delta) \leqslant C \|\mathbf{F}\|_\infty \sum_{i=1}^{J} r_i^{N-1} \leqslant \delta C \|\mathbf{F}\|_\infty.$$

Now, since $\chi_{\Omega_\delta} \to \chi_A$ pointwise in $\mathcal{D}$ as $\delta \to 0$ (recall that A is compact), one has

$$|\operatorname{div} \mathbf{F}|(A) = \operatorname{div} \mathbf{F}(A) = 0.$$

This completes the proof. $\square$

Now, Proposition 8.5 immediately implies the following proposition.

PROPOSITION 8.9. *Let* $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$. *If* $E \Subset \mathcal{D}$ *is a set of finite perimeter in* $\mathcal{D}$, *then*

$$\operatorname{div}(\chi_E \mathbf{F}) = \bar{\chi}_E \operatorname{div} \mathbf{F} + \overline{\mathbf{F} \cdot \nabla \chi_E}, \tag{8.28}$$

where $\overline{\mathbf{F}\cdot\nabla\chi_E} = w-\lim_{\varepsilon\to 0}\mathbf{F}\cdot\nabla(\chi_E)^{\varepsilon}$ *for* $(\chi_E)^{\varepsilon}=\chi_E*\omega^{\varepsilon}$. *Furthermore, the measure* $\overline{\mathbf{F}\cdot\nabla\chi_E}$ *is absolutely continuous with respect to the measure* $|\nabla\chi_E|$.

THEOREM 8.3. *Let* $\mathbf{F}\in\mathcal{D}M^{\infty}(\mathcal{D})$. *If* $E\Subset\mathcal{D}$ *is a bounded set of finite perimeter, then there exists an* $\mathcal{H}^{N-1}$*-integrable function* (*denoted by*) $\mathbf{F}\cdot\mathbf{n}\in L^{\infty}(\partial^s E;\mathcal{H}^{N-1})$ *such that*

$$\int_{E^1}\operatorname{div}\mathbf{F} = -\int_{\partial^s E}\overline{\mathbf{F}\cdot\nabla\chi} = -\int_{\partial^s E}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{N-1}. \tag{8.29}$$

Then we have the following Gauss–Green formula.

THEOREM 8.4 (Gauss–Green formula). *Let* $\mathbf{F}\in\mathcal{DM}^{\infty}(\mathcal{D})$. *Let* $E\Subset\mathcal{D}$ *be a bounded set of finite perimeter. Then there exists an* $\mathcal{H}^{N-1}$*-integrable function*

$$\mathbf{F}\cdot\mathbf{n}\quad\text{on } \partial^s E$$

such that, for any $\phi\in C_0^1(\mathbb{R}^N)$,

$$\int_{E^1}\phi\operatorname{div}\mathbf{F} = -\int_{\partial^s E}\mathbf{F}\cdot\mathbf{n}\phi\,\mathrm{d}\mathcal{H}^{N-1} - \int_{E^1}\mathbf{F}\cdot\nabla\phi\,\mathrm{d}\mathbf{y}.$$

THEOREM 8.5. *Let* $\Omega\Subset E\Subset\mathcal{D}$ *be bounded open sets where* E *is a set of finite perimeter in* $\mathbb{R}^N$. *Let* $\mathbf{F}_1\in\mathcal{D}M^{\infty}(\mathcal{D})$ *and* $\mathbf{F}_2\in\mathcal{D}M^{\infty}(\mathbb{R}^N-\overline{\Omega})$. *Then*

$$\mathbf{F}(\mathbf{y}) = \begin{cases}\mathbf{F}_1(\mathbf{y}), & \mathbf{y}\in E,\\ \mathbf{F}_2(\mathbf{y}), & \mathbf{y}\in\mathbb{R}^N-\overline{E},\end{cases} \tag{8.30}$$

belongs to $\mathcal{D}M^{\infty}(\mathbb{R}^N)$, *and*

$$\begin{aligned}&\|\mathbf{F}\|_{\mathcal{D}M^{\infty}(\mathbb{R}^N)}\\ &\quad\leqslant \|\mathbf{F}_1\|_{\mathcal{D}M^{\infty}(E)} + \|\mathbf{F}_2\|_{\mathcal{D}M^{\infty}(\mathbb{R}^N-\overline{E})} + \|\mathbf{F}_1\cdot\mathbf{n}-\mathbf{F}_2\cdot\mathbf{n}\|_{L^1(\partial^s E;\mathcal{H}^{N-1})}.\end{aligned}$$

The normal trace over a surface of finite perimeter, introduced by Chen and Torres [83], can be understood as the weak-star limit of the normal traces in Theorem 8.1 by Chen and Frid [67] over the Lipschitz deformation surfaces of the surface, which implies their consistency.

Some entropy methods based on the theory of divergence-measure fields presented above have been developed and applied for solving various nonlinear problems for conservation laws and related nonlinear equations. These problems especially include

(1) stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics in [69,70,86],

(2) decay of periodic entropy solutions for hyperbolic conservation laws in [65],

(3) initial and boundary layer problems for hyperbolic conservation laws in [67,82, 83,329],

(4) rigorous derivation of systems of balance laws from the physical principle of balance law and the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation in [84],

(5) nonlinear degenerate parabolic–hyperbolic equations in [37,73,81,248].

One of the entropy methods is to identify Lyapunov-type functionals and employ the Gauss–Green formula to establish the uniqueness and stability of entropy solutions; see [69,70,86]. In this regard, some related Lyapunov-type functionals have been identified for small *BV* solutions obtained by the Glimm scheme, the wave-front tracking scheme, and the vanishing viscosity method; see [20,33,111,167,204,210] and the references cited therein for the details. It would be interesting to apply the theory of divergence-measure fields to developing more efficient entropy methods for solving more various problems in partial differential equations and related areas whose solutions are only measures or L^p functions.

For more details, see [67,69,83,84].

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CHAPTER 2

Blow-up of Solutions of Supercritical Parabolic Equations

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Introduction

For some nonlinear parabolic equations, solutions may not exist globally for $t \geqslant 0$ but may become unbounded in finite time. This phenomenon is called "blow-up" and it has been intensively studied in connection with various fields of science such as plasma physics, combustion theory and population dynamics.

Early studies of blow-up problems, including the pioneering works of Kaplan [58], Fujita [29,30] and Levine [66], were mainly devoted to finding sufficient conditions under which blow-up occurs. Since the middle of 1980s, researchers started to pay more attention to the structure of singularities that appear as solutions blow up (see [28,33,34,39,40,110] for earlier works in this direction).

The equation which has been studied most extensively is

$$u_t = \Delta u + |u|^{p-1}u, \quad p > 1,$$

where $x \in \Omega$ and Ω is a bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. If Ω is bounded then the Dirichlet boundary condition $u = 0$ is usually imposed on $\partial\Omega$. By blow-up in finite time we mean that there is a $T \in (0, \infty)$ such that

$$\lim_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

The critical Sobolev exponent,

$$p_S = \begin{cases} \frac{N+2}{N-2} & \text{if } N > 2, \\ \infty & \text{if } N \leqslant 2, \end{cases}$$

plays a crucial role here. The blow-up behavior is much better understood in the subcritical case $p < p_S$, see, for example, [49–54,105–108] for results on the blow-up profile. In the critical and supercritical cases ($p = p_S$ and $p > p_S$, respectively), many important questions are still open although this has been a very active area of research recently. Our main aim here is to review these recent results, many of which have not appeared in journals yet (at the time of writing).

Let us briefly point out the differences in blow-up behavior in the subcritical and supercritical cases.

(i) If $p < p_S$ and Ω is bounded then blow-up of positive solutions is complete in the sense that there is no weak continuation beyond blow-up, see Section 1.2. On the other hand, for $p > p_S$ there are many examples of solutions which blow up in finite time but continue to exist globally in the weak sense, see Sections 1.1, 3, 7, 8.7 and 9. Continuation beyond blow-up is the main theme of this chapter.

(ii) In the case $p < p_S$, solutions blow up with the same rate (in time) as solutions of the corresponding ordinary differential equation (ODE) $u' = |u|^{p-1}u$, while for

$$p > 1 + \frac{4}{N - 4 - 2\sqrt{N-1}}, \quad N > 10,$$

there are counterexamples, and for $p = p_S$ there are formal counterexamples, see Section 5.

(iii) Global solutions are bounded by a constant which depends only on the L^∞ norm of initial data if Ω is bounded and $p < p_S$, see [97]. This is not true if $p \geqslant p_S$ and Ω is star shaped. We shall discuss global unbounded solutions in Sections 10.1 and 10.2.

It is well known that the critical Sobolev exponent plays also a crucial role in the existence of positive stationary solutions and in the existence of nonconstant backward self-similar solutions, see Section 2.

Since the embedding of the Sobolev space $W^{1,2}(\Omega)$ in the Lebesgue space $L^{p+1}(\Omega)$ is not valid if $p > p_S$, it is hard to apply functional analysis. The maximum principle and its more sophisticated one-dimensional version – intersection comparison or "zero number" play a very important role in the study of supercritical blow-up. We shall illustrate that in Section 8 in detail. Because of the use of the intersection comparison method, most results reviewed here are restricted to radially symmetric solutions. Another fruitful technique is the method of matched asymptotics, see Section 10.2. It yields the correct result formally but it also serves as a reliable guiding principle in finding a rigorous proof.

For the exponential equation

$$u_t = \Delta u + \lambda e^u, \quad \lambda > 0,$$

the cases $N \leqslant 2$, $N > 2$ are in some sense similar to $p < p_S$, $p > p_S$, respectively. A significant part of this chapter is devoted to this equation when $N > 2$.

1. Beyond blow-up

1.1. *Global L^1-solutions*

We begin with the definition of L^1-solutions of the problem

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{1.1}$$

here Ω is a bounded domain in $\mathbb{R}^N$.

DEFINITION 1.1. By an L^1-solution of (1.1) on $[0, T]$ we mean a function $u \in C([0, T]; L^1(\Omega))$ such that $f(u) \in L^1(Q_T)$, $Q_T := \Omega \times (0, T)$, and the equality

$$\int_\Omega [u\Psi]_\tau^t \,dx - \int_\tau^t \int_\Omega u\Psi_t \,dx\,ds = \int_\tau^t \int_\Omega \bigl(u\Delta\Psi + f(u)\Psi\bigr)\,dx\,ds$$

holds for any $0 \leqslant \tau < t \leqslant T$ and $\Psi \in C^2(\overline{Q}_T)$, $\Psi = 0$ on $\partial\Omega \times [0, T]$. By a global L^1-solution we mean an L^1-solution which exists on $[0, T]$ for every $T > 0$.

For $f(u) = u^p$ it was shown in [88] that a global unbounded (in L^∞) positive L^1-solution exists if Ω is convex and $(N-2)p \geqslant N+2$. We now explain the reason why such a solution exists. By the Pohozaev identity (cf. [92]), there is no positive equilibrium. The equilibrium $u = 0$ is stable. If we choose an initial function $u_0 \in C(\overline{\Omega})$, $u_0 \geqslant 0$, $u_0 \not\equiv 0$, and denote by $u(\cdot, t; \lambda u_0)$ the solution of (1.1) with $u(\cdot, 0) = \lambda u_0$ then

$$\lambda^* := \sup\bigl\{\lambda > 0\colon\ u(\cdot, t; \lambda u_0) \text{ is a global classical solution} \\ \text{such that } u(\cdot, t; \lambda u_0) \to 0 \text{ as } t \to \infty\bigr\}$$

is positive and finite since for λ large the solution $u(\cdot, t; \lambda u_0)$ blows up in finite time. Now, $u(\cdot, t; \lambda^* u_0)$ cannot be global and bounded since otherwise its ω-limit set would have to contain a nonnegative equilibrium. But the only nonnegative equilibrium is zero and its domain of attraction is open in any reasonable topology. Hence $u(\cdot, t; \lambda^* u_0)$ cannot converge to zero as $t \to \infty$. On the other hand, if we take a sequence $\{\lambda_n\}$ such that $\lambda_n \nearrow \lambda^*$ then $u(\cdot, t; \lambda_n u_0)$ is a monotone sequence of global solutions, and the monotone convergence theorem can be used in order to pass to the limit in a suitable weak formulation of (1.1) and show that $u(\cdot, t; \lambda^* u_0)$ is a global L^1-solution (see [88] for more details).

For a long time it had been an open problem whether or not $u(\cdot, t; \lambda^* u_0)$ is classical for all $t > 0$ and becomes unbounded only as $t \to \infty$. An answer was given in [35] in the case when $\Omega = B_R(0) := \{x \in \mathbb{R}^N\colon\ |x| < R\}$ and u_0 is radially symmetric. It is shown in [35] that if $(N-2)p = N+2$ and u_0 is radially decreasing then $u(\cdot, t; \lambda^* u_0)$ is indeed classical for all $t > 0$; while for

$$\frac{N+2}{N-2} < p \quad \left(< 1 + \frac{6}{N-10} \text{ if } N > 10\right), \tag{1.2}$$

$u(\cdot, t; \lambda^* u_0)$ blows up in finite time and continues to exist globally only as an L^1-solution. It follows from [35] and a recent result in [83] that $u(\cdot, t; \lambda^* u_0)$ blows up in finite time for all $p > (N+2)/(N-2)$, $N > 2$.

1.2. *Complete blow-up*

Under several circumstances a solution that blows up at a finite time T cannot be continued as an L^1-solution beyond T. This phenomenon is called complete blow-up. To describe it, let us recall a result from [2], as applied to (1.1) with $f(u) = u^p$.

Let $f_n(u) := \min\{u^p, n\}$. Let u_n be the unique global classical solution of

$$\begin{aligned} &(u_n)_t = \Delta u_n + f_n(u_n), && x \in \Omega, t > 0, \\ &u_n = 0, && x \in \partial\Omega, t > 0, \\ &u_n(\cdot, 0) = u_0 \geqslant 0, && x \in \Omega, u_0 \in L^\infty(\Omega). \end{aligned}$$

Suppose that one of the following holds:

(a) $u_0 \in W_0^{1,1}(\Omega)$ and $\Delta u_0 + f(u_0) \geqslant 0$ in $\mathcal{D}'(\Omega)$,

(b) $(N-2)p < N+2$.

Assume that the solution u of (1.1) with $f(u) = u^p$ blows up at the time $T \in (0, \infty)$. Then u blows up completely in the sense that

(i) $\lim_{n\to\infty} u_n(x,t) = u(x,t)$ for all $(x,t) \in \Omega \times [0,T)$,

(ii) $\lim_{n\to\infty} u_n(x,t) = \infty$ for all $(x,t) \in \Omega \times (T,\infty)$.

It was shown later in [35] that if $\Omega = B_R(0)$ and u_0 is radially symmetric then blow-up is complete in the above sense also for $p = (N+2)/(N-2)$, $N > 2$. Other results on complete blow-up can also be found, for example, in [35,62,68].

The notion of complete blow-up can be defined in the same manner for a more general nonlinearity $f(u)$, including the case $f(u) = \mathrm{e}^u$, see [68].

A different but equivalent definition of complete blow-up was used in [62]. There problem (1.1) was reformulated by seeking the minimal solution to the integral equation

$$u(x,t) = \lambda \int_0^t \int_\Omega G(x,y,t-\tau) f\big(u(y,\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}y + \int_\Omega G(x,y,t) u_0(y)\,\mathrm{d}y, \tag{1.3}$$

where G is the Green function for the heat equation in Ω with the Dirichlet boundary condition on $\partial\Omega$. The true time of existence t_c ($t_\mathrm{c} \geqslant T$) is defined as

$$t_\mathrm{c} := \sup\big\{t\colon\ u(\cdot,t) < \infty \text{ almost everywhere in } \Omega\big\}.$$

Blow-up is then called complete if $t_\mathrm{c} = T$. It was observed in [2] that $\lim_{n\to\infty} u_n$ is the minimal solution of (1.3).

2. Self-similar solutions

2.1. *Backward self-similar solutions for the power*

By a backward self-similar solution of the equation

$$u_t = u_{rr} + \frac{N-1}{r} u_r + |u|^{p-1}u, \quad r > 0,\ p > 1, \tag{2.1}$$

we mean a solution of the form

$$u(r,t) = (T-t)^{-1/(p-1)} \psi(y), \quad y = \frac{r}{\sqrt{T-t}},\ T \in \mathbb{R},\ t < T,$$

where ψ is a solution of the ODE

$$\psi_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right)\psi_y + |\psi|^{p-1}\psi - \frac{1}{p-1}\psi = 0, \quad y > 0. \tag{2.2}$$

Bounded solutions of this equation satisfy the initial conditions

$$\psi(0) = \alpha, \qquad \psi_y(0) = 0. \tag{2.3}$$

In the case $p \leqslant p_S$, the only bounded solutions of (2.2) are the constants

$$\psi \equiv 0, \qquad \psi \equiv \pm\kappa, \quad \kappa := (p-1)^{-1/(p-1)},$$

see [39]. On the other hand, for $p_S < p < p^*$,

$$p^* := \begin{cases} \infty & \text{if } N \leqslant 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{if } N > 10, \end{cases} \tag{2.4}$$

there exists an increasing sequence $\{\alpha\}_{n=1}^{\infty}$, $\alpha_n \to \infty$, such that the solution ψ_n of (2.2), (2.3) with $\alpha = \alpha_n$ satisfies

$$\psi_n(y) > 0 \quad \text{for } y > 0, \qquad \psi_n(y) \to 0 \quad \text{as } y \to \infty,$$

see [6,64,103]. For

$$p^* \leqslant p < p_L := 1 + \frac{6}{N-10},$$

there are at least finitely many solutions of (2.2), (2.3), see [65]. If $p_S < p < p_L$ then all nonconstant positive bounded solutions of (2.2) intersect the explicit singular solution

$$\varphi_\infty(y) := K y^{-2/(p-1)}, \quad K := \left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{1/(p-1)}, \tag{2.5}$$

at least twice, see [6,64,103] and [65].

For $p = 2$, $6 < N < 16$, there is an explicit solution of (2.2), (2.3) of the form

$$\psi(y) = \frac{A}{(a+y^2)^2} + \frac{B}{a+y^2},$$

where

$$a := 2\bigl(10D - (N+14)\bigr) > 0,$$
$$A := 24a,$$
$$B := 24(D-2) > 0, \quad D := \sqrt{1+\frac{N}{2}},$$

see [32].

Matos [72] showed that for every solution of (2.2), (2.3) there is a constant $C > 0$ such that

$$\psi(y) = \pm C y^{-2/(p-1)}\bigl(1 - C_1 y^{-2} - C_2 y^{-4} + o\bigl(y^{-4}\bigr)\bigr) \quad \text{as } y \to \infty,$$

where

$$C_1 := C^{p-1} - \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right), \qquad C_2 := -pC_1\left(\frac{2}{p-1} + \frac{C_1}{2}\right).$$

Matano and Merle gave a classification of all solutions of (2.2) satisfying

$$\left|\psi(y)\right| \leqslant C_0\left(1 + y^{-2/(p-1)}\right), \quad y > 0,$$

where $C_0 > 0$ may depend on ψ. They proved in [69] that for $p > p_S$ every such solution ψ either satisfies (2.3) for some $\alpha > 0$ or $\psi(y) = \pm\varphi_\infty(y)$.

In [78], Mizoguchi showed the nonexistence of positive bounded solutions of (2.2) which intersect φ_∞ at least twice for

$$p > 1 + \frac{7}{N-11}, \quad N > 11.$$

A numerical study of Plecháč and Šverák [91] suggests that this is true if $p > p_L$, $N > 10$.

2.2. *Forward self-similar solutions for the power*

By a forward self-similar solution of (2.1) we mean a solution of the form

$$u(r,t) = (t-T)^{-1/(p-1)}\theta(y), \quad y = \frac{r}{\sqrt{t-T}}, T \in \mathbb{R}, t > T,$$

where θ is a solution of the ODE

$$\theta_{yy} + \left(\frac{N-1}{y} + \frac{y}{2}\right)\theta_y + |\theta|^{p-1}\theta + \frac{1}{p-1}\theta = 0, \quad y > 0. \tag{2.6}$$

Equation (2.6) with the initial conditions

$$\theta(0) = \beta, \qquad \theta_y(0) = 0, \tag{2.7}$$

was first studied by Haraux and Weissler in [47]. It was shown there that if θ_β is the solution of (2.6), (2.7) then

$$L(\beta) := \lim_{y\to\infty} y^{2/(p-1)}\theta_\beta(y)$$

exists and is a locally Lipschitz continuous function of $\beta \in \mathbb{R}$, if $p \leqslant 1 + 2/N$ then there is no positive solution, if $p \geqslant p_S$ and $\beta > 0$ then $\theta_\beta(y) > 0$ for $y > 0$ and $L(\beta) > 0$.

The main goal of [47] was to prove that if $1 + 2/N < p < p_S$ then $L(\beta_0) = 0$ for some $\beta_0 > 0$ such that $\theta_{\beta_0}(y) > 0$ for $y > 0$. Moreover, for such p there exist infinitely many pairs β_1, β_2 such that $0 < \beta_1 < \beta_2 < \beta_0$, $L(\beta_1) = L(\beta_2)$ and both $\theta_{\beta_1}, \theta_{\beta_2}$ are positive everywhere. Also, $\theta_\beta(y) > 0$ for $y > 0$ if $0 < \beta < \beta_0$. Furthermore, $L(\beta) > 0$ if $\beta > 0$ is sufficiently small.

Dohmen and Hirose [12] proved that if $N/(N-2) \leqslant p < p_S$ and $\beta > \beta_0$ then θ_β assumes negative values. In fact, they showed uniqueness of β_0, see [112] for the same result in the case $N = 1$ and [113] in the case $p \leqslant N/(N-2)$.

Souplet and Weissler showed in [102] that $L(\beta)$ oscillates around K (see (2.5)) infinitely many times as $\beta \to \infty$ if $p_S < p < p^*$, while $L(\beta) = K$ for at least two values of β if $N/(N-2) < p \leqslant p_S$.

2.3. *Backward self-similar solutions for the exponential*

By a backward self-similar solution of the equation

$$u_t = u_{rr} + \frac{N-1}{r} u_r + e^u, \quad r > 0, p > 1, \tag{2.8}$$

we mean a solution of the form

$$u(r,t) = \log(T-t) + \psi(y), \quad y = \frac{r}{\sqrt{T-t}}, T \in \mathbb{R}, t < T,$$

where ψ is a solution of the ODE

$$\psi_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right)\psi_y + e^\psi - 1 = 0, \quad y > 0. \tag{2.9}$$

We are interested in solutions of (2.9) which satisfy

$$\psi(0) = \alpha \geqslant 0, \qquad \psi_y(0) = 0 \tag{2.10}$$

and

$$\lim_{y\to\infty}\left(1 + \frac{y}{2}\psi_y(y)\right) = 0. \tag{2.11}$$

Condition (2.11) arises naturally for formal (physical) reasons (see [4]) and it means in particular that if u is a backward self-similar solution of (2.8) with ψ satisfying (2.11) then $\lim_{t\to T} u(r,t)$ exists and is finite for $r > 0$.

In the case $N = 1, 2$, there is no solution of (2.9)–(2.11), see [4,15]. On the other hand, for $3 \leqslant N \leqslant 9$, there exists an increasing sequence $\{\alpha\}_{n=1}^\infty$, $\alpha_n \to \infty$, such that the solution ψ_n of (2.9), (2.10) satisfies (2.11), see [16]. Lacey and Tzanetis [63] proved that there is a solution ψ_α of (2.9)–(2.11) such that

$$\lim_{y\to\infty}\left(\psi_\alpha(y) + \log\frac{y^2}{2}\right) = -C, \quad C > 0. \tag{2.12}$$

2.4. *Forward self-similar solutions for the exponential*

Forward self-similar solutions of (2.8) are solutions of the form

$$u(r,t) = \log(t-T) + \theta(y), \quad y = \frac{r}{\sqrt{t-T}}, T \in \mathbb{R}, t > T,$$

where ψ is a solution of the ODE

$$\theta_{yy} + \left(\frac{N-1}{y} + \frac{y}{2}\right)\theta_y + \mathrm{e}^{\theta} + 1 = 0, \quad y > 0. \tag{2.13}$$

For $N = 3$ it was shown by Lacey and Tzanetis [63] that if θ_β is a solution of (2.9) with initial conditions

$$\theta(0) = \beta, \qquad \theta_y(0) = 0, \tag{2.14}$$

then

$$L(\beta) := -\lim_{y\to\infty}\left(\theta_\beta(y) + \log\frac{y^2}{2}\right)$$

exists and is a continuous function of $\beta \in \mathbb{R}$; $L(\beta) \to \infty$ as $\beta \to -\infty$ and there exists β_0 such that $L(\beta_0) = 0$. It is easy to see that the results from [63] can be generalized to $3 \leqslant N \leqslant 9$. Vázquez proved in [104] that $-\infty < \min L(\beta) < 0$.

3. Examples of peaking solutions for the Cauchy problem

It was shown in [63] that for the equation

$$u_t = \Delta u + \mathrm{e}^u$$

in $\mathbb{R}^3$ there exist backward and forward self-similar solutions which coincide at the time T where single point blow-up occurs. Namely, if we choose α such that (2.12) holds then there exists β such that $L(\beta) = C$. Thus ψ_α and θ_β can be glued together to form a "peaking solution" for $t \in \mathbb{R}$. The uniqueness of the continuation was left open. The above construction of a global peaking solution yields an L^1-solution. It was shown in [35] that a similar construction can be done for

$$u_t = \Delta u + u^p$$

in $\mathbb{R}^N$ provided $N > 2$ and p satisfies (1.2). For each positive integer k there is α large enough such that ψ_α has at least k different forward self-similar continuations beyond blow-up, see [20].

In [80], Mizoguchi constructed for $p > p^*$ minimal L^1-solutions u_i, $i = 1, 2, 3$, of

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

which blow up at $t = T_i < \infty$, become regular for $t > T_i$ and behave as follows:

(i) $\|u_1(\cdot, t)\|_{L^\infty} \to 0$ as $t \to \infty$,

(ii) $0 < \liminf_{t\to\infty} \|u_2(\cdot, t)\|_{L^\infty} \leqslant \limsup_{t\to\infty} \|u_2(\cdot, t)\|_{L^\infty} < \infty$,

(iii) $\|u_3(\cdot, t)\|_{L^\infty} \to \infty$ as $t \to \infty$.

Ideas from [56] and [96] play an important role in this construction.

An explicit example of a solution which blows up arbitrarily many times was given by Pierre [90]. He observed that

$$u(x, t) := \frac{1}{|x|^2 + \psi(t)}$$

is an L^1-solution of the equation

$$u_t = \Delta u + g\big(|x|, t\big)u^2, \quad g(r, t) := 2N - \psi'(t) - \frac{8r^2}{r^2 + \psi(t)},$$

provided ψ is a nonnegative C^1-function and $N > 4$. Obviously, u blows up at each time t such that $\psi(t) = 0$.

4. Boundedness of global solutions

Consider problem (1.1) with $f(u) = |u|^{p-1}u$, where $p > 1$ and $u_0 \in L^\infty(\Omega)$.

The study of boundedness of global solutions was initiated by Ni, Sacks and Tavantzis [88]. Their result can be expressed roughly in the following way. If $p < (N + 2)/N$, Ω is convex, $u_0 \geqslant 0$ and u is a global (classical) solution then

$$u(x, t) \leqslant C(u_0, \Omega, p), \quad x \in \Omega, t > 0,$$

where the constant $C(u_0, \Omega, p) > 0$ depends on the shape of u_0 near $\partial\Omega$.

An improvement of the result from [88] was given by Cazenave and Lions [7]. They removed the assumptions on convexity of Ω and nonnegativity of u_0 and showed that global solutions are uniformly bounded if $p < p_S$ (without giving any explicit dependence of the bound on the data). An a priori bound of the form

$$\big|u(x, t)\big| \leqslant C\big(\|u_0\|_{L^\infty(\Omega)}, \Omega, p\big), \quad x \in \Omega, t > 0, \tag{4.1}$$

for any global solution u was established in [7] for $p(3N - 4) < 3N + 8$.

Later Giga [38] derived an a priori bound of the form

$$u(x, t) \leqslant C\big(\|u_0\|_{L^\infty(\Omega)}, \Omega, p\big), \quad x \in D, t > 0,$$

for any nonnegative global solution u when $p < p_S$.

The results from [7] and [38] were improved by Quittner [97] who proved that (4.1) holds for the whole subcritical range $p < p_S$.

The first universal bound (independent of initial data) was established in [24]. More precisely, if $(N-1)p < N+1$, $\tau > 0$ and u is a global nonnegative solution then

$$u(x,t) \leqslant C(\tau, \Omega, p), \quad x \in \Omega, t \geqslant \tau. \tag{4.2}$$

Slightly later, Quittner derived a universal bound of this kind in [98] for $p < p_S$ and $N \leqslant 3$. Recently, Quittner, Souplet and Winkler have obtained (4.2) in [99] if either $p < p_S$ and $N \leqslant 4$ or $(N-3)p < N-1$. They also studied the dependence of $C(\tau, \Omega, p)$ on τ.

Using intersection comparison with a backward self-similar solution, Galaktionov and Vázquez showed in [35] that if Ω is a ball then any global classical radial solution is uniformly bounded if $p_S < p < p_L$. For a recent result in the case $p > p^*$ see Theorem 4.1.

Consider next problem (3.1). When $p < p_S$ and u is a global solution with u_0 satisfying

$$\int_{\mathbb{R}^N} \left(u_0^2(x) + \left|\nabla u_0(x)\right|^2\right) \mathrm{e}^{|x|^2/4}\, \mathrm{d}x < \infty,$$

then the estimate

$$\left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant Ct^{-1/(p-1)}, \quad t > 0, \tag{4.3}$$

was obtained by Kavian [59] with a constant C depending on u_0. In [101], Souplet proved that when $p < p_S$, any global nonnegative solution of (3.1) with $u_0 \in L^2 \cap L^\infty$ satisfies

$$\lim_{t\to\infty} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} = 0.$$

In [74], Matos and Souplet showed that (4.3) holds with a constant C independent of u_0 provided $p < p_S$, $N \leqslant 3$, $u_0(x) = U_0(|x|) \geqslant 0$ and U_0 is a bounded nonincreasing function.

Mizoguchi proved in [77] that if $p_S < p < p_L$, $u_0(x) = U_0(|x|) \geqslant 0$, $U_0 \in C^1([0,\infty))$ is compactly supported, the set of strict local minima of U_0 is bounded away from zero and u is a global (classical) solution then $u(\cdot,t) \to 0$ locally uniformly as $t \to \infty$.

The main result of [83] is the following theorem.

THEOREM 4.1. *Let $p > p^*$.*

(i) *Let u be a global classical solution of* (1.1), *where Ω is a ball, $f(u) = u^p$, $u_0(x) = U_0(|x|) \geqslant 0$. Then u is uniformly bounded.*

(ii) *Assume u is a global classical solution of* (3.1), *$u_0(x) = U_0(|x|) \geqslant 0$ and there are $c, R > 0$, $c < K$, such that*

$$U_0(r) \leqslant cr^{-2/(p-1)}, \quad r > R.$$

Then u is uniformly bounded.

Consider problem (1.1) with $f(u) = \lambda e^u$, $\lambda > 0$. For $N = 1, 2$ it was shown in [18] that every global classical solution is uniformly bounded. On the other hand, for $N \geqslant 10$, $\lambda = 2(N-2)$, global unbounded classical solutions do exist (cf. [61,89]) when $\Omega = B_1(0) = \{x \in \mathbb{R}^N\colon |x| < 1\}$. In [23] it was shown that every global classical solution is uniformly bounded if $3 \leqslant N \leqslant 9$, $\lambda > 0$, $\Omega = B_1(0)$ and $u_0(x) = U_0(|x|)$.

5. Blow-up rate

5.1. *Definition of type I and type II blow-up*

Let u be a solution of the equation

$$u_t = \Delta u + |u|^{p-1}u$$

which blows up at a finite time T. A simple comparison argument (cf. [28]) shows that

$$\big\|u(\cdot,t)\big\|_{L^\infty} \geqslant \big((p-1)(T-t)\big)^{-1/(p-1)} \quad \text{for } 0 \leqslant t < T.$$

Here $((p-1)(T-t))^{-1/(p-1)}$ is a solution of the corresponding ordinary differential equation $u' = |u|^{p-1}u$.

DEFINITION 5.1. We say that blow-up is of type I if $(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty}$ stays bounded as $t \to T$. Blow-up is called type II if it is not of type I. This means that blow-up is of type II if and only if there is a sequence $\{t_n\}$, $t_n \to T$, such that $(T-t_n)^{1/(p-1)}\|u(\cdot,t_n)\|_{L^\infty} \to \infty$.

Type II blow-up is sometimes called "fast blow-up" but as pointed out in [69], there is also a "slow" aspect of type II blow-up. Namely, if we denote $m(t) = \|u(\cdot,t)\|_{L^\infty}$ then $m'(t) = \mathrm{O}(m^p(t))$ for type I blow-up, while $m'(t_n) = \mathrm{o}(m^p(t_n))$ for type II blow-up. Hence, type II blow-up solutions frequently grow at a rate slower than m^p.

For the exponential equation

$$u_t = \Delta u + \lambda e^u, \quad \lambda > 0,$$

blow-up is of type I if $\log(T-t) + \|u(\cdot,t)\|_{L^\infty}$ is bounded and of type II otherwise.

5.2. *Type I blow-up*

Let us consider problem (1.1) with $f(u) = |u|^{p-1}u$, $p > 1$, or problem (3.1). We assume that $u_0 \in L^\infty$.

The first result on type I blow-up was established by Weissler in [111] under the assumptions that $p < p_S$, Ω is a ball, u is radially decreasing, $u, u_t \geqslant 0$ and $u_t(\cdot,t)$ achieves its maximum at zero. An earlier example of type I blow-up can be found in [110].

Using the maximum principle, Friedman and McLeod showed in [28] that blow-up is of type I if Ω is bounded and $u, u_t \geqslant 0$.

Giga and Kohn proved in [39,40] that blow-up is of type I provided Ω is a bounded convex domain or $\Omega = \mathbb{R}^N$ and either $u_0 \geqslant 0$, $p < p_S$ or $(3N-4)p < 3N+8$. Recently, Giga, Matsui and Sasayama have replaced the technical assumption $(3N-4)p < 3N+8$ by $p < p_S$ in [42,43]. All these results were established by energy methods. In [34], Galaktionov and Posashkov obtained that blow-up is of type I for $N = 1$ using a different method. Merle and Zaag [75,76] refined the above results by proving that

$$\big((p-1)(T-t)\big)^{1/(p-1)} \big\|u(\cdot,t)\big\|_{L^\infty} \to 1 \quad \text{as } t \to T.$$

In the critical case $p = p_S$, Filippas, Herrero and Velázquez proved in [27] that blow-up is of type I if the initial function is positive, radially symmetric and decreasing in r. In [69], Matano and Merle removed the assumption that the initial function is decreasing in r.

In the case $p_S < p < p^*$, it was shown in [69] that blow-up is of type I provided Ω is a ball and u is radially symmetric. A similar result for $\Omega = \mathbb{R}^N$ can also be found in [69] under the additional assumption that there is some $t_0 \in [0,T)$ such that the functions $|u(\cdot,t_0)| - \varphi_\infty$ and $u_t(\cdot,t_0)$ change sign only finitely many times in $[0,\infty)$.

In [71], Matos proved that unfocused blow-up of radially symmetric solutions (Ω is a ball or $\Omega = \mathbb{R}^N$) is of type I for any $p > 1$. By unfocused blow-up at time $T > 0$ we mean that there exist $r_0 > 0$, $r_n \to r_0$, $t_n \to T$ such that $|u(r_n,t_n)| \to \infty$.

For the exponential equation, blow-up is of type I if Ω is bounded and $u, u_t \geqslant 0$, see [28].

5.3. *Type II blow-up*

In [55,56], Herrero and Velázquez gave an example of a positive radial solution u of (3.1) with type II blow-up for $p > p^*$.

To indicate how type II blow-up occurs in [55,56], and to explain the role of the assumption $p > p^*$, we first introduce the change of variables

$$w(y,s) = (T-t)^{1/(p-1)} u(r,t), \qquad y = \frac{r}{\sqrt{T-t}}, s = -\log(T-t). \tag{5.1}$$

Then w satisfies the equation

$$w_s = w_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right) w_y + w^p - \frac{1}{p-1} w, \qquad y > 0,$$

and if $w(\cdot,s)$ converges to the singular stationary solution φ_∞ as $s \to \infty$ then u exhibits type II blow-up. To show that such solution w exists one linearizes around φ_∞ by setting

$$\Psi(y,s) := w(y,s) - \varphi_\infty(y).$$

Then Ψ satisfies (for $y > 0$) the equation

$$
\begin{aligned}
\Psi_s = {} & \Psi_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right)\Psi_y + p\frac{K^{p-1}}{y^2}\Psi - \frac{\Psi}{p-1} \\
& + \left(\left(\varphi_\infty(y) + \Psi\right)^p - \left(\varphi_\infty(y)\right)^p - p\frac{K^{p-1}}{y^2}\Psi\right) \\
=: {} & -A\Psi + f(\Psi).
\end{aligned}
$$

Let

$$
L_w^2 = \left\{ g \in L^2_{\mathrm{loc}}\colon \int_0^\infty g^2(y) y^{N-1} \mathrm{e}^{-y^2/4}\,\mathrm{d}y < \infty \right\}
$$

and

$$
H_w^1 = \left\{ g \in H^1_{\mathrm{loc}}\colon g, g' \in L_w^2 \right\}.
$$

To prove that $\Psi(s, y) \to 0$ as $s \to \infty$ in a suitable way, a key point consists in showing that the linear operator A with domain $D(A) = \{g \in H_w^1\colon Ag \in L_w^2\}$ has a self-adjoint extension. This can only be done if $p > p^*$.

A new shorter proof of the existence of a type II blow-up solution for $p > p^*$ can be found in [79]. The main ideas in [79] are similar as in [56].

In [78], Mizoguchi used her result on nonexistence of positive bounded backward self-similar solutions intersecting φ_∞ at least twice (cf. Section 2.1) to show that for $N \geqslant 24$ there is $p_N > 1$ such that type II blow-up of a positive radial solution occurs provided Ω is a ball and $p \geqslant p_N$.

Using her result on boundedness of global solutions (Theorem 4.1(i)), Mizoguchi established in [83] the existence of a type II blow-up solution when Ω is a ball and $p > 1 + 7/(N-11)$, $N > 11$.

In the critical case $p = p_S$, $3 \leqslant N \leqslant 6$, Filippas, Herrero and Velázquez constructed formal examples of type II blow-up for sign-changing solutions by matched asymptotics in [27].

No examples of type II blow-up are available for the exponential equation.

6. Convergence to backward self-similar solutions

Let us consider problem (1.1) with $f(u) = |u|^{p-1}u$, $p > 1$, or problem (3.1).

By the blow-up set $B(u_0)$ of a solution u which blows up at $t = T < \infty$ we mean the set of all $x \in \overline{\Omega}$ such that there exist sequences $\{x_n\} \subset \Omega$, $\{t_n\} \subset (0, T)$, $x_n \to x$, $t_n \to T$, for which $|u(x_n, t_n)| \to \infty$.

Assume that Ω is a ball centered at the origin or $\Omega = \mathbb{R}^N$ and u_0 is radially symmetric. If $\Omega = \mathbb{R}^N$ assume further that $u_0 \in H^1(\mathbb{R}^N)$. Let u exhibit type I blow-up. Then one of the following holds (see [69,71]):

(1) $B(u_0) = \{0\}$,

(2) $B(u_0)$ is the union of finitely many spheres centered at the origin,

(3) $B(u_0)$ is the union of finitely many spheres centered at the origin and $\{0\}$.

If $u_0 \notin H^1(\mathbb{R}^N)$ when $\Omega = \mathbb{R}^N$ then $B(u_0)$ may contain infinitely many spheres or it may be empty. (Blow-up may occur only at $|x| = \infty$, see [44].)

Concerning the local blow-up profile near a point $a \in B(u_0)$, Matano and Merle proved the following theorem in [69].

THEOREM 6.1. *Let $\Omega = \{x \in \mathbb{R}^N \mid |x| < R\}$, $R > 0$, let u_0 be radially symmetric and $p > p_S$, $N > 2$.*

(i) *If $a \in B(u_0)$, $a \neq 0$, then*

$$\lim_{t \to T} (T-t)^{1/(p-1)} u\big(a + y\sqrt{T-t}, t\big) = \pm (p-1)^{-1/(p-1)} \quad \textit{locally uniformly in } y \in \mathbb{R}^N.$$

(ii) *If $0 \in B(u_0)$ and if blow-up is of type I then*

$$\lim_{t \to T} (T-t)^{1/(p-1)} u\big(a + y\sqrt{T-t}, t\big) = \psi\big(|y|\big) \quad \textit{locally uniformly in } y \in \mathbb{R}^N, \tag{6.1}$$

where ψ is a solution of (2.2), (2.3) *with $\alpha \neq 0$.*

(iii) *If $0 \in B(u_0)$ and if blow-up is of type II then*

$$\lim_{t \to T} (T-t)^{1/(p-1)} u\big(a + y\sqrt{T-t}, t\big) = \tilde{\psi}\big(|y|\big) \quad \textit{locally uniformly in } y \in \mathbb{R}^N,$$

where $\tilde{\psi}$ is as in (ii) *or $\tilde{\psi} = \pm \phi^*$.*

Giga and Kohn [39–41] showed that for $p < p_S$ one has

$$\lim_{t \to T} (T-t)^{1/(p-1)} u\big(a + y\sqrt{T-t}, t\big) = (p-1)^{-1/(p-1)} \quad \text{locally uniformly in } y \in \mathbb{R}^N,$$

provided Ω is a convex bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$ and $u_0 \geqslant 0$.

Part (i) of Theorem 6.1 was proved earlier in [71] under the additional assumption that blow-up is of type I and part (ii) in [72] for positive solutions. In [73] it was shown that if a solution blows up at a finite time T but continues to exist as an L^1-solution for $t > T$ then either blow-up is of type II or (6.1) holds with a nonconstant solution of (2.2), (2.3). By the result from [69] mentioned in Section 5.2, blow-up is of type I if $p_S < p < p^*$. Therefore, for solutions which can be continued beyond blow-up (6.1) holds with a nonconstant solution of (2.2), (2.3) when $p_S < p < p^*$.

For solutions of

$$\begin{aligned}
&u_t = u_{xx} + \mathrm{e}^u, && |x| < 1, t > 0,\\
&u(\pm 1, t) = 0, && t > 0,\\
&u(x, 0) = u_0(x), && |x| \leqslant 1,
\end{aligned}$$

which blow up at $x = 0$ and $t = T$ it was shown in [3] that

$$\lim_{t\to T} \big(u\big(y\sqrt{T-t}, t\big) + \log(T-t)\big) = 0 \quad \text{locally uniformly in } y \in \mathbb{R},$$

under suitable assumptions on u_0.

The same is true for solutions of

$$\begin{aligned}
&u_t = u_{xx} + \mathrm{e}^u, && x \in \mathbb{R}, t > 0,\\
&u(x, 0) = u_0(x), && x \in \mathbb{R},
\end{aligned}$$

if u_0 is continuous nonnegative and bounded, see [52].

In higher space dimension, analogous results are known for radial solutions which are decreasing in $|x| = r$ and nondecreasing in t, see [4].

7. Connecting equilibria by blow-up solutions

In the qualitative theory of one-dimensional parabolic equations, much effort has been devoted to the study of the connection problem – determining which equilibria are connected by heteroclinic orbits (see [17] for recent results and references).

A traditional notion of a connection from an equilibrium $\phi^-(x)$ to an equilibrium $\phi^+(x)$ means a classical solution $u(x, t)$ of an underlying parabolic equation which is defined for $t \in (-\infty, \infty)$ and satisfies

$$u(\cdot, t) \to \phi^\pm \quad \text{as } t \to \pm\infty. \tag{7.1}$$

However, the paper [23] revealed the possibility of connecting equilibria by nonclassical solutions which we call L^1-connections. Considering L^1-connections as well as classical ones may provide more comprehensive understanding of the global dynamics of the underlying parabolic equation. By an L^1-connection we mean a function $u(\cdot, t)$ which is a classical solution on the interval $(-\infty, T)$ for some $T \in \mathbb{R}$ and blows up at $t = T$, but continues to exist as a weak solution on $[T, \infty)$ and satisfies (7.1) in a suitable sense. The term L^1-connections that we adopt here, is derived from the notion of an L^1-solution, which is conventionally used (cf. [88]) for weak solutions that can be viewed as continuous $L^1(\Omega)$-valued functions of t (see Definition 7.2). Abusing the language slightly, we exclude classical solutions when using this term: by an L^1-connection we always mean a nonclassical L^1-connection.

The study of L^1-connections was initiated in [23] and further developed in [19] and [21]. We consider the problem

$$\begin{cases} u_t = \Delta u + \lambda e^u, & x \in B_1(0),\ t > 0, \\ u = 0, & x \in \partial B_1(0),\ t > 0, \\ u(x,0) = u_0\big(|x|\big), & x \in B_1(0), \end{cases} \tag{E}$$

where $B_1(0) = \{x \in \mathbb{R}^N\colon |x| < 1\}$, u_0 is a continuous function on $[0,1]$ vanishing at $r = 1$, λ is a positive parameter and

$$3 \leqslant N \leqslant 9.$$

To describe the results on L^1-connections we have to recall some known properties of equilibria of (E). The stationary problem corresponding to (E) is equivalent to

$$\begin{cases} \phi_{rr} + \frac{N-1}{r}\phi_r + \lambda e^{\phi} = 0, & r \in (0,1), \\ \phi_r(0) = 0, \qquad \phi(1) = 0. \end{cases} \tag{SE}$$

Note that even without the assumption on the radial symmetry of $u(\cdot,0)$, problem (SE) describes all equilibrium solutions of (E), since they are positive hence radially symmetric due to the general result of [37]. By the same result, each equilibrium ϕ is decreasing in r. In particular, we have

$$\|\phi\|_\infty = \phi(0)$$

for the L^∞ norm of ϕ.

THEOREM 7.1 ([36,57], see Figure 1). *Denote by S the solution set of the parameterized problem* (SE),

$$S = \big\{(\phi,\lambda)\colon \lambda \in \mathbb{R}^+ \textit{ and } \phi \textit{ is a solution of } (\mathrm{SE})\big\}.$$

There exists a smooth curve

$$s \mapsto \big(\phi(s), \lambda(s)\big) : (0,\infty) \to C\big(\overline{B}_1(0)\big) \times (0,\infty)$$

such that $S = \{(\phi(s), \lambda(s))\colon s > 0\}$ and

$$\sup_{x \in B_1(0)} \phi(s)(x) = \phi(s)(0) = s.$$

Moreover, the following holds:

(a) $\lim_{s\to 0} \lambda(s) = 0$, $\lim_{s\to\infty} \lambda(s) = \lambda_\infty := 2(N-2)$;

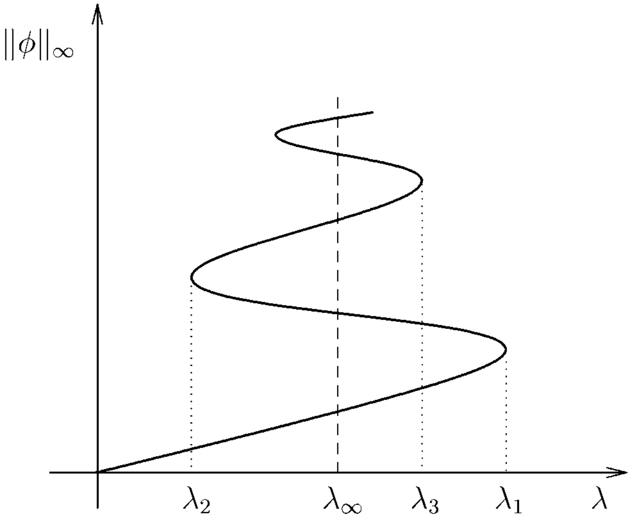

Fig. 1. Bifurcation diagram for equilibria.

(b) $\lim_{s\to 0}\phi(s) = 0$ *in* $C^2([0,1])$ *and*

$$\lim_{s\to\infty}\phi(s)(r) = \phi_\infty(r) := -2\ln r \quad \textit{in } C^2_{\mathrm{loc}}\big((0,1]\big); \tag{7.2}$$

(c) $\lambda(s)$ *is a Morse function, that is,*

$$\textit{if } \lambda'(s) = 0 \textit{ then } \lambda''(s) \neq 0;$$

(d) *the set of all zeros of* $\lambda'(\cdot)$ *is given by a sequence* $0 < s_1 < s_2 < s_3 < \cdots \to \infty$ *and the critical values* $\lambda_j = \lambda(s_j)$, $j = 1, 2, 3, \dots,$ *satisfy*

$$\lambda_1 > \lambda_3 > \cdots > \lambda_{2j+1} \searrow \lambda_\infty, \qquad \lambda_2 < \lambda_4 < \cdots < \lambda_{2j+2} \nearrow \lambda_\infty;$$

(e) *for each* $\lambda \leqslant \lambda_1$ *define*

$$\phi_i^\lambda = \phi(\tilde{s}_i), \quad i = 0, 1, \dots,$$

where $\tilde{s}_0 < \tilde{s}_1 < \cdots$ *is the sequence of all points* s *with* $\lambda(s) = \lambda$*. This sequence is finite if* $\lambda \neq \lambda_\infty$ *and infinite if* $\lambda = \lambda_\infty$*. In the latter case we have*

$$\phi_i^\lambda(r) \to \phi_\infty(r) \quad \textit{as } i \to \infty.$$

By using standard bifurcation techniques and transversality arguments, it is not difficult to establish classical connections between these equilibria. As shown in [23] (see also Proposition 3.3 in [21]), if λ is different from any of the critical values $\lambda_1, \lambda_2, \dots,$ then a classical connection from ϕ_k^λ to ϕ_j^λ exists if and only if $k > j$.

Now we introduce the notion of an L^1-connection.

DEFINITION 7.2. By a (nonclassical) L^1-connection from ϕ_k^λ to ϕ_j^λ we mean a function $u^*(r,t)$ such that

(i) u^* is a classical solution of

$$\begin{aligned} &u_t = \Delta u + \lambda e^u, && x \in B_1(0), \infty < t < T,\\ &u = 0, && x \in \partial B_1(0), -\infty < t < T, \end{aligned}$$

for some $T \in \mathbb{R}$;

(ii) u^* blows up at $t = T$ which means that

$$\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(B_1(0))} = \infty;$$

(iii) for any $\tau < T$, $u^*(x, t+\tau)$ is a global L^1-solution of (E) with $u_0 = u^*(\cdot,\tau)$;

(iv) $u^*(\cdot,t) \to \phi_k^\lambda$ in $C^1([0,1])$ as $t \to -\infty$;

(v) $u^*(\cdot,t) \to \phi_j^\lambda$ in $C^1_{\mathrm{loc}}((0,1])$ as $t \to \infty$;

(vi) for any $\tau < T$ and $u_0 = u^*(\cdot,\tau)$ there is a sequence $\{u_{0,n}\} \subset C([0,1])$ such that $u_{0,n}(1) = 0$, $u_{0,n} \to u_0$ in $C([0,1])$, and $\{u(r,t;u_{0,n})\}$ is a sequence of global classical solutions.

The last condition (vi) in this definition requires some explanation. It implies that u^* can be approximated, in the sense of pointwise convergence, by global classical solutions on $(0,1] \times (-\infty,\infty)$. Thanks to this approximation property, we can derive various useful properties of u^*. It is known that condition (vi) is automatically satisfied if the L^1-continuation beyond the blow-up time T is unique, but whether this latter property always holds or not remains an open question.

Note also that condition (vi) is slightly weaker than the corresponding condition in [19], where it is required that $\{u_{0,n}\}$ be an increasing sequence. It follows that the L^1-continuation beyond the blow-up time in the sense of [19] is minimal among all possible L^1-continuations. As mentioned above, we do not yet know whether the continuation beyond the blow-up time is unique. If it is not unique, it is an intriguing question to ask whether all the possible continuations connect ϕ_k^λ to the same equilibrium ϕ_j^λ.

In [23] we found an L^1-connection from ϕ_2^λ to ϕ_0^λ and in [19] we proved that an L^1-connection from ϕ_k^λ to ϕ_0^λ exists provided that either $k \geqslant 2$ is an even integer and $\lambda \in (\lambda_k, \lambda_{k+1})$ or $k > 2$ is odd and $\lambda \in (\lambda_{k+1}, \lambda_k)$.

We have the following necessary condition on the existence of L^1-connections.

THEOREM 7.3. *If there exists an L^1-connection from ϕ_k^λ to ϕ_j^λ, then $k \geqslant j+2$.*

Although this was proved in [19] under the more restrictive definition of L^1-connections, the proof is easily seen to extend to the present more general definition. However, the proof of nonexistence of a homoclinic L^1-connection from ϕ_k^λ to ϕ_k^λ was not completely correct in [19] but it can be fixed easily. In Lemma 5.4 there, one should replace ϕ_∞ by ϕ_n^λ with n large enough and show that the number of intersections with ϕ_n^λ drops before the blow-up time. An alternative proof is offered by Lemma 2.3(ii) in [21].

We conjectured in [19] that the converse of Theorem 7.3 is also true and we proved the following theorem in [21].

THEOREM 7.4. *An L^1-connection from ϕ_k^λ to ϕ_j^λ exists provided $k \geqslant j+2$.*

We thus gave a complete answer to the question of determining which classical equilibria are connected by L^1-solutions. Note that no information is lost by restricting our considerations to radial solutions. In fact, any solution that is classical on an interval $(-\infty, T)$ and approaches an equilibrium as $t \to -\infty$ must be radially symmetric in space (see [94]).

8. Immediate regularization after blow-up

In this section we review the main results and ideas from [22]. We consider the problem

$$\begin{cases} u_t = \Delta u + f(u), & x \in B_1, t > 0, \\ u = 0, & x \in \partial B_1, t > 0, \\ u(x,0) = u_0(x) \ \big(= U_0\big(|x|\big)\big), & x \in B_1, \end{cases} \tag{P}$$

where $B_1 = \{x \in \mathbb{R}^N\colon\ |x| < 1\}$, $U_0 \in C([0,1])$, $U_0 \geqslant 0$ with $U_0(1) = 0$, and either

$$f(u) = \lambda \mathrm{e}^u, \quad \lambda > 0, 3 \leqslant N \leqslant 9, \tag{8.1}$$

or

$$f(u) = u^p, \quad \frac{N+2}{N-2} < p < p^*, N \geqslant 3, \tag{8.2}$$

where p^* is defined in (2.4).

In the case of the exponential nonlinearity (8.1), we shall further assume that $U_0(r)$ is a nonincreasing function on $[0,1]$.

We shall show that if a solution blows up in a finite time $t = T < \infty$ but continues to exist as a weak solution for $t > T$, then this extended solution becomes regular immediately after the blow-up time T, that is, it possesses no singularity in the time interval $T < t < T^*$ for some $T^* \in (T, \infty]$. Here, by an extended solution we mean the minimal L^1-solution in the sense of the following definition.

DEFINITION 8.1. By a *limit L^1-solution* we mean a global L^1-solution (cf. Definition 1.1) which can be approximated by global classical solutions in the following way: There is a sequence $\{u_{0,n}\}$ in $C(\overline{B}_1)$ such that

$$u_{0,n} \to u_0 \quad \text{in } C\big(\overline{B}_1\big) \tag{8.3}$$

and that the solution u_n of (P) with $u(\cdot,0) = u_{0,n}$ exists globally for $t \geqslant 0$ and satisfies

$$\begin{aligned} u_n(\cdot,t) &\to u(\cdot,t) && \text{in } L^1(B_1) \text{ for every } t > 0, \\ f(u_n) &\to f(u) && \text{in } L^1\big(B_1 \times (0,t)\big) \text{ for every } t > 0. \end{aligned} \tag{8.4}$$

We refer to any such sequence $\{u_n\}$ as an *approximating sequence* for u. We call a limit L^1-solution a *minimal L^1-solution* if it has an approximating sequence that is nondecreasing in n.

Now, given a minimal L^1-solution u, we define the set of *regular time moments* by

$$\mathcal{R} := \{t_0 > 0\colon\ u \text{ is a classical solution on some time interval around } t = t_0\}$$

and the *set of singular time moments* by $\mathcal{S} = (0,\infty) \setminus \mathcal{R}$. By Lemma 2.16 in [22], we see that $\mathcal{S} = \mathcal{B}$, where

$$\mathcal{B} := \Big\{t_0 > 0\colon\ \lim_{t\to t_0} \big\|u(\cdot,t)\big\|_{L^\infty(B_1)} = \infty\Big\}.$$

Our main aim is to show that $\mathcal{B}$ is a finite set. This in particular implies that the solution recovers smoothness immediately after blow-up, and it remains smooth until $t = \infty$ or until the next blow-up occurs. We also give an example of a limit L^1-solution for which $\mathcal{B}$ is a singleton.

The next propositions show that the convergence of approximating classical solutions to a limit L^1-solution takes place in a much stronger topology than (8.4).

PROPOSITION 8.2. *Assume that*

$$f(u) = \lambda e^u, \qquad N > 2,$$

and that $u_0(x) = U_0(|x|)$ *with*

$$U_0 \in C^1\big([0,1]\big), \qquad U_0' \leqslant 0, \qquad U_0(1) = 0.$$

Let u be a limit L^1-solution and let $\{u_n\}$ be an approximating sequence for u. Then

(i) *for each* $t > 0$,

$$u_n(\cdot,t) \to u(\cdot,t) \quad \textit{in } H_0^1(B_1), \qquad e^{u_n(\cdot,t)} \to e^{u(\cdot,t)} \quad \textit{in } L^q(B_1)$$

for any $1 \leqslant q < N/2$. *Moreover, for each* $1 \leqslant q < N/2$ *there is a constant* $M_q > 0$ *such that*

$$\|u\|_{L^q(B_1)} \leqslant M_q, \qquad \big\|\lambda e^u\big\|_{L^q(B_1)} \leqslant M_q,$$

and for each $\delta > 0$ *there is a constant* $C_\delta > 0$ *such that*

$$\big\|u(\cdot,t)\big\|_{H_1} \leqslant C_\delta, \qquad \delta \leqslant t < \infty;$$

(ii) $u \in C((0,\infty); H_0^1(B_1))$;

(iii) *for each* $t > 0$,

$$u_n(\cdot,t) \to u(\cdot,t) \quad \textit{in } C^2_{\mathrm{loc}}\big(\overline{B}_1 \setminus \{0\}\big);$$

(iv) $$J[u_n] \to J[u] \quad \textit{as } n \to \infty,$$

hence $J[u(\cdot,t)]$ *is monotone nonincreasing in* t, *where*

$$J[u] = \int_{B_1} \left(\frac{1}{2}|\nabla u|^2 - \lambda \mathrm{e}^u\right) \mathrm{d}x.$$

For the power nonlinearity we have the following proposition.

PROPOSITION 8.3. *Assume*

$$f(u) = u^p, \quad N > 2,\ p > \frac{N+2}{N-2},$$

and that $U_0 \in C[0,1]$. *Let* u *be a limit* L^1-*solution and let* $\{u_n\}$ *be an approximating sequence for* u. *Then for each* $t > 0$,

$$u_n(\cdot,t) \to u(\cdot,t) \quad \textit{in } H_0^1(B_1) \cap L^q(B_1) \cap C^2_{\mathrm{loc}}\big(\overline{B}_1 \setminus \{0\}\big)$$

for any $1 \leqslant q < N(p-1)/2$. *Furthermore,*

$$u \in C\big((0,\infty); H_0^1(B_1)\big)$$

and $J[u(\cdot,t)]$ *is monotone nonincreasing in* t, *where*

$$J[u] = \int_{B_1} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) \mathrm{d}x.$$

8.1. *Statement and remarks*

Let I be an interval (open, halfopen or closed) with endpoints a, b, $-\infty \leqslant a < b \leqslant \infty$, and let f be a continuous function on I. We define the *zero number* of f by

$$\begin{aligned} \mathcal{Z}_I(f) = \sup\big\{n \in \mathbb{N}\colon & \text{ there are } a < x_0 < x_1 < \cdots < x_n < b \\ & \text{ such that } f(x_i) f(x_{i+1}) < 0 \text{ for } 0 \leqslant i < n\big\} \end{aligned}$$

if f changes sign in I and $\mathcal{Z}_I(f) = 0$ otherwise.

THEOREM 8.4. *Let u be a minimal L^1-solution of problem* (P) *which blows up in a finite time T, and let either* (8.1) *or* (8.2) *be satisfied. Assume that the initial data $U_0(|x|)$ satisfies*

$$U_0 \in C\big([0,1]\big), \qquad U_0(r) \geqslant 0, \quad 0 \leqslant r \leqslant 1.$$

In the case (8.1), *assume further that $U_0(r)$ is nonincreasing in $0 \leqslant r \leqslant 1$. Then there exists a positive integer k such that*

$$\mathcal{B} = \{t_i\}_{i=1}^k, \quad t_1 = T < t_2 < \cdots < t_k < \infty.$$

Consequently, the solution is regular except at $t = t_i$, $i = 1, 2, \ldots, k$. Moreover, the following estimate holds

$$2k - 1 \leqslant j := \min_{0<t<T} \mathcal{Z}_{[0,1]}\big(u_t(\cdot,t)\big). \tag{8.5}$$

In particular, if $U_0 \in C^2([0,1])$ then

$$2k - 1 \leqslant j_0 := \mathcal{Z}_{[0,1]}\left(U_0'' + \frac{N-1}{r}U_0' + f(U_0)\right). \tag{8.6}$$

REMARK 8.5. Since $w := u_t$ satisfies a parabolic equation of the form

$$w_t = w_{rr} + \frac{N-1}{r} w_r + a(r,t)w,$$

and $w(1,t) = 0$, the zero number $\mathcal{Z}_{[0,1]}(u_t(\cdot,t))$ is nonincreasing in t and is finite for every $0 < t < T$. See [10] for details. Consequently, the minimum on the right-hand side of (8.5) is well defined and is a finite integer. It is also clear that $j \leqslant j_0$.

REMARK 8.6. In order for the solution u to be a global L^1-solution, it should necessarily hold that $j > 0$. Indeed, if $j = 0$, this means that the solution satisfies $u_t(r,t) \geqslant 0$, $0 \leqslant r \leqslant 1$, for t close to T. By the result of [2], this means a complete blow-up, therefore the solution cannot be continued as an L^1-solution beyond the blow-up time T.

REMARK 8.7. The above theorem means that the solution recovers smoothness immediately after the blow-up time. Note that this result does not follow from standard parabolic estimates. Indeed, at the time of blow-up, some of the solutions may have a singularity of the form $\log(1/|x|^2) + C$ (in the case (8.1)) or of the form $C|x|^{-2/(p-1)}$ (in the case (8.2)), as exemplified by certain self-similar solutions. When such singularities occur, the solution profile $u(x,T)$ does no longer belong to the space where (P) is well posed (for example, $L^q(B_1)$ with $q > (N-1)p/2$ in the case (8.2)), therefore parabolic regularization alone cannot bring the solution back to the space where (P) is well posed. Thus smoothness does not follow automatically. Indeed the singular stationary solution φ_∞ defined below is an example of a weak solution that never becomes regular. (Since φ_∞ is not a minimal L^1-solution, there is no contradiction with the above theorem.)

REMARK 8.8. We have that $\lim_{t\nearrow t_0}\|u(\cdot,t)\|_{L^\infty}=\infty$ for every $t_0\in\mathcal{B}$. However we do not know whether or not this always implies that $\|u(\cdot,t_0)\|_{L^\infty}=\infty$. Since our equation has a supercritical nonlinearity, some subtle behavior may occur near the origin at the time of blow-up.

8.2. *Singular stationary solutions*

In the proof of Theorem 8.4, a singular stationary solution plays an important role. The equation

$$u_{rr}+\frac{N-1}{r}u_r+f(u)=0, \quad r>0, \tag{8.7}$$

has an explicit singular solution φ_∞ if either $f(u)=\lambda e^u$, $N\geqslant 3$, or $f(u)=u^p$, $N\geqslant 3$, $p>N/(N-2)$. Namely,

$$\varphi_\infty(r)=\log\frac{2(N-2)}{\lambda r^2} \tag{8.8}$$

in the former case, and φ_∞ was defined in (2.5) in the latter case. The assumption that $N\leqslant 9$ in (8.1) or $p<p^*$ in (8.2) guarantees the existence of a forward self-similar solution of the problem

$$\begin{cases} u_t=u_{rr}+\frac{N-1}{r}u_r+f(u), & r,t>0,\\ u_r(0,t)=0, & t>0,\\ u(r,0)=\varphi_\infty(r), & r>0, \end{cases} \tag{S}$$

which is regular for $r\geqslant 0$, $t>0$ (cf. [35,102,104]). This forward self-similar solution is needed in our proof of Theorem 8.4.

Another notable feature of the critical dimension $N=9$ for $f(u)=\lambda e^u$ and the critical power $p=p^*$ for $f(u)=u^p$ is that under the assumption (8.1) or (8.2), it is well known that the graph of any smooth solution of (8.7) intersects with the graph of the singular solution φ_∞ infinitely many times, while this is not the case if $N>9$ (in the exponential case) or $p>p^*$ (in the power case). This property will also be used in our proof of Theorem 8.4.

8.3. *Zero number properties for singular solutions*

It is well known that if u and v are classical solutions of (P), then $\mathcal{Z}_{[0,1]}(u(\cdot,t)-v(\cdot,t))$ is a nonincreasing function of t. This is because $w:=u-v$ satisfies a parabolic equation of the form

$$w_t=w_{rr}+\frac{N-1}{r}w_r+a(r,t)w.$$

Moreover, each time the function $r \mapsto u(r,t) - v(r,t)$ develops a degenerate zero somewhere in $[0,1]$, the above zero number drops at least by 1. See [10] for details. It is easily seen that the same is true if u is a classical solution and $v = \varphi_\infty$, since $u(r,t) - \varphi_\infty(r)$ always have the same sign (i.e., negative) near $r = 0$. However, if u is an L^1-solution, then both u and φ_∞ may have a singularity at $r = 0$, and this makes the situation a bit more complicated. Nonetheless, a slightly weaker version of the above property still holds.

LEMMA 8.9. *Let $u(r,t)$ be a limit L^1-solution of* (P) *and let $\varphi_\infty(r)$ be the singular stationary solution. Let $t^* > 0$ and suppose that there exists a sequence $0 < \tau_1 < \cdots < \tau_k < t^*$ such that $u(r,\tau_i) - \varphi_\infty(r)$ has a degenerate zero in $(0,1]$ for $i = 1,2,\ldots,k$. Then*

$$\mathcal{Z}_{[0,1]}\bigl(u\bigl(\cdot,t^*\bigr) - \varphi_\infty\bigr) \leqslant \mathcal{Z}_{[0,1]}\bigl(u(\cdot,0) - \varphi_\infty\bigr) - k. \tag{8.9}$$

Here we understand that $k = 0$ if there is no such τ_i in the interval $(0,t^*)$.

PROOF. Let u_n be an approximating sequence for u, and let $t_0 \in (0,T)$ be such that $t_0 < \tau_1$ and that $u(r,t_0) - \varphi_\infty(r)$ has no degenerate zero in the interval $[0,1]$. Such t_0 exists since u is a classical solution for $0 \leqslant t < T$ therefore the function $r \mapsto u(r,t) - \varphi_\infty(r)$ can have a degenerate zero at most for a discrete set of values of t (see [10]). Then, since we have

$$u_n(\cdot,t_0) \to u(\cdot,t_0) \quad \text{in } C^2\bigl(\overline{B}_1\bigr),$$

the simplicity of the zeros of $u(r,t_0) - \varphi_\infty(r)$ implies

$$\mathcal{Z}_{[0,1]}\bigl(u_n(\cdot,t_0) - \varphi_\infty\bigr) = \mathcal{Z}_{[0,1]}\bigl(u(\cdot,t_0) - \varphi_\infty\bigr) \tag{8.10}$$

for n sufficiently large. On the other hand, by Propositions 8.2 and 8.3, we have the convergence

$$u_n\bigl(\cdot,t^*\bigr) \to u\bigl(\cdot,t^*\bigr) \quad \text{in } C^2_{\mathrm{loc}}\bigl(\overline{B}_1 \setminus \{0\}\bigr).$$

Consequently,

$$\mathcal{Z}_{[0,1]}\bigl(u_n\bigl(\cdot,t^*\bigr) - \varphi_\infty\bigr) \geqslant \mathcal{Z}_{[0,1]}\bigl(u\bigl(\cdot,t^*\bigr) - \varphi_\infty\bigr) \tag{8.11}$$

for n sufficiently large.

Now let $r_i \in (0,1]$, $i = 1,2,\ldots,k$, be such that the function $r \mapsto u(r,\tau_i) - \varphi_\infty(r)$ has a degenerate zero at $r = r_i$. Suppose first that $r_i < 1$, $i = 1,2,\ldots,k$. (This is always true if $f(u) = u^p$, $p > 1$, or if $f(u) = \lambda \mathrm{e}^u$, $\lambda > 0$, $\lambda \neq 2(N-2)$.) Choose a_i, b_i such that $0 < a_i < r_i < b_i < 1$ and that

$$u(a_i,\tau_i) - \varphi_\infty(a_i) \neq 0, \qquad u(b_i,\tau_i) - \varphi_\infty(b_i) \neq 0 \quad \text{for } i = 1,\ldots,k.$$

Next choose $\varepsilon > 0$ sufficiently small so that $u(r,t) - \varphi_\infty(r) \neq 0$ for $r = a_i$, $\tau_i - \varepsilon \leqslant t \leqslant \tau_i + \varepsilon$ and $r = b_i$, $\tau_i - \varepsilon \leqslant t \leqslant \tau_i + \varepsilon$. We denote these two line segments by $\gamma_i, \tilde{\gamma}_i$. Since

$u(r,t) - \varphi_\infty(r)$ has a degenerate zero at $(r,t) = (r_i, \tau_i)$ and since this function does not vanish on $\gamma_i, \tilde{\gamma}_i$, we have

$$\mathcal{Z}_{[a_i,b_i]}\big(u(\cdot, \tau_i - \varepsilon) - \varphi_\infty\big) > \mathcal{Z}_{[a_i,b_i]}\big(u(\cdot, \tau_i + \varepsilon) - \varphi_\infty\big).$$

Here we can choose $\varepsilon > 0$ in such a way that the functions $u(\cdot, t_i \pm \varepsilon) - \varphi_\infty$ have only simple zeros in the interval $[a_i, b_i]$ and that the intervals $[\tau_i - \varepsilon, \tau_i + \varepsilon]$, $i = 1, \dots, k$, are mutually disjoint. This is possible since degenerate zeros can occur only at a discrete set of time t. Then

$$\mathcal{Z}_{[a_i,b_i]}\big(u_n(\cdot, \tau_i \pm \varepsilon) - \varphi_\infty\big) = \mathcal{Z}_{[a_i,b_i]}\big(u(\cdot, \tau_i \pm \varepsilon) - \varphi_\infty\big)$$

for n sufficiently large, hence

$$\mathcal{Z}_{[a_i,b_i]}\big(u_n(\cdot, \tau_i - \varepsilon) - \varphi_\infty\big) > \mathcal{Z}_{[a_i,b_i]}\big(u_n(\cdot, \tau_i + \varepsilon) - \varphi_\infty\big).$$

Moreover, since $u(r,t) - \varphi_\infty(r)$ does not vanish on $\gamma_i, \tilde{\gamma}_i$, the same is true for $u_n(r,t) - \varphi_\infty(r)$ for n sufficiently large. This and the above inequality imply that the function $r \mapsto u_n(r,t) - \varphi_\infty(r)$ has a degenerate zero in $[a_i, b_i]$ for some $t \in (\tau_i - \varepsilon, \tau_i + \varepsilon)$. Consequently at least k degenerate zeros occur in the time interval $[t_0, t^*]$, hence

$$\mathcal{Z}_{[0,1]}\big(u_n(\cdot, t^*) - \varphi_\infty\big) \leqslant \mathcal{Z}_{[0,1]}\big(u_n(\cdot, t_0) - \varphi_\infty\big) - k.$$

Combining this inequality with (8.10) and (8.11), we obtain

$$\mathcal{Z}_{[0,1]}\big(u(\cdot, t^*) - \varphi_\infty\big) \leqslant \mathcal{Z}_{[0,1]}\big(u(\cdot, t_0) - \varphi_\infty\big) - k.$$

Since u is a classical solution for $0 \leqslant t < T$, we have

$$\mathcal{Z}_{[0,1]}\big(u(\cdot, t_0) - \varphi_\infty\big) \leqslant \mathcal{Z}_{[0,1]}\big(u(\cdot, 0) - \varphi_\infty\big).$$

This and the previous inequality prove the lemma if $r_i < 1$, $i = 1, 2, \dots, k$.

When $r_i = 1$ for some $i \in \{1, 2, \dots, k\}$ (which can happen only if $f(u) = \lambda e^u$, $\lambda = 2(N-2)$), then one can use the Hopf boundary lemma to proceed similarly as before. □

REMARK 8.10. Since $\mathcal{Z}_{[0,1]}(u(\cdot, t_0) - \varphi_\infty) < \infty$, the left-hand side of (8.9) is finite even if

$$\mathcal{Z}_{[0,1]}\big(u(\cdot, 0) - \varphi_\infty\big) = \infty.$$

REMARK 8.11. Note that the left-hand side of (8.9) is not necessarily monotone nonincreasing in t. This is because some intersection points between the graph of $r \mapsto u(r,t)$ and that of $\varphi_\infty(r)$ may escape to infinity (at $r = 0$) and later emerge from infinity repeatedly.

8.4. *Rescaled equations*

As usual, rescaling arguments provide useful information about the behavior of solutions near the blow-up point. In the case of the exponential nonlinearity (8.1), we use the rescaling

$$w(y,s) = w^\theta(y,s) = u(r,t) + \log(\theta - t), \quad y = \frac{r}{\sqrt{\theta - t}}, s = -\log(\theta - t), \tag{8.12}$$

where θ is any positive number. Then (P) is converted into the following problem

$$\begin{cases} w_s = \frac{1}{\rho}(\rho w_y)_y + \lambda e^w - 1, & 0 < y < e^{s/2}, s > -\log\theta, \\ w_y(0,s) = 0, \qquad w\big(e^{s/2}, s\big) = -s, & s > -\log\theta, \\ w(y, -\log\theta) = u_0\big(\sqrt{\theta} y\big) + \log\theta, & 0 \leqslant y \leqslant \theta^{-1/2}, \end{cases} \tag{$\mathrm{R_e}$}$$

where

$$\rho(y) = y^{N-1} e^{-y^2/4}.$$

In the case of the power nonlinearity (8.2), we use the rescaling

$$w^\theta(y,s) = (\theta - t)^{1/(p-1)} u(r,t), \tag{8.13}$$

with y and s as before. Then (P) is converted into

$$\begin{cases} w_s = \frac{1}{\rho}(\rho w_y)_y + w^p - \frac{1}{p-1} w, & 0 < y < e^{s/2}, s > -\log\theta, \\ w_y(0,s) = 0, \qquad w\big(e^{s/2}, s\big) = 0, & s > -\log\theta, \\ w(y, -\log\theta) = \theta^{1/(p-1)} u_0\big(\sqrt{\theta} y\big), & 0 \leqslant y \leqslant \theta^{-1/2}. \end{cases} \tag{$\mathrm{R_p}$}$$

Note that, in contrast with the usual setup, we do not assume that $\theta = T$, the blow-up time of solution u. In fact, in our later argument, we shall need to consider the case $\theta > T$, thus w may possess singularity in finite time. However, for the time being we assume that w is a classical solution of ($\mathrm{R_e}$) or ($\mathrm{R_p}$) and shall later use a limiting argument to deal with singular solutions.

Energy functionals corresponding to ($\mathrm{R_e}$) and ($\mathrm{R_p}$) are, respectively, the following equations

$$E_\mathrm{e}[w](s) := \int_0^{e^{s/2}} \left(\frac{1}{2} w_y^2 - \lambda e^w + w\right) \rho \, dy,$$

$$E_\mathrm{p}[w](s) := \int_0^{e^{s/2}} \left(\frac{1}{2} w_y^2 - \frac{1}{p+1} w^{p+1} + \frac{1}{2(p-1)} w^2\right) \rho \, dy.$$

As is well known, E_{p} is a Lyapunov functional for $(\mathrm{R_p})$. More precisely,

$$\frac{\mathrm{d}}{\mathrm{d}s}E_{\mathrm{p}}[w](s) \leqslant -\int_0^{\mathrm{e}^{s/2}} w_s^2 \rho \,\mathrm{d}y, \quad s > -\log\theta, \tag{8.14}$$

hence $E_{\mathrm{p}}[w](s)$ is monotone decreasing in s. The same is true of E_{e}, at least for large s, as shown in the following lemma.

LEMMA 8.12. *Let w be a global classical solution of* $(\mathrm{R_e})$. *Then there is $s_0 < 1$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}s}E_{\mathrm{e}}[w](s) \leqslant -\int_0^{\mathrm{e}^{s/2}} w_s^2 \rho \,\mathrm{d}y, \quad s \geqslant s_0. \tag{8.15}$$

PROOF. By direct computations we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}E_{\mathrm{e}}[w](s)\\&\quad = -\int_0^{\mathrm{e}^{s/2}} w_s^2 \rho \,\mathrm{d}y - \left(w_y + \frac{1}{4}\mathrm{e}^{s/2}w_y^2 + \frac{\lambda}{2}\mathrm{e}^{-s/2} + \frac{1}{2}s\mathrm{e}^{s/2}\right)\rho\bigg|_{y=\mathrm{e}^{s/2}}.\end{aligned}$$

Since

$$w_y + \frac{1}{4}\mathrm{e}^{s/2}w_y^2 \geqslant -\mathrm{e}^{-s/2},$$

it suffices to choose s_0 such that $s_0\mathrm{e}^{s_0} > 2 - \lambda$. □

In the case of the power nonlinearity (8.2), the global existence of w^θ for all large s implies that $E_{\mathrm{p}}[w^\theta](s) \geqslant 0$ (see [31]). Integrating (8.14) with respect to s, we obtain

$$\int_{s_1}^{s_2}\int_0^{\mathrm{e}^{s/2}} \left(w_s^\theta\right)^2 \rho \,\mathrm{d}y\,\mathrm{d}s \leqslant E_{\mathrm{p}}\left[w^\theta\right](s_1) - E_{\mathrm{p}}\left[w^\theta\right](s_2).$$

Letting $s_2 \to \infty$ and using the boundedness of $E_{\mathrm{p}}[w^\theta]$, we obtain

$$\int_{s_1}^{\infty}\int_0^{\mathrm{e}^{s/2}} \left(w_s^\theta\right)^2 \rho \,\mathrm{d}y\,\mathrm{d}s < \infty \tag{8.16}$$

(cf. Proposition 7.1 of [69]). As a matter of fact, the same estimate holds for any limit L^1-solutions. More precisely, we have the following lemma.

LEMMA 8.13. *Let u be a limit L^1-solution of* (P) *for the power nonlinearity* (8.2) *and let w^θ be as in* (8.13) *for some $\theta > 0$. Then* (8.16) *holds for any $s_1 > -\log\theta$.*

PROOF. Let u_n be an approximating sequence for u. Then w_n^θ satisfies the same estimate as (8.16) for $n = 1, 2, 3, \ldots$ where the bound does not depend on n. Letting $n \to \infty$ and using Fatou's lemma, we obtain (8.16). $\square$

COROLLARY 8.14. *Let u and w^θ be as in Lemma* 8.13. *Then $w^\theta(y, s)$ approaches stationary solutions of* ($\mathrm{R_p}$) *as $s \to \infty$ locally uniformly in $y > 0$. More precisely, the ω limit set of w^θ is contained in the set of solutions of*

$$\frac{1}{\rho}(\rho w_y)_y + w^p - \frac{1}{p-1} w = 0, \quad y > 0. \tag{8.17}$$

PROOF. Corollary 3.3 and Remark 3.5 in [69] show that any limit L^1-solution satisfies

$$\left|u(r,t)\right| \leqslant C r^{-2/(p-1)} \quad \text{for } 0 \leqslant t < \infty,$$

for some constant $C > 0$. This estimate yields

$$w^\theta(y, s) \leqslant C y^{-2/(p-1)}. \tag{8.18}$$

This pointwise bound and parabolic regularization imply that, for any $M > 0$, the derivatives of w^θ are uniformly Hölder continuous in y in the region

$$y \in I_M := \left[M^{-1}, M\right], \qquad s \geqslant s_1$$

for s_1 sufficiently large. Consequently, $w^\theta(\cdot, s)$ remains in a compact set of $C^2(I_M)$ as s varies over $[s_1, \infty)$. Furthermore, the uniform Hölder continuity of w_s^θ and (8.16) imply that

$$w_s^\theta(y, s) \to 0 \quad \text{as } s \to \infty$$

uniformly in $y \in I_M$. The conclusion of the lemma now follows immediately. $\square$

In the case of the exponential nonlinearity (8.1), pointwise estimates of w^θ are much more difficult to obtain. However, under certain special circumstances we have an analogue of the above corollary.

LEMMA 8.15. *Let u be a limit L^1-solution of* (P) *for the exponential nonlinearity* (8.1) *and let w^θ be as in* (8.12). *Suppose that for some $0 < r_0 \leqslant 1$, $C_1 > 0$ and $0 < t_1 < t_2$,*

$$u(r, t) \geqslant \log \frac{1}{r^2} - C_1 \quad \textit{for } 0 < r \leqslant r_0, t_1 \leqslant t \leqslant t_2. \tag{8.19}$$

Then, for any $\theta \in (t_1, t_2)$, the solution $w^\theta(y, s)$ approaches stationary solutions of ($\mathrm{R_e}$) *as $s \to \infty$ locally uniformly in $y > 0$. More precisely, the ω limit set of w^θ is contained in the set of solutions of*

$$\frac{1}{\rho}(\rho w_y)_y + \lambda \mathrm{e}^w - 1 = 0, \quad y > 0. \tag{8.20}$$

PROOF. By choosing a suitable constant $a > 0$, we see that

$$u(r,t) \geqslant \log \frac{1}{r^2} - C_1, \quad t_0 \leqslant t \leqslant t_0 + ar^2,$$

for any $0 < r \leqslant r_0,\ t_1 \leqslant t_0 \leqslant \theta$. By Lemma 2.15 in [22], we have

$$u(r,t) \leqslant \log \frac{1}{r^2} + \alpha \quad \text{for } 0 < r \leqslant r_0, t_1 \leqslant t \leqslant \theta.$$

(In Lemma 2.15 in [22], the assumption that $u(r,t)$ is nonincreasing in r is used. This lemma is not needed in the power case.)

Now, w^θ satisfies

$$\log \frac{1}{y^2} - C_1 \leqslant w^\theta(y,s) \leqslant \log \frac{1}{y^2} + \alpha \tag{8.21}$$

for $0 < y \leqslant r_0 e^{s/2}$ and for all large s. Thus, once we have the estimate (8.16), then the same argument as in the proof of Corollary 8.14 will yield the conclusion of the lemma.

In order to prove (8.16), it suffices to show that $E_e[w^\theta](s)$ is bounded from below as $s \to \infty$. Observe that (8.21) implies

$$\begin{aligned} E_e\big[w^\theta\big](s) &\geqslant \int_0^{e^{s/2}} \big(-\lambda e^{w^\theta} + w^\theta\big)\rho \, dy \\ &\geqslant \int_0^{e^{s/2}} \left(-\lambda e^{\log(1/y^2)+\alpha} + \log \frac{1}{y^2} - C_1\right)\rho \, dy \\ &= \int_0^{e^{s/2}} \left(-\lambda e^\alpha \frac{1}{y^2} + \log \frac{1}{y^2} - C_1\right) y^{N-1} e^{-y^2/4} \, dy. \end{aligned}$$

It is easily seen that the above integral remains bounded as $s \to \infty$. The lemma is proved. $\square$

8.5. *Singular stationary solutions for the rescaled equation*

The singular stationary solution φ_∞ defined in (8.8) (resp. (2.5)) is also a singular stationary solution for the rescaled equation (8.20) (resp. (8.17)). In Corollary 8.14 and Lemma 8.15, we are not excluding the possibility that w^θ approaches a singular stationary solution as $s \to \infty$.

The following lemmas show that there is no singular stationary solution that lies above φ_∞.

LEMMA 8.16. *Let φ_∞ be as in* (8.8) *and $\rho(y) = y^{N-1} e^{-y^2/4}$. If ψ is a solution of* (8.20) *satisfying $\psi \geqslant \varphi_\infty$, then $\psi = \varphi_\infty$.*

PROOF. Suppose $\Phi := \psi - \varphi_\infty > 0$ for $0 < y < \infty$. Then $(\rho\Phi_y)_y < 0$ because Φ satisfies

$$(\rho\Phi_y)_y + \lambda\rho\left(e^{\psi} - e^{\varphi_\infty}\right) = 0.$$

Define a new space variable $z = z(y)$ by

$$z = \int_1^y \frac{d\xi}{\rho(\xi)}.$$

Then $(\rho\Phi_y)_y < 0$, $y > 0$, is equivalent to $\Phi_{zz} < 0$, $-\infty < z < \infty$. Therefore Φ is a strictly concave function of z on $\mathbb{R}$, but this is impossible since $\Phi > 0$. □

LEMMA 8.17. *Let φ_∞ be as in* (2.5). *If ψ is a solution of* (8.17) *satisfying $\psi \geqslant \varphi_\infty$, then $\psi = \varphi_\infty$.*

PROOF. Suppose $\Phi := \psi/\varphi_\infty > 1$ for $y \in (0, \infty)$. Then $(\sigma\Phi_y)_y < 0$ because Φ satisfies

$$(\sigma\Phi_y)_y + K^{p-1}\frac{\sigma}{y^2}\left(\Phi^p - \Phi\right) = 0,$$

where

$$\sigma(y) := y^{-4/(p-1)}\rho(y) = y^{N-1-4/(p-1)}e^{-y^2/4}.$$

The rest of the proof is the same as in the proof of Lemma 8.16 since our assumption $p > (N+2)/(N-2)$ implies $2 - N + 4/(p-1) < 0$. □

8.6. *Proof of Theorem 8.4*

We begin with the following lemma.

LEMMA 8.18. *Under the assumptions of Theorem* 8.4, *the set $\mathcal{B}$ is decomposed into a disjoint union of finitely many closed intervals*

$$\mathcal{B} = \bigcup_{i=1}^{k} A_i,$$

where $A_i = [t_i^1, t_i^2]$ or $A_i = \{t_i\}$ for $1 \leqslant i \leqslant k$, and A_k may also be of the form $[t_k, \infty)$.

PROOF. By the definition of $\mathcal{B}$, it is clear that this set is closed. What we have to show is that the number of connected components of $\mathcal{B}$ does not exceed $(j+1)/2$.

Suppose $\mathcal{B}$ has at least k connected components. Then, considering that $u(x,t)$ is a classical solution for $t \notin \mathcal{B}$, we can find a sequence of numbers

$$0 < \tau_1 < t_1 = T < \tau_2 < t_2 < \cdots < \tau_k < t_k$$

such that u is regular in the time interval $[\tau_i, t_i)$ and that

$$\lim_{t \nearrow t_i} u(0, t) = \infty.$$

Therefore there exists $\tilde{\tau}_i \in (\tau_i, t_i)$ such that

$$u(0, \tau_1) < u(0, \tilde{\tau}_1) > u(0, \tau_2) < u(0, \tilde{\tau}_2) > \cdots < u(0, \tilde{\tau}_k).$$

Now let u_n be an approximating sequence for u. Then we have

$$u_n(0, \tau_1) < u_n(0, \tilde{\tau}_1) > u_n(0, \tau_2) < u_n(0, \tilde{\tau}_2) > \cdots < u_n(0, \tilde{\tau}_k)$$

for n sufficiently large. It follows that $(u_n)_t(0, t)$ changes sign at least $2(k-1)$ times. Since $\mathcal{Z}_{[0,1]}((u_n)_t(\cdot, t))$ drops at least by 1 each time $(u_n)_t(0, t)$ changes sign, we have

$$\mathcal{Z}_{[0,1]}\big((u_n)_t(\cdot, \tilde{\tau}_k)\big) \leqslant \mathcal{Z}_{[0,1]}\big((u_n)_t(\cdot, \tau_1)\big) - 2(k-1).$$

Letting $n \to \infty$ we obtain

$$\mathcal{Z}_{[0,1]}\big(u_t(\cdot, \tilde{\tau}_k)\big) \leqslant \mathcal{Z}_{[0,1]}\big(u_t(\cdot, \tau_1)\big) - 2(k-1). \tag{8.22}$$

Since τ_1 can be chosen arbitrarily close to $t_1 := T$, the right-hand side of (8.22) can be replaced by $j - 2(k-1)$. Moreover, since u does not blow up completely at $t = t_k$, we have

$$\mathcal{Z}_{[0,1]}\big(u_t(\cdot, \tilde{\tau}_k)\big) \geqslant 1$$

(cf. Remark 8.6). Combining this and (8.22), we obtain

$$1 \leqslant j - 2(k-1).$$

The lemma is proved. □

REMARK 8.19. In obtaining (8.22), we have used the fact that a pointwise convergence $v_n(r) \to v(r)$ implies

$$\liminf_{n \to \infty} \mathcal{Z}_{(0,1]}(v_n) \geqslant \mathcal{Z}_{(0,1]}(v),$$

and that the equality holds if $v(r)$ has only simple zeros in the interval $0 \leqslant r \leqslant 1$. Thus (8.22) holds if $u_t(r, \tau_1)$ has only simple zeros in $0 \leqslant r \leqslant 1$. Since $u_t(r, t)$ can have a degenerate zero only for a discrete set of values of $t \in (0, T)$, we can always assume without loss of generality that $\tau_1 \in (0, T)$ has the above property.

Now we are ready to prove the main theorem of this section.

PROOF OF THEOREM 8.4. We show by contradiction that $A_1 = \{T\}$. Suppose $A_1 \supset [T, T+\delta]$ for some $\delta > 0$. By Lemma 8.9, $u(r,t) - \varphi_\infty(r)$ can have a degenerate zero in $0 < r \leqslant 1$ only at finitely many values of t. Therefore we can find an interval $[\tau_1, \tau_2] \subset [T, T+\delta]$ (with $\tau_1 < \tau_2$) such that $u(r,t) - \varphi_\infty(r)$ has no degenerate zero for any $t \in [\tau_1, \tau_2]$. This means that the graph of $r \mapsto u(r,t)$ and that of $\varphi_\infty(r)$ always intersect transversally as t varies over the interval $[\tau_1, \tau_2]$, hence these intersection points are smooth functions of t. The number of the intersection points is uniformly bounded as we see in Remark 8.10, but it may not be constant since some intersection points may escape into $r = 0$ (where $\varphi_\infty = \infty$) or emerge from $r = 0$ as t varies (see Remark 8.11). Nonetheless, by replacing $[\tau_1, \tau_2]$ by its suitable subinterval if necessary, we may assume without loss of generality that the number of the intersection points is constant as t varies over $[\tau_1, \tau_2]$. Consequently, there exists $r_0 > 0$ such that either

$$u < \varphi_\infty, \quad r \in (0, r_0], t \in [\tau_1, \tau_2], \tag{8.23}$$

or

$$u > \varphi_\infty, \quad r \in (0, r_0], t \in [\tau_1, \tau_2]. \tag{8.24}$$

If (8.23) holds, the approximating sequence u_n satisfies

$$u_n(r, \tau_1) < \varphi_\infty(r), \qquad r \in (0, r_0],$$
$$u_n(r_0, t) \leqslant u(r_0, t), \quad t \in [\tau_1, \tau_2],$$

for $n = 1, 2, 3, \ldots$ since $u_1 < u_2 < u_3 < \cdots \to u$. Let $\tilde{u}$ be the solution of the initial value problem (S) introduced in Section 8.2. Then since both u and $\tilde{u}$ are smooth outside $r = 0$, there exists some $\delta_0 > 0$ such that $u(r_0, t) < \tilde{u}(r_0, t - \tau_1)$ for $t \in [\tau_1, \tau_1 + \delta_0]$. By the comparison principle we have

$$u_n(r,t) < \tilde{u}(r, t - \tau_1), \quad r \in (0, r_0], t \in [\tau_1, \tau_1 + \delta_0],$$

for $n = 1, 2, 3, \ldots$. Letting $n \to \infty$ we obtain

$$u(r,t) < \tilde{u}(r, t - \tau_1), \quad r \in (0, r_0], t \in [\tau_1, \tau_1 + \delta_0].$$

This implies that u is regular for $\tau_1 < t \leqslant \tau_1 + \delta_0$, contradicting our assumption that $[T, T+\delta] \subset \mathcal{B}$.

Next we consider the case where (8.24) holds. We fix $\theta \in (\tau_1, \tau_2)$ arbitrarily and rescale u using the backward self-similar variables as in (8.12) (for the exponential case (8.1)) or as in (8.13) (for the power case (8.2)). Then the rescaled solution $w^\theta(y,s)$ satisfies (R_e) or (R_p), depending on the nonlinearity.

By Lemma 8.15 (in the case of the exponential nonlinearity) or by Corollary 8.14 (in the case of the power nonlinearity), $w^\theta(y,s)$ must approach stationary solutions as $s \to \infty$

locally uniformly in $y > 0$. Inequality (8.24) implies that w^θ can only approach stationary solutions that lie above φ_∞. However, Lemmas 8.16 and 8.17 state that there is no stationary solution strictly above φ_∞. Therefore

$$w^\theta(y,s) \to \varphi_\infty(y) \quad \text{as } s \to \infty \tag{8.25}$$

locally uniformly in $y > 0$. We shall show that this convergence cannot occur. We consider the exponential case and the power case separately.

First, let us consider the exponential case (8.1). Fix $t_0 \in (0, T)$ such that

$$\mathcal{Z}_{(0,1]}\big(u(\cdot,t_0) - \varphi_\infty\big) =: m_0 < \infty \tag{8.26}$$

and such that the function $r \mapsto u(r,t_0) - \varphi_\infty(r)$ has no degenerate zero in $0 < r \leqslant 1$. For each $a > 0$, let $\varphi_a(r)$ be the solution of

$$\begin{cases} \varphi'' + \frac{N-1}{r}\varphi' + \lambda e^\varphi = 0, & r > 0, \\ \varphi_a'(0) = 0, & \varphi_a(0) = a. \end{cases} \tag{8.27}$$

In other words, $\varphi_a(|x|)$ is a stationary solution of (8.1) in $\mathbb{R}^N$. It is known that, under the assumption $3 \leqslant N \leqslant 9$,

$$\varphi_a(r) \to \varphi_\infty(r) \quad \text{as } a \to \infty \tag{8.28}$$

locally uniformly in $r > 0$ and that

$$\mathcal{Z}_{(0,\infty)}(\varphi_a - \varphi_\infty) = \infty, \tag{8.29}$$

see [57]. The convergence (8.28) and (8.26) yield

$$\big|\mathcal{Z}_{[0,1]}\big(u(\cdot,t_0) - \varphi_a\big) - m_0\big| \leqslant 1 \tag{8.30}$$

for all large a. On the other hand, by (8.29), we can choose $s = s_1$ large enough so that

$$\mathcal{Z}_{(0,\eta(s_1)]}(\varphi_1 - \varphi_\infty) > m_0,$$

where $\eta(s) = e^{s/2}$. Then this and the convergence (8.25) imply that

$$\mathcal{Z}_{(0,\eta(s_1)]}\big(w^\theta(\cdot,s) - \varphi_1\big) > m_0$$

for all large s, hence

$$\mathcal{Z}_{(0,\eta(s)]}\big(w^\theta(\cdot,s) - \varphi_1\big) > m_0.$$

Fix $s_2 \geqslant s_1$ large enough so that the above inequality holds and $t_2 := \theta - e^{-s_2} > t_0$. Then the above inequality can be rewritten as

$$\mathcal{Z}_{(0,\mu]}\big(u^\mu(\cdot,t_2) - \varphi_1\big) > m_0,$$

where

$$u^{\mu}(r,t) = u(\sqrt{\mu}r, t) + \log\mu, \quad \mu = \theta - t_2.$$

Applying the rescaling $v(r) \mapsto v(r/\sqrt{\mu}) - \log\mu$ to both u^{μ} and φ_1 and using the fact that

$$\varphi_1\left(\frac{r}{\sqrt{\mu}}\right) - \log\mu = \varphi_\nu(r), \quad \nu := 1 - \log\mu,$$

we obtain

$$\mathcal{Z}_{[0,1]}\big(u(\cdot, t_2) - \varphi_\nu\big) > m_0.$$

This, however, contradicts (8.30) since $\mathcal{Z}_{[0,1]}(u(\cdot,t) - \varphi_a)$ is monotone nonincreasing in t. This contradiction proves the assertion $A_1 = \{T\}$ for the exponential case (8.1).

In the power case (8.2), the argument goes completely parallel to the above. The only difference is that φ_∞ has now the form (2.5) instead of (8.8), φ_a is the solution of the problem

$$\begin{cases} \varphi'' + \frac{N-1}{r}\varphi' + \varphi^p = 0, & r > 0, \\ \varphi_a'(0) = 0, & \varphi_a(0) = a, \end{cases}$$

and rescaling that we use is

$$u^{\mu}(r,t) = \mu^{1/(p-1)} u(\sqrt{\mu}r, t).$$

The rest of the proof is the same as in the exponential case. Thus we have $A_1 = \{T\}$ both for the power case and the exponential case.

The same argument shows that the sets $A_2, \ldots, A_k$ are all singletons. $\square$

8.7. *Example of a peaking solution on a ball*

In this subsection we consider (P) with the exponential nonlinearity (8.1) and discuss the existence of a minimal L^1-solution that blows up exactly once.

THEOREM 8.20. *There exists an initial function u_0 such that the assumptions of Theorem* 8.4 *are satisfied in the case* (8.1) *with* $j = 1$. *This means that* $\mathcal{B} = \{T\}$.

To provide an example of such single point blow-up (in time–space) we use the following proposition.

PROPOSITION 8.21. *Let u be the minimal L^1-connection from ϕ_2 to ϕ_0 (cf. Section* 7). *Then*

$$\frac{u_t(\cdot,t)}{\|u_t(\cdot,t)\|_{C^1(B_1)}} \to \psi_2 \quad \text{in } C^1(B_1) \quad \text{as } t \to -\infty,$$

where ψ_2 is a normalized eigenfunction of

$$\begin{aligned} &\Delta\psi + \lambda e^{\phi_2(|x|)}\psi + \mu\psi = 0, && x \in B_1, \\ &\psi = 0, && x \in \partial B_1, \end{aligned}$$

corresponding to the second eigenvalue μ_2.

For the proof of this proposition see the proof of Lemma 3.6 (in particular, (3.14)) in [23].

Theorem 8.20 follows now from Proposition 8.21 because $\mathcal{Z}_{[0,1]}(\psi_2) = 1$.

9. Multiple blow-up

In this section we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x, 0) = u_0\big(|x|\big), & x \in \mathbb{R}^N, \end{cases} \tag{9.1}$$

where $p > p^*$ (see (2.4)) and u_0 is a nonnegative bounded continuous function. We review Mizoguchi's construction of solutions of (9.1) which become singular k-times for any positive integer k. Her examples are in some sense complementary to the main result of the previous section, although there the domain was a ball and $p < p^*$.

The following theorem was shown in [82].

THEOREM 9.1. *Assume that $p > p^*$. Then, for every integer $k > 1$, there exist T_i, $i = 1, 2, \ldots, k$, $0 < T_1 < T_2 < \cdots < T_k < \infty$, and a minimal L^1-solution u defined on $[0, T_k]$ such that $u(0, T_i) = \infty$, $i = 1, 2, \ldots, k$, and $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ for $t \in (0, T_1) \cup (T_1, T_2) \cup \cdots \cup (T_{k-1}, T_k)$.*

In the special case $k = 2$, this result was established before in [81]. In order to sketch the proof for $k = 2$, we recall three lemmas from [81].

LEMMA 9.2 [81]. *Let $p > p^*$. Assume that there are positive constants δ, c_i, R_i, $i = 1, 2$, $c_1 < K < c_2$, $R_1 < R_2$, and u_0 such that $\mathcal{Z}_{(0,\infty)}(u_0 - \varphi_\infty) = j + 1$ for some even integer j and*

(a) $u_0(r) = c_1 r^{-2/(p-1)}$ *for* $r \in [R_1 - 2\delta, R_1 + 2\delta]$,

(b) $u_0(r) = c_2 r^{-2/(p-1)}$ *for* $r \in [R_2 - 2\delta, \infty)$,

(c) $u_0(r) \leqslant c_2 r^{-2/(p-1)}$ *for* $r \in (0, \infty)$.

Let $u(\cdot, t)$ be a classical solution for $t \in (0, T)$ such that there are $\varepsilon, R_3 > 0$, $t_1 \in (0, T]$ for which

$$\mathcal{Z}_{(0, R_3\sqrt{T-t})}\big(u(\cdot, t) - \varphi_\infty\big) = j, \quad t \in [0, t_1),$$

$$u(r, t) \leqslant \left(c_2 + \frac{\varepsilon}{2}\right) r^{-2/(p-1)}, \qquad (r, t) \in \big(0, R_3\sqrt{T-t}\,\big) \times (0, t_1).$$

If T is sufficiently small then

(i) $u(r,t) < (c_1+\varepsilon)r^{-2/(p-1)}$ *for* $(r,t) \in [R_1-\delta, R_1+\delta] \times [0,t_1]$,
(ii) $u(r,t) > (c_2-\varepsilon)r^{-2/(p-1)}$ *for* $(r,t) \in [R_2-\delta, \infty) \times [0,t_1]$,
(iii) $u(r,t) < (c_2+\varepsilon)r^{-2/(p-1)}$ *for* $(r,t) \in (0,\infty) \times [0,t_1]$.

LEMMA 9.3 [81]. *Let $p > p^*$ and let u_0 be decreasing. If there are $c, r_0 > 0$ such that*

$$u_0(r_0) = \varphi_\infty(r_0), \qquad u_0'(r_0) + \frac{2}{(p-1)r_0} u_0(r_0) \neq 0,$$
$$u_0(r) \leqslant cr^{-2/(p-1)}, \quad r > 0, \qquad u_0(r) < \varphi_\infty(r), \quad 0 < r < r_0,$$

then the minimal L^1-solution $u(\cdot,t)$ emanating from u_0 is regular for $t \in (0,t_1)$ for some $t_1 > 0$.

Before we state the next lemma we recall that the spectrum of the operator A defined in Section 5.3 consists of eigenvalues

$$\lambda_j = -\frac{\mu}{2} + \frac{1}{p-1} + j, \quad j = 0,1,2,\ldots,$$

where

$$\mu := \frac{1}{2}\left(N-2-\sqrt{(N-2-2m)^2 - 8(N-2-m)}\right), \quad m := \frac{2}{p-1},$$

see [56] or [79]. The corresponding eigenfunction ϕ_j satisfies

$$\phi_j(y) = C_j y^{-\mu} + \mathrm{o}\left(y^{-\mu}\right) \quad \text{as } y \to 0, \tag{9.2}$$

here $C_j > 0$ is taken so that ϕ_j is normalized in L^2_w.

It is also known that there is a constant $c_* > 0$ which depends only on N and p such that for every global classical radially decreasing solution u we have

$$u(r,t) \leqslant c_* r^{-2/(p-1)}, \quad r > 0, t \geqslant 0, \tag{9.3}$$

see [77]. For more general solutions, a bound of this type can be found in [69].

LEMMA 9.4 [81]. *Let $p > p^*$. Assume that j is an even integer such that $\lambda_j > 0$. Then there exists a radially decreasing function u_0 such that the corresponding solution blows up at a finite time T and the following holds*:

(i) $\mathcal{Z}_{(0,\infty)}(u_0 - \varphi_\infty) = j+1$;
(ii) u_0 *satisfies* (a)–(c) *in Lemma* 9.2 *with* $c_1 < K \leqslant c_* < c_2$;
(iii) *let*

$$m = \frac{2}{p-1}, \qquad \eta = -\frac{\lambda_j}{\mu+m}, \qquad \gamma = \eta m,$$

let φ_α and φ_β be radial stationary solutions of (9.1) *such that $a(\alpha) > C_j > a(\beta)$, here a is from* (10.2) *and C_j is from* (9.2) *then for the rescaled solution w* (*see* (5.1)) *it holds that*

$$\mathrm{e}^{\gamma s}\varphi_\alpha\big(\mathrm{e}^{\eta s}y\big) < w(y,s) < \mathrm{e}^{\gamma s}\varphi_\alpha\big(\mathrm{e}^{\eta s}y\big)$$

for $y \in [0, M\mathrm{e}^{-\eta s}]$ and $s \geqslant -\log T$ with some $M > 0$;

(iv) *for sufficiently small $\varepsilon > 0$, the rescaled solution w satisfies*

$$\left|w(y,s) - \varphi_\infty(y) + \mathrm{e}^{-\lambda_j s}\phi_j(y)\right| \leqslant \varepsilon \mathrm{e}^{-\lambda_j s}\big(y^{-\mu} + y^{2\lambda_j - m}\big)$$

for $y \in [M\mathrm{e}^{-\eta s}, \mathrm{e}^{\sigma s}]$, $s \in [-\log T, \infty)$, with some $\sigma > 0$.

Now we sketch the proof of Theorem 9.1. Let u be a solution obtained in Lemma 9.4. Applying Lemma 9.2 we see that j intersections between u and φ^* vanish at the blow-up time T at $r = 0$ and the last intersection remains in $(R_1 + \delta, R_2 - \delta)$ at $t = T$. At the intersection point r_0 we have

$$u_r(r_0, T) + \frac{2}{(p-1)r_0}u(r_0, T) > 0.$$

According to Lemma 9.3, the minimal L^1-solution emanating from $u(\cdot, T)$ is regular for $t \in (T, t_1)$ with some $t_1 > 0$.

Let U be the solution of the heat equation with the initial function

$$U(r,0) = \begin{cases} 0, & r \in [0, R_2], \\ (c_2 - \varepsilon)r^{-2/(p-1)}, & r \in (R_2, \infty), \end{cases}$$

where $\varepsilon > 0$ is sufficiently small and R_2 is from Lemma 9.3. Then one can show that

$$U(r,t) \geqslant (c_2 - 2\varepsilon)r^{-2/(p-1)}, \quad (r,t) \in [R_2, \infty) \times (0, t_2),$$

for sufficiently small $t_2 > 0$. Therefore

$$u(r,t) \geqslant (c_2 - 2\varepsilon)r^{-2/(p-1)}, \quad (r,t) \in [R_2, \infty) \times (0, t_2).$$

According to (9.3), u cannot exist for all $t > T$ by the choice of c_2.

10. Grow-up rate

10.1. *Grow-up rate for Dirichlet problems*

Dold, Galaktionov, Lacey and Vazquez [14] studied the Dirichlet problem

$$\begin{aligned} &u_t = \Delta u + u^p, && x \in B_R, t > 0, \\ &u = \phi_\infty(R), && x \in \partial B_R, t > 0, \\ &u(x,0) = u_0(x), && x \in B_R, \end{aligned}$$

where B_R is a ball in $\mathbb{R}^N$ with its center at the origin and radius R. Note that φ_∞ is a singular steady state of this problem which is stable from below. It was proved in [14] that for $p > p^*$, if

$$0 \leqslant u_0(x) \leqslant \varphi_\infty\big(|x|\big), \quad x \in B_R,$$

then

$$\log \|u\|_{L^\infty(B_R)} = C\big(t + \mathrm{o}(1)\big) \quad \text{as } t \to \infty$$

with $C = C(N, p)$. Namely, all positive solutions grow up exponentially with the common exponent.

Recently, the growth rate of global unbounded solutions of the problem

$$\begin{aligned} &u_t = \Delta u + u^p, && x \in B_R, t > 0,\\ &u = 0, && x \in \partial B_R, t > 0,\\ &u(x,0) = u_0\big(|x|\big) \geqslant 0, && x \in B_R, \end{aligned}$$

with $p = p_S$, $N \geqslant 3$, has been found by Galaktionov and King in [31]. (See Section 1.1 for the existence of global unbounded (classical) solutions of this problem.) In [31] it is shown that these solutions behave as $t \to \infty$ as follows:

$$\begin{aligned} &\log\big\|u(\cdot,t)\big\|_{L^\infty(B_R)} = \frac{\pi^2}{4} t\big(1 + \mathrm{o}(1)\big) && \text{for } N = 3,\\ &\log\big\|u(\cdot,t)\big\|_{L^\infty(B_R)} = 2\sqrt{t}\big(1 + \mathrm{o}(1)\big) && \text{for } N = 4,\\ &\big\|u(\cdot,t)\big\|_{L^\infty(B_R)} = \gamma_0(N) t^{(N-2)/(2(N-4))}\big(1 + \mathrm{o}(1)\big) && \text{for } N \geqslant 5. \end{aligned}$$

An explicit value of $\gamma_0(N)$ is calculated in a formal way.

10.2. *Grow-up rate for a Cauchy problem*

This subsection is concerned with the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N, t > 0,\\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{10.1}$$

where $p > 1$ and u_0 is a nonnegative continuous function on $\mathbb{R}^N$ that decays to zero as $|x| \to \infty$.

Concerning the existence of positive steady states, it is well known that the Sobolev exponent p_S plays a crucial role. Namely, there is a family of positive radial solutions of

$$\Delta\varphi + \varphi^p = 0 \quad \text{on } \mathbb{R}^N$$

if and only if $p \geqslant p_S$. We denote the solution by $\varphi = \varphi_\alpha(|x|)$, $\alpha > 0$, where $\varphi_\alpha(0) = \alpha$. For each $\alpha > 0$, the solution φ_α is strictly decreasing in $|x|$ and satisfies $\varphi(|x|) \to 0$ as $|x| \to \infty$.

Another important value of the exponent p is p^* defined in (2.4). It is known that for $p_S \leqslant p < p^*$, any positive steady state intersects with other positive steady states, see [109]. For $p \geqslant p^*$, Wang [109] showed that the family of positive steady states $\{\varphi_\alpha; \alpha > 0\}$, is ordered, that is, φ_α is strictly increasing in α for each x and satisfies

$$\lim_{\alpha \to 0} \varphi_\alpha\big(|x|\big) = 0, \qquad \lim_{\alpha \to \infty} \varphi_\alpha\big(|x|\big) = \varphi_\infty\big(|x|\big),$$

where φ_∞ is the singular steady state given by (2.5). Moreover, each positive steady state satisfies

$$\varphi_\alpha\big(|x|\big) = K|x|^{-m} - a|x|^{-m-\lambda_1} + \text{h.o.t.}, \quad |x| \simeq \infty, m = \frac{2}{p-1}, \tag{10.2}$$

where λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N, p) := \frac{N - 2 - 2m - \sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

and $a = a(\alpha, N, p)$ is a positive number that is monotone decreasing in α. We note that for $p > p^*$, λ_1 is the smaller root of the quadratic equation

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0. \tag{10.3}$$

The larger root of this equation is given by

$$\lambda_2 = \lambda_2(N, p) := \frac{N - 2 - 2m + \sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

which will also play an important role below.

Using the ordering property, it was proved by Gui, Ni and Wang [45,46] that $p = p^*$ is a critical exponent where a change in stability properties of the positive steady states occurs. For $p < p^*$ all steady states $u = \varphi_\alpha$ are unstable in “any reasonable sense” (indeed, for each $u_0 \geqslant \varphi_\alpha$, $u_0 \not\equiv \varphi_\alpha$, the solution of (10.1) blows up in finite time), whereas for $p \geqslant p^*$ they are stable under perturbations in some weighted L^∞ spaces.

Recently, building on the results in [45,46], Poláčik and Yanagida [95] obtained global attractivity properties of steady states for $p > p^*$. Let us consider an initial function u_0 given by

$$u_0 = \varphi_\alpha + v_0,$$

where v_0 is a continuous (not necessarily small or radial) perturbation satisfying

$$0 \leqslant \varphi_\alpha + v_0 \leqslant \varphi_\infty \quad \text{on } \mathbb{R}^N \setminus \{0\},$$

and

$$\lim_{|x|\to\infty} |x|^{m+\lambda_1}\big|v_0(x)\big| = 0.$$

Then the solution u of (10.1) exists globally in time and satisfies

$$\big\|u(\cdot,t) - \varphi_\alpha\big\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty.$$

In other words, the asymptotic behavior in space of the initial data u_0 determines the asymptotic behavior in time of the solution.

As an application of the global stability result, among other things, the following result was obtained in [95].

THEOREM 10.1. *Let $p \geqslant p^*$. Suppose that u_0 is any continuous function on $\mathbb{R}^N$ such that*

$$0 \leqslant u_0(x) \leqslant \varphi_\infty\big(|x|\big) \quad \text{on } \mathbb{R}^N \setminus \{0\}, \tag{10.4}$$
$$\lim_{|x|\to\infty} |x|^{m+\lambda_1}\big(\varphi_\infty\big(|x|\big) - u_0(x)\big) = 0.$$

Then the solution of (10.1) *exists globally in time and satisfies*

$$\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \to \infty$$

as $t \to \infty$.

The proof of this theorem roughly goes as follows. By the global stability result and the comparison theorem, if the initial data decay to 0 more slowly than all steady states, then the solution is shown to be unbounded. On the other hand, since the solution is bounded above by the singular steady state, the solution does not blow up in finite time. Therefore, the solution approaches the singular steady state from below as $t \to \infty$. See also [96] for other properties of the global unbounded solutions.

It is shown in [25] that the grow-up rate depends on how close the initial data are to the singular steady state near $|x| = \infty$. The following theorem is the main result of [25].

THEOREM 10.2. *Let $p > p^*$. Suppose that u_0 is a continuous function on $\mathbb{R}^N$ satisfying* (10.4) *and*

$$K|x|^{-m} - C_1|x|^{-l} \leqslant u_0(x) \leqslant K|x|^{-m} - C_2|x|^{-l}, \quad |x| > R,$$

with some constants $l \in (m+\lambda_1, m+\lambda_2]$ and $C_1, C_2, R > 0$. Then there exist positive constants C_3, C_4 and T such that the solution of (10.1) *satisfies*

$$C_3 t^{m(l-m-\lambda_1)/(2\lambda_1)} \leqslant \big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant C_4 t^{m(l-m-\lambda_1)/(2\lambda_1)}$$

for all $t > T$.

We note that the grow-up rate depends on l, and it becomes arbitrarily slow as $l \downarrow m+\lambda_1$. For $l > m+\lambda_2$, the method from [25] does not work. However, we were able to show there that for any initial data satisfying $0 \leqslant u_0(x) \leqslant \varphi_\infty(|x|)$ on $\mathbb{R}^N$, there is a universal upper bound for the grow-up rate.

THEOREM 10.3. *Let $p \geqslant p^*$. Suppose that the initial data satisfy* (10.4). *Then there exist constants $C > 0$ and $T > 0$ such that the solution of* (10.1) *satisfies*

$$\left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{m(N-m-\lambda_1)/(2\lambda_1)}$$

for all $t > T$.

This theorem implies that the grow-up rate obtained in Theorem 10.2 cannot be extended to the case of large l. Mizoguchi has shown recently in [84] that Theorem 10.3 is not optimal, see Section 10.6. It is a natural and interesting question to ask what is happening for $l > m+\lambda_2$.[1]

The rate of convergence to the regular stationary solutions φ_α has been studied in [26]. One of the main results in [26] reads as following theorem.

THEOREM 10.4. *Let $p > p^*$ and $m+\lambda_1 < l < m+\lambda_2$. Suppose that u_0 satisfies*

$$-\varphi_\infty\big(|x|\big) \leqslant u_0(x) \leqslant \varphi_\infty\big(|x|\big), \quad x \in \mathbb{R}^N \setminus \{0\},$$

and

$$\left|u_0(x) - \varphi_\alpha(x)\right| \leqslant c\big(1+|x|\big)^{-l}, \quad x \in \mathbb{R}^N,$$

with some $\alpha \in \mathbb{R}$ and $c > 0$. Then there exists $C > 0$ such that

$$\left\|u(\cdot,t) - \varphi_\alpha(\cdot)\right\|_{L^\infty} \leqslant C(1+t)^{-(l-m-\lambda_1)/2}$$

for all $t > 0$, and this estimate is optimal for $\alpha \neq 0$.

In the following subsections, we carry out a formal asymptotic expansion of grow-up solutions. This formal analysis suggests that the solution behaves in a self-similar way near the origin and infinity respectively, but these two self-similar structures are different. Therefore, we need to match these two asymptotic expansions, and the matching condition leads to the grow-up rate. Based on this formal analysis, we constructed in [25] upper and lower solutions which yield the rigorous proof of Theorem 10.2.

[1] *Note added in proof.* It has been shown recently by M. Fila, J. King, M. Winkler and E. Yanagida that Theorem 10.2 holds for $p \geqslant p^*$ and $l \in (m+\lambda_1, m+\lambda_2+2)$.

10.3. *Inner expansion*

Any radial solution $u = u(r,t)$, $r = |x|$, of (10.1) satisfies

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r} u_r + u^p, & r > 0,\ t > 0, \\ u(r,0) = u_0(r), & r > 0. \end{cases} \tag{10.5}$$

We consider the formal expansion of global unbounded solutions of this equation.

Set

$$u(r,t) = \sigma(t) f(\xi, t),$$

where $\sigma(t) := u(0,t)$ and $\xi = \xi(r,t)$. Substituting

$$u_t = \sigma_t f + \sigma(f_t + \xi_t f_\xi),$$

$$u_r = \sigma f_\xi \xi_r,$$

$$u_{rr} = \sigma f_{\xi\xi} \xi_r^2 + \sigma f_\xi \xi_{rr}$$

in (10.5), we have

$$f_{\xi\xi} + \frac{N-1}{r\xi_r} f_\xi + \frac{\sigma^{p-1}}{\xi_r^2} f^p = \frac{\sigma_t}{\sigma \xi_r^2} f + \frac{1}{\xi_r^2} f_t + \frac{\xi_t}{\xi_r^2} f_\xi - \frac{\xi_{rr}}{\xi_r^2} f_\xi. \tag{10.6}$$

Here we take ξ such that $r\xi_r = \xi$ and $\sigma^{p-1}/\xi_r^2 = 1$, that is,

$$\xi = r\sigma^{(p-1)/2}. \tag{10.7}$$

Note that for each $r > 0$, $\xi \to \infty$ as $t \to \infty$ if $\sigma(t) \to \infty$ as $t \to \infty$. By (10.7), (10.6) is rewritten as

$$f_{\xi\xi} + \frac{N-1}{\xi} f_\xi + f^p = \frac{1}{\sigma^{p-1}} f_t + \frac{\sigma_t}{\sigma^p} \left(f + \frac{p-1}{2} \xi f_\xi \right). \tag{10.8}$$

Assuming that

$$\left| \frac{1}{\sigma^{p-1}} f_t \right| \ll \left| \frac{\sigma_t}{\sigma^p} \right| \ll 1,$$

we may put

$$f = \psi(\xi) + \frac{\sigma_t}{\sigma^p} \Phi(\xi, t), \tag{10.9}$$

where

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi} \psi_\xi + \psi^p = 0, & \xi > 0, \\ \psi(0) = 1, \qquad \psi'(0) = 0. \end{cases} \tag{10.10}$$

We note that this expansion was used first by Galaktionov and King [31]. By (10.2), $\psi(\xi)$ satisfies

$$\psi(\xi) = K\xi^{-m} - a\xi^{-m-\lambda_1} + \text{h.o.t.}, \quad \xi \simeq \infty, \tag{10.11}$$

with some constant $a > 0$.

Substituting (10.9) in (10.8), we have

$$\begin{aligned}
&\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \frac{\sigma_t}{\sigma^p}\left(\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi\right) + \left(\psi + \frac{\sigma_t}{\sigma^p}\Phi\right)^p \\
&\quad = \frac{\sigma_t}{\sigma^p}\left(\psi + \frac{p-1}{2}\xi\psi_\xi\right) + \text{o}\left(\left|\frac{\sigma_t}{\sigma^p}\right|\right), \quad \xi \simeq \infty.
\end{aligned}$$

Here, by (10.10),

$$\begin{aligned}
\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \left(\psi + \frac{\sigma_t}{\sigma^p}\Phi\right)^p &= -\psi^p + \left(\psi + \frac{\sigma_t}{\sigma^p}\Phi\right)^p \\
&= p\psi^{p-1}\frac{\sigma_t}{\sigma^p}\Phi + \text{o}(\Phi), \quad \xi \simeq \infty.
\end{aligned}$$

Hence Φ satisfies

$$\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi + p\psi^{p-1}\Phi + \text{o}(\Phi) = \psi + \frac{p-1}{2}\xi\psi_\xi.$$

Here

$$p\psi^{p-1} = \frac{pK^{p-1}}{\xi^2} + \text{h.o.t.}, \quad \xi \simeq \infty,$$

and

$$\begin{aligned}
\psi + \frac{p-1}{2}\xi\psi_\xi &= -a\xi^{-m-\lambda_1} + \frac{a}{m}(m+\lambda_1)\xi^{-m-\lambda_1} + \text{h.o.t.} \\
&= \frac{a\lambda_1}{m}\xi^{-m-\lambda_1} + \text{h.o.t.}, \quad \xi \simeq \infty.
\end{aligned}$$

This implies that Φ is expanded as

$$\Phi = L\xi^{2-m-\lambda_1} + \text{h.o.t.}, \quad \xi \simeq \infty,$$

where L is a constant determined from

$$\begin{aligned}
&L\big\{(2-m-\lambda_1)(1-m-\lambda_1) + (N-1)(2-m-\lambda_1) + (m+2)(N-2-m)\big\} \\
&\quad = \frac{a\lambda_1}{m}.
\end{aligned}$$

After some computation, we obtain

$$L = \frac{a\lambda_1}{2m(N - 2m - 2\lambda_1)} > 0.$$

Thus, for each $r > 0$, the formal expansion near the origin is written as

$$\begin{aligned} u &= \sigma\left(\psi + \frac{\sigma_t}{\sigma^p}\Phi\right) \\ &= \sigma\left(K\xi^{-m} - a\xi^{-m-\lambda_1} + L\frac{\sigma_t}{\sigma^p}\xi^{2-m-\lambda_1} + \text{h.o.t.}\right), \quad \xi \simeq \infty. \end{aligned}$$

10.4. *Outer expansion*

Next, we consider the formal expansion of global unbounded solutions near $r = \infty$. Setting

$$u = Kr^{-m} - v,$$

we have

$$v_t = v_{rr} + \frac{N-1}{r}v_r + \frac{pK^{p-1}}{r^2}v + \text{h.o.t.}, \quad r \simeq \infty.$$

We will find a solution which behaves in a self-similar way near $r = \infty$

$$v(r,t) = t^{-\beta}V(y), \quad y = t^{-1/2}r.$$

By noting

$$\begin{aligned} v_t &= -\beta t^{-\beta-1}V - \frac{1}{2}t^{-\beta-1}yV_y, \\ \frac{N-1}{r}v_r &= t^{-\beta-1}\frac{N-1}{y}V_y, \\ v_{rr} &= t^{-\beta-1}V_{yy}, \end{aligned}$$

V must satisfy

$$V_{yy} + \left(\frac{N-1}{y} + \frac{y}{2}\right)V_y + \beta V + \frac{pK^{p-1}}{y^2}V = 0.$$

From this results, we have an expansion

$$V(y) = c_1\big(y^{-l} + c_2y^{-l-2} + \text{h.o.t.}\big), \quad y \simeq \infty,$$

where $l = 2\beta$, $c_1 > 0$ is an arbitrary constant and c_2 is computed as

$$c_2 = l^2 - (N-2)l + (m+2)(N-2-m).$$

We note that

$$c_2 \begin{cases} < 0 & \text{if } m+\lambda_1 < l < m+\lambda_2, \\ = 0 & \text{if } l = m+\lambda_2, \\ > 0 & \text{if } l > m+\lambda_2, \end{cases}$$

where $\lambda_2 > 0$ is a larger root of (10.3). In fact, putting $l = m + \lambda$ we have

$$l^2 - (N-2)l + (m+2)(N-2-m) = \lambda^2 - (N-2-m)\lambda + 2(N-2-m).$$

Thus, we obtain an expansion

$$\begin{aligned} u &= Kr^{-m} - c_1\big(y^{-l} + c_2 y^{-l-2} + \text{h.o.t.}\big)t^{-l/2} \\ &= Kr^{-m} - c_1\big(r^{-l} + c_2 r^{-l-2} t\big) + \text{h.o.t.}, \quad r \simeq \infty. \end{aligned}$$

10.5. *Matching of inner and outer expansions*

By the above argument, we have obtained an inner expansion

$$u = \sigma\left(K\xi^{-m} - a\xi^{-m-\lambda_1} + L\frac{\sigma_t}{\sigma^p}\xi^{2-m-\lambda_1} + \text{h.o.t.}\right), \quad \xi = r\sigma^{(p-1)/2} \simeq \infty,$$

and an outer expansion

$$u = Kr^{-m} - c_1\big(y^{-l} + c_2 y^{-l-2} + \text{h.o.t.}\big)t^{-l/2}, \quad y = t^{-1/2} r \simeq \infty.$$

Notice that the leading terms of these expansions are the same. Equating the second- and third-order terms, we have

$$a\sigma\xi^{-m-\lambda_1} = c_1 y^{-l} t^{-l/2} \tag{10.12}$$

and

$$L\frac{\sigma_t}{\sigma^{p-1}}\xi^{2-m-\lambda_1} = -c_1 c_2 y^{-l-2} t^{-l/2}, \tag{10.13}$$

respectively. From (10.12), r is computed as

$$r = A\sigma^{\lambda_1/m(l-m-\lambda_1)}, \quad A := \left(\frac{c_1}{a}\right)^{1/(l-m-\lambda_1)}.$$

Substituting this term in (10.13), we have

$$L\sigma_t\sigma^{-1+4\lambda_1/m(l-m-\lambda_1)} = -c_1c_2A^{m+\lambda_1-l-4}t.$$

Integrating this equation, we see that the formal asymptotic expansions suggest a grow-up rate as in Theorem 10.2.

10.6. *An optimal upper bound*

In [84], Mizoguchi has shown the following theorem.

THEOREM 10.5. *Let $p > p^*$. Then for each nonnegative even integer n there exists a global solution u_n of* (10.1) *with n intersections with φ_∞ such that*

$$t^{-a_n}\|u_n(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 1$$

as $t \to \infty$, where

$$a_n = \bigl(N + 2(n - m - \lambda_1)\bigr)\frac{m}{2\lambda_1}.$$

The idea of the proof is to rescale the problem using the forward self-similar change of variables and then linearize around φ_∞ (which is invariant under this rescaling). The exponents a_n correspond to the eigenvalues of the linearization in an appropriate weighted space.

From this theorem, the following sharp upper bound follows easily, see [84] for details.

COROLLARY 10.6. *Let $p > p^*$. Then, for any global solution u of* (10.1) *with initial data u_0 satisfying* (10.4), *there are $C, T > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leqslant t^{a_0}$$

for $t > T$.

11. Oscillating grow-up solutions and grow-up set

In [95], Poláčik and Yanagida constructed a global positive solution of (3.1) with $p \geqslant p^*$ such that

$$\liminf_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = 0, \qquad \limsup_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

In [96], they found a solution of (3.1) with $p \geqslant p^*$ which first develops a peak at some position then the peak disappears and appears again later at a different position. This process is repeated with the positions and heights of the peaks prescribed arbitrarily. In particular, the heights can become unbounded as $t \to \infty$. More precisely, one has the following theorem.

THEOREM 11.1 [96]. *Assume $p \geqslant p^*$. For any infinite sequence $\{(\alpha_i, \xi_i, \varepsilon_i)\}$ with $\alpha_i \in \mathbb{R} \setminus \{0\}$, $\xi \in \mathbb{R}^N$ and $\varepsilon_i > 0$ there exists $u_0 \in C_0(\mathbb{R}^N)$ such that the solution u of* (3.1) *satisfies the following*:

(i) *$u(x,t)$ exists globally in time and decays to 0 as $|x| \to \infty$ for each t;*

(ii) *there exists an increasing sequence of positive numbers $\{s_i\}$ such that $\|u(\cdot, s_i)\|_{L^\infty(\mathbb{R}^N)} < \varepsilon_i$;*

(iii) *there exists an increasing sequence of positive numbers $\{t_i\}$, $t_i \in (s_i, s_{i+1})$ such that*

$$\left\| u(\cdot, t_i) - \varphi_{\alpha_i}(\cdot - \xi_i) \right\|_{L^\infty(\mathbb{R}^N)} < \varepsilon_i;$$

(iv) *if $\alpha_i > 0$ for all $i = 1, 2, \ldots,$ then $u(x,t) > 0$ for all $x \in \mathbb{R}^N$ and $t > 0$;*

(v) *if the sequence $\{\xi_i\}$ is bounded then $u(x,t) \to 0$ as $|x| \to \infty$ uniformly with respect to $t \geqslant 0$.*

For global unbounded solutions it is natural to ask on which set they become unbounded. We say that $\xi \in \mathbb{R}^N$ is a *grow-up point* if there is a sequence $\{t_i\}$, $t_i \to \infty$, such that $|u(\xi, t_i)| \to \infty$ as $i \to \infty$. The set of all grow-up points is called the *grow-up set* of u. The following result says that the grow-up set can be prescribed arbitrarily.

THEOREM 11.2 [96]. *Assume $p \geqslant p^*$. Given any closed subsets G^+ and G^- of $\mathbb{R}^N$. There exists u_0, with $u_0 \geqslant 0$ if $G^- = \emptyset$, such that the solution of* (3.1) *is global and satisfies*

$$\limsup_{t\to\infty} u(x,t) = \begin{cases} +\infty, & x \in G^+, \\ K\big(\operatorname{dist}(x, G^+)\big)^{-2/(p-1)}, & x \notin G^+, \end{cases}$$

$$\liminf_{t\to\infty} u(x,t) = \begin{cases} -\infty, & x \in G^-, \\ -K\big(\operatorname{dist}(x, G^-)\big)^{-2/(p-1)}, & x \notin G^-, \end{cases}$$

where K is the constant from (2.5).

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CHAPTER 3

The Boltzmann Equation and Its Hydrodynamic Limits

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1. Introduction

The classical models of fluid dynamics, such as the Euler or Navier–Stokes equations, were first established by applying Newton's second law of motion to each infinitesimal volume element of the fluid considered, see, for instance, Chapter 1 of [74]. While this method has the advantage of being universal – indeed, all hydrodynamic models can be obtained in this way – it has one major drawback: equations of state and transport coefficients (such as the viscosity or heat conductivity) are given as phenomenological or experimental data, and are not related to microscopic data (essentially, to the laws governing molecular interactions). As a matter of fact, a microscopic theory of liquids is most likely too complex to be of any use in deriving the macroscopic models of fluid mechanics. In the case of gases or plasmas, however, molecular interactions are on principle much more elementary, so that one can hope to express thermodynamic functions and transport coefficients in terms of purely mechanical data concerning collisions between gas molecules.

In fact, the subject of hydrodynamic limits goes back to the work of the founders J. Clerk Maxwell and L. Boltzmann, of the kinetic theory of gases. Both checked the consistency of their new – and, at the time, controversial – theory with the well-established laws of fluid mechanics. Interestingly, while the very existence of atoms was subject to heated debates, kinetic theory would provide estimates on the size of a gas molecule from macroscopic data such as the viscosity of the gas.

Much later, D. Hilbert formulated the question of hydrodynamic limits as a mathematical problem, as an example in his 6th problem on the axiomatization of physics [67]. In Hilbert's own words "[...] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua". Some years later, Hilbert himself attacked the problem in [68], as an application of his own fundamental work on integral equations.

There is an ambiguity in Hilbert's formulation. Indeed, what is meant by "the atomistic view" could designate two very different theories. One is molecular dynamics (i.e., the N-body problem of classical mechanics with elastic collisions, assuming for simplicity all bodies to be spherical and of equal mass). The other possibility is to start from the kinetic theory of gases, and more precisely from the Boltzmann equation, which is what Hilbert himself did in [68]. However, one should be aware that the Boltzmann equation is not itself a "first principle" of physics, but a low density limit of molecular dynamics. In the days of Maxwell and Boltzmann, and maybe even at the time of Hilbert's own papers on the subject, this may not have been so clear to everyone. In particular, much of the controversy on irreversibility could perhaps have been avoided with a clear understanding of the relations between molecular gas dynamics and the kinetic theory of gases.

In any case, the problem of hydrodynamic limits is to obtain rigorous derivations of macroscopic models such as the fundamental partial differential equations (PDEs) of fluid mechanics from a microscopic description of matter, be it molecular dynamics or the kinetic theory of gases. The situation can be illustrated by the following diagram.

$$\begin{tikzcd} \boxed{\text{MOLECULAR DYNAMICS}} \arrow[r] \arrow[dr] & \boxed{\text{KINETIC THEORY}} \arrow[d] \\ & \boxed{\text{HYDRODYNAMICS}} \end{tikzcd}$$

Throughout the present chapter, we are concerned with only the vertical arrow in the diagram above. As a matter of fact, this is perhaps the part of the subject that is best understood so far, at least according to the mathematical standards of rigor.

The other arrows in this diagram correspond with situations that are only partially understood, and where certain issues are still clouded with mystery. Before starting our discussion of the hydrodynamic limits of the kinetic theory of gases, let us say a few words on these other limits and direct the interested reader at the related literature.

Although beyond the scope of this chapter, the horizontal arrow is of considerable interest to our discussion, being a justification of the kinetic theory of gases on the basis of the molecular gas dynamics (viewed as a first principle of classical, nonrelativistic physics). A rigorous derivation of the Boltzmann equation from molecular dynamics on short time intervals was obtained by Lanford [76]; see also the very nice rendition of Lanford's work in the book [28]. Hence, although not a first principle itself, the Boltzmann equation is rigorously derived from first principles and therefore has more physical legitimacy than phenomenological models (such as lattice gases or stochastic Hamiltonian models). Besides, the Boltzmann equation is currently used by engineers in aerospace industry, in vacuum technology, in nuclear engineering, as well as several other applied fields, a more complete list of those being available in the Proceedings of the Rarefied Gas Dynamics Symposia.

On the other hand, "formal" derivations of the Euler system for compressible fluids from molecular dynamics were proposed by Morrey [97]. Later on, S.R.S. Varadhan and his collaborators studied the same limit, however with a different method. Instead of taking molecular dynamics as their starting point, they modified slightly the N-body Hamiltonian by adding an arbitrarily small noise term to the kinetic energy; they also cut off high velocities at a threshold compatible with the maximum speed observed on the macroscopic system. Starting from this stochastic variant of molecular gas dynamics, they derived the Euler system of compressible fluids for short times (before the onset of singularities such as shock waves); see for instance [119] and the references therein, notably [103], see also [33], and the more recent reference [44]. The role of the extra noise term in their derivation is to guarantee some form of the ergodic principle, i.e., that the only invariant measure for the Hamiltonian in the limit of infinitely many particles is a local Gibbs state (parametrized by macroscopic quantities). At the time of this writing, deriving the Euler system of compressible fluids from molecular gas dynamics without additional noise terms as in [103] and for all positive times seems beyond reach.

For these reasons, we have limited our discussion to only the derivation of hydrodynamic models from the kinetic theory of gases, i.e., from the Boltzmann equation. For a more general view of the subject of hydrodynamic limits, the reader is advised to read the excellent survey article by Esposito and Pulvirenti [41], whose selection of topics is quite different from ours.

This chapter is organized as follows: in Section 2 we review the classical models of fluid mechanics. Section 3 introduces the Boltzmann equation and discusses its structure and main formal properties. In Section 4 we discuss the dimensionless form of the Boltzmann equation and introduce its main scaling parameters. Sections 5 and 6 explain in detail the formal derivation of the most classical PDEs of fluid mechanics from the Boltzmann equation by several different methods. Section 7 recalls the known mathematical results on the Cauchy problem for the PDEs of fluid mechanics. In Section 8 we review the state of the art on the existence theory for the Boltzmann equation. Sections 9–11 sketch the mathematical proofs of the formal derivations described in Sections 6 and 7; here again, we present three different methods for establishing these hydrodynamic limits and discuss their respective merits.

We have chosen to emphasize compactness methods, leading to global results, and especially the derivation of global weak solutions of the incompressible Navier–Stokes equations from renormalized solutions of the Boltzmann equation. There is more than a simple matter of taste in this choice. Indeed, it is a nontrivial question to decide whether these hydrodynamic limits are intrinsic properties of the microscopic versus macroscopic models governing the dynamics of gases, or simply an illustration of more or less standard techniques in asymptotic analysis. The second viewpoint leads to derivations of hydrodynamic models that fall short of describing any singular behavior beyond isolated shock waves in compressible gas dynamics. The first viewpoint uses the specific structure of the Boltzmann equation to design convergence proofs that are based on only the a priori estimates on this equation that have an intrinsic physical meaning; these convergence proofs are insensitive to whether singularities appear in finite time on the limiting hydrodynamic model.

2. Fluid dynamics: A presentation of models

Usually, one thinks of a fluid – more generally, a continuous medium – as a set of material points which, at any given time t, fill a smooth domain in the Euclidean space $\mathbb{R}^N$, where $N = 1, 2, 3$ are the dimensions of physical interest.

The purpose of fluid dynamics is to describe the state of the fluid at any instant of time with a *small* number of fields – such as the velocity or temperature fields – defined on the domain filled by the fluid.

These fields are governed by several partial differential equations that share a common structure which we briefly recall below.

Consider the motion of a continuous medium, and denote by

$$X(t, s; a) \in \mathbb{R}^N$$

the position at time t of the material point which occupied position a at time s.

The kinematics of such a medium is based on the *parallel transport* along the family of curves $t \mapsto X(t,s;a)$ indexed by a (s, being the origin of times, is kept fixed). The infinitesimal description of this parallel transport involves the first-order differential operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + u(t,x)\cdot\nabla_x = \partial_t + \sum_{j=1}^{N} u_j(t,x)\,\partial_{x_j},$$

where the velocity field $u(t,x)$ is defined in terms of the particle paths $X(t,s;a)$ by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t,s;a) = u\big(t, X(t,s;a)\big).$$

The operator $\frac{\mathrm{D}}{\mathrm{D}t}$ is usually called *the material derivative,* and using it allows one to eliminate the trajectories $X(t,s;a)$. In other words, instead of following the motion of each material point, one looks at any fixed point in the Euclidean space $\mathbb{R}^N$, say x, and observes, at any given time t, the velocity $u(t,x)$ of the material point that is located at the position x at time t. This is called the *Eulerian description* of a continuous medium, whereas the description in terms of $X(t,s;a)$ is called the *Lagrangian description*. Interestingly, the connections between the kinetic theory of gases (or plasmas) and fluid dynamics are always formulated in terms of the Eulerian, instead of the Lagrangian description, although the latter may seem more natural when dealing with the motion of a gas at the atomic or molecular level.

Fluid dynamics rests on three fundamental laws – or equations:

- the continuity equation,
- the motion equation, and
- the energy balance equation.

The continuity equation states that the density ρ of the fluid is transported by the flow, i.e., that the measure $\rho(t,x)\,\mathrm{d}x$ is the image of the measure $\rho(s,a)\,\mathrm{d}a$ under the map $a \mapsto X(t,s;a)$. The infinitesimal formulation of this fact is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\,\mathrm{div}_x\,u. \tag{2.1}$$

The motion equation states that each portion of the fluid obeys Newton's second law of motion (i.e., $\frac{\mathrm{d}}{\mathrm{d}t}$(momentum) = force). The acceleration is computed in terms of the material derivative, and the infinitesimal formulation of the motion equation is

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{div}_x\,S + \rho\mathbf{f}, \tag{2.2}$$

where $\mathbf{f}$ is the external force field (e.g., gravity, Lorentz force in the case of a plasma ...) and S is the stress tensor. The meaning of S is as follows: at any given time t, isolate a smooth domain Ω in the fluid, denote by $\partial\Omega$ its boundary, by n_x the unit normal field on $\partial\Omega$ pointing toward the outside of Ω, and by $\mathrm{d}\sigma(x)$ the surface element on $\partial\Omega$. Then, the force exerted by the fluid outside Ω on the fluid inside Ω is

$$\int_{\partial\Omega} S(t,x)n_x\,\mathrm{d}\sigma(x).$$

Finally, the energy balance equation involves the internal energy of the fluid per unit of mass E; the total energy per unit of mass is $\frac{1}{2}|u|^2+E$ (the sum of the kinetic energy and the internal energy). It states that the material derivative of the total energy of any portion of fluid is the sum of the works of the stresses and of the external force $\mathbf{f}$, minus the heat flux lost by that portion of fluid. Its infinitesimal formulation is

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}|u|^2+E\right)=-\mathrm{div}_x\,Q+\mathrm{div}_x(Su)+\rho\mathbf{f}\cdot u, \tag{2.3}$$

where Q is the heat flux.

In the motion and energy balance equations, $\mathbf{f}$ is a given vector field, while the density ρ, the velocity field u, the internal energy E, the stress tensor S and the heat flux Q are unknown. However, these quantities are usually not independent, but are related by *equations of state* that depend on the fluid considered.

Equations (2.1)–(2.3) are *Galilean invariant*. Specifically, let $v\in\mathbb{R}^3$; define the Galilean transformation

$$x'=x+vt,\qquad u'(t,x')=u(t,x)+v,\qquad \phi'(t,x')=\phi(t,x)$$

for $\phi=\rho,S,\mathbf{f},E,Q$. Then, setting

$$\frac{\mathrm{D}'}{\mathrm{D}t}=\partial_t+u'\cdot\nabla_{x'}$$

one deduces from (2.1)–(2.3) that

$$\begin{aligned}
&\frac{\mathrm{D}'}{\mathrm{D}t}\rho'=-\rho'\,\mathrm{div}_{x'}\,u',\\
&\rho'\frac{\mathrm{D}'}{\mathrm{D}t}u'=\mathrm{div}_{x'}\,S'+\rho'\mathbf{f}',\\
&\rho'\frac{\mathrm{D}'}{\mathrm{D}t}\left(\frac{1}{2}|u'|^2+E'\right)=-\mathrm{div}_{x'}\,Q'+\mathrm{div}_{x'}(S'u')+\rho'\mathbf{f}'\cdot u'.
\end{aligned}$$

2.1. *The compressible Euler system*

An *ideal fluid* is one where the effects of viscosity and thermal conductivity can be neglected. In this case, $Q=0$ and the stress tensor is of the form

$$S=-pI,$$

where the scalar p is the pressure. Hence the system consisting of the continuity equation the motion equation and the energy balance equation becomes

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\
&\rho\big(\partial_t u + (u \cdot \nabla_x) u\big) = -\nabla_x p + \rho \mathbf{f}, \\
&\rho\big(\partial_t E + (u \cdot \nabla_x) E\big) = -p \operatorname{div}_x u.
\end{aligned} \tag{2.4}$$

Thus, the unknowns are the density ρ, the velocity field u, the pressure p and the internal energy E. However, the quantities ρ, p and E are not independent, but are related by *equations of state*.

Choosing the density ρ and the temperature θ as independent thermodynamic variables, these equations of state are relations that express the pressure p and the internal energy E in terms of ρ and θ

$$p \equiv p(\rho, \theta), \qquad E \equiv E(\rho, \theta). \tag{2.5}$$

Hence (2.4) is a system of $N + 2$ partial differential equations for the unknowns ρ, u and θ; notice that there are in fact $N + 2$ scalar unknowns, ρ and θ, plus the N components of the vector field u.

The case of a perfect gas is of particular importance for the rest of this chapter. In this case, the equations of state are

$$p(\rho, \theta) = k\rho\theta, \qquad e(\rho, \theta) = \frac{k\theta}{\gamma - 1}, \tag{2.6}$$

where k is the Boltzmann constant ($k = 1.38 \cdot 10^{-23}$ J K^{-1}) and $\gamma > 1$ is a constant called the adiabatic exponent. For a perfect gas whose molecules have n degrees of freedom

$$\gamma = 1 + \frac{2}{n}.$$

For instance, in the case of a perfect monatomic gas, each molecule has 3 degrees of freedom (the coordinates of its center of mass); hence $\gamma = 5/3$. In the case of a diatomic gas, each molecule has 5 degrees of freedom (the coordinates of its center of mass and the direction of the line passing through the centers of both atoms); hence $\gamma = 7/5$.

From now on, we choose a temperature scale such that $k = 1$.

Adding the continuity equation to the motion equation and to the energy balance equation, one can recast (2.4) in the form

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\
&\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = \rho \mathbf{f}, \\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{1}{\gamma - 1}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{\gamma}{\gamma - 1}\theta\right)\right) = \rho \mathbf{f} \cdot u.
\end{aligned} \tag{2.7}$$

In the absence of external force, i.e., when $\mathbf{f} = 0$, (2.7) is a hyperbolic system of conservation laws.

2.2. *The compressible Navier–Stokes system*

If the fluid considered is not ideal, the viscous forces and heat conduction must be taken into account.

In the case of moderate temperature gradients in the fluid, heat conduction is usually modeled with Fourier's law: the heat flux Q is proportional to the temperature gradient, i.e.,

$$Q = -\kappa \nabla_x \theta,$$

where the coefficient κ is called the *heat conductivity*. Usually, κ is a function of the pressure and the temperature. Because of the equation of state for the pressure, one has equivalently $\kappa \equiv \kappa(\rho, \theta) > 0$.

The viscous forces are modeled by adding a correction term to the pressure in the stress tensor S. In the case where the gradient of the velocity field is not too large, this correction term is linear in the gradient of the velocity field – by analogy with Fourier's law. Usually, the fluid under consideration is isotropic, and this implies that this correcting term is a linear combination of the scalar tensor $(\mathrm{div}_x\, u)I$ and of the traceless part of the symmetrized gradient of the velocity field

$$D(u) = \nabla_x u + \nabla_x u^{\mathrm{T}} - \frac{2}{N}(\mathrm{div}_x\, u)I.$$

In other words, the stress tensor takes the form

$$S = -pI + \mu(\mathrm{div}_x\, u)I + \lambda D(u),$$

where λ and μ are two positive scalar quantities referred to as the viscosity coefficients. Again, λ and μ are functions of the pressure and temperature, which, by the equation of state for the pressure, can be transformed into $\lambda \equiv \lambda(\rho, \theta)$ and $\mu \equiv \mu(\rho, \theta)$.

Inserting this form of the stress tensor in the motion and energy balance equation, one finds the system of Navier–Stokes equations for compressible fluids

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}_x(\rho u) = 0,\\
&\rho\big(\partial_t u + (u\cdot\nabla_x)u\big) = -\nabla_x p(\rho,\theta)\\
&\qquad\qquad + \rho\mathbf{f} + \mathrm{div}_x\big(\lambda(\rho,\theta)D(u)\big) + \nabla_x\big(\mu(\rho,\theta)\,\mathrm{div}_x(u)\big), \qquad (2.8)\\
&\rho\big(\partial_t E(\rho,\theta) + (u\cdot\nabla_x)E(\rho,\theta)\big) = -p(\rho,\theta)\,\mathrm{div}_x\, u + \mathrm{div}_x\big(\kappa(\theta)\nabla_x\theta\big)\\
&\qquad\qquad + \frac{1}{2}\lambda(\rho,\theta)D(u) : D(u) + \mu(\rho,\theta)(\mathrm{div}_x\, u)^2.
\end{aligned}$$

This is a degenerate parabolic system of partial differential equations in the unknowns ρ, u and θ. Observe that there is no diffusion term in the first equation, which is clear on physical grounds. Indeed, the meaning of the continuity equation is purely geometric – namely, the fact that the measure $\rho\,\mathrm{d}x$ is transported by the fluid flow – and cannot be affected by physical assumptions on the fluid (such as whether the fluid is ideal or not).

2.3. *The acoustic system*

The acoustic waves in an ideal fluid are small amplitude disturbances of a constant equilibrium state. Therefore the propagation of acoustic waves is governed by the linearization at a constant state $(\bar\rho, \bar u, \bar\theta)$ of the compressible Euler system. Without loss of generality, one can assume by Galilean invariance that $\bar u = 0$. The density, velocity and temperature fields are written as

$$\rho = \bar\rho + \tilde\rho, \qquad u = \tilde u, \qquad \theta = \bar\theta + \tilde\theta,$$

where the letters adorned with tildes designate small disturbances of the background equilibrium state $(\bar\rho, 0, \bar\theta)$. In the case of a perfect gas, and in the absence of external force (i.e., for $\mathbf{f} = 0$), the acoustic system takes the form

$$\begin{aligned}
&\partial_t \tilde\rho + \bar\rho\,\mathrm{div}_x\,\tilde u = 0,\\
&\partial_t \tilde u + \frac{\bar\theta}{\bar\rho}\nabla_x \tilde\rho + \nabla_x \tilde\theta = 0,\\
&\frac{1}{\gamma - 1}\,\partial_t \tilde\theta + \bar\theta\,\mathrm{div}_x\,\tilde u = 0.
\end{aligned} \tag{2.9}$$

By combining the first and the last equation in the system above, one can put it in the form

$$\begin{aligned}
&\partial_t \left(\frac{\tilde\rho}{\bar\rho} + \frac{\tilde\theta}{\bar\theta}\right) + \gamma\,\mathrm{div}_x\,\tilde u = 0,\\
&\partial_t \tilde u + \bar\theta\nabla_x \left(\frac{\tilde\rho}{\bar\rho} + \frac{\tilde\theta}{\bar\theta}\right) = 0.
\end{aligned} \tag{2.10}$$

Splitting the fluctuation of velocity field $\tilde u$ as the sum of a gradient field and of a *solenoidal* (i.e., divergence-free) field

$$\tilde u = -\nabla_x \varphi + \tilde u_{\mathrm{s}}, \qquad \mathrm{div}_x\, u_{\mathrm{s}} = 0,$$

one deduces from the system (2.10) – together with boundary conditions, or conditions at infinity, or else conditions on the mean value of the fields, whose detailed description does not belong here – that

$$
\begin{aligned}
&\big(\partial_{tt} - \gamma\bar{\theta}\Delta_x\big)\left(\frac{\tilde{\rho}}{\bar{\rho}} + \frac{\tilde{\theta}}{\bar{\theta}}\right) = 0,\\
&\big(\partial_{tt} - \gamma\bar{\theta}\Delta_x\big)\varphi = 0,\\
&\partial_t u_{\mathrm{s}} = 0.
\end{aligned}
\tag{2.11}
$$

In other words, the acoustic system can be reduced to two independent wave equations for $(\tilde{\rho}/\bar{\rho} + \tilde{\theta}/\bar{\theta})$ (the relative pressure fluctuation) and φ (the fluctuating stream function), while the solenoidal part of the velocity fluctuation u_{s} is a constant of motion.

2.4. *The incompressible Euler equations*

Consider next the case of an incompressible, homogeneous ideal fluid. The evolution of such a fluid is governed by the system (2.4) with $\rho = \text{const}$. The continuity and motion equations in (2.4) reduce to

$$
\begin{aligned}
&\mathrm{div}_x\, u = 0,\\
&\partial_t u + (u\cdot\nabla_x)u = -\nabla_x\pi + \mathbf{f},
\end{aligned}
\tag{2.12}
$$

where $\pi = p/\rho$. At variance with the compressible Euler system, there is no need of an equation of state to determine π. Indeed, taking the divergence of both sides of the motion equation leads to

$$
-\Delta_x\pi = \mathrm{div}_x(u\cdot\nabla_x u) - \mathrm{div}_x\,\mathbf{f} = \mathrm{trace}\big((\nabla_x u)^2\big) - \mathrm{div}_x\,\mathbf{f},
$$

so that π can be expressed in terms of u by solving the Laplace equation. In other words, π must be thought of as the Lagrange multiplier associated to the constraint $\mathrm{div}_x\, u = 0$.

The incompressible Euler equations also arise in a different context, namely in the description of *incompressible flows* of compressible fluids (such as perfect gases, for instance).

The dimensionless number that monitors the compressibility is the *Mach number*, i.e., the ratio of the length of the velocity field to the speed of sound. In the case of a perfect gas with adiabatic exponent γ, our discussion of the acoustic system above shows that the speed of sound in the gas at a temperature θ is $c = \sqrt{\gamma\theta}$, so that the Mach number in that case is

$$
Ma = \frac{|u|}{\sqrt{\gamma\theta}}.
\tag{2.13}
$$

With this definition, the Mach number is a local quantity, since u and θ are in general functions of x and t. But one can replace $|u|$ and θ in the definition above by constant quantities of the same order of magnitude, for instance by averages of $|u|$ and θ over large spatial and temporal domains.

Flows of perfect gases are incompressible in the small Mach number limit. Setting $\varepsilon = \sqrt{Ma} \ll 1$, consider the rescaled density, velocity and temperature fields defined by

$$\begin{aligned}
\rho_\varepsilon(t,x) &= \rho\left(\frac{t}{\varepsilon}, x\right),\\
u_\varepsilon(t,x) &= \frac{1}{\varepsilon} u\left(\frac{t}{\varepsilon}, x\right),\\
\theta_\varepsilon(t,x) &= \theta\left(\frac{t}{\varepsilon}, x\right),
\end{aligned} \tag{2.14}$$

assuming (ρ, u, θ) is a solution of the compressible Euler system (2.7), with $\mathbf{f} \equiv 0$ for simplicity. Hence $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ satisfies

$$\begin{aligned}
&\partial_t \rho_\varepsilon + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon) = 0,\\
&\rho_\varepsilon\big(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon\big) + \frac{1}{\varepsilon^2} \nabla_x(\rho_\varepsilon \theta_\varepsilon) = 0,\\
&\partial_t \theta_\varepsilon + u_\varepsilon \cdot \nabla_x \theta_\varepsilon + (\gamma - 1)\theta_\varepsilon \operatorname{div}_x u_\varepsilon = 0.
\end{aligned} \tag{2.15}$$

The leading-order term in the momentum equation is the gradient of the pressure field, which suggests that, in the limit as $\varepsilon \to 0$, $\rho_\varepsilon \theta_\varepsilon \simeq C(t)$; then, combining the continuity and temperature equations above leads to

$$\gamma \operatorname{div}_x u_\varepsilon = -\partial_t \ln(\rho_\varepsilon \theta_\varepsilon) - u_\varepsilon \cdot \nabla_x \ln(\rho_\varepsilon \theta_\varepsilon) \simeq \frac{\mathrm{d}}{\mathrm{d}t}\big(\ln C(t)\big).$$

In many situations – for instance, if the spatial domain is a periodic box, or in the case of a bounded domain Ω with the usual boundary condition $u_\varepsilon \cdot n_x = 0$ on $\partial\Omega$ – integrating in x both sides of this equality leads to the incompressibility condition

$$\operatorname{div}_x u_\varepsilon \simeq 0 \quad \text{in the limit as } \varepsilon \to 0.$$

Hence the continuity equation reduces to

$$\partial_t \rho_\varepsilon + u_\varepsilon \cdot \nabla_x \rho_\varepsilon \simeq 0$$

so that, if the initial data for ρ_ε is a constant $\bar{\rho}$, then

$$\rho_\varepsilon(t,x) \simeq \bar{\rho} \quad \text{in the limit as } \varepsilon \to 0.$$

Then, the momentum equation reduces to

$$\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon \simeq -\frac{1}{\varepsilon^2} \nabla_x \theta_\varepsilon = \text{gradient field}$$

in the limit as $\varepsilon \to 0$. This discussion suggests that, in the small Mach number limit, flows of a compressible fluid such as a perfect gas are well described by the incompressible Euler equations.

2.5. *The incompressible Navier–Stokes equations*

Next, we start from the compressible Navier–Stokes system (2.8), and assume that the density ρ is a constant. As above, the continuity equation in (2.8) reduces to the incompressibility condition $\operatorname{div}_x u = 0$. Moreover, assuming that the viscosity λ is a constant, we find that the momentum equation reduces to

$$\rho\big(\partial_t u + (u\cdot\nabla_x)u\big) + \nabla_x p = \rho\mathbf{f} + \lambda\Delta_x u.$$

Defining the kinematic viscosity to be

$$\nu = \frac{\lambda}{\rho}$$

and setting $\pi = p/\rho$, we arrive at the incompressible Navier–Stokes equations

$$\begin{aligned} &\operatorname{div}_x u = 0,\\ &\partial_t u + (u\cdot\nabla_x)u + \nabla_x\pi = \mathbf{f} + \nu\Delta_x u. \end{aligned} \tag{2.16}$$

We leave it to the reader to verify that the incompressible Navier–Stokes equations can be viewed as the small Mach number limit of the compressible Navier–Stokes system, as was done in the case of the incompressible Euler system. The scaling law is slightly different from the Euler case: for $\varepsilon = \sqrt{Ma}$, set

$$\begin{aligned} \rho_\varepsilon(t,x) &= \rho\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right),\\ u_\varepsilon(t,x) &= \frac{1}{\varepsilon}u\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right),\\ \theta_\varepsilon(t,x) &= \theta\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right), \end{aligned} \tag{2.17}$$

where (ρ,u,θ) is a solution to the Navier–Stokes system (2.8), with $\mathbf{f}\equiv 0$. Then, to leading order as $\varepsilon \to 0$, u_ε satisfies (2.16) with $\mathbf{f}\equiv 0$.

So far, we have said nothing about the temperature field in incompressible flows; this will be the subject matter of the next subsection.

2.6. *The temperature equation for incompressible flows*

Going back to the Navier–Stokes system (2.8) for a perfect gas with adiabatic exponent γ, we see that, in the incompressible case where $\rho = \text{const}$, the third equation reduces to

$$\frac{1}{\gamma-1}\rho(\partial_t\theta + u\cdot\nabla_x\theta) = \operatorname{div}_x\big(\kappa(\theta)\nabla_x\theta\big) + \frac{1}{2}\lambda D(u) : D(u). \tag{2.18}$$

On the right-hand side of (2.18), the first term represents the divergence of the heat flux due to thermal conduction, as described by Fourier's law, while the second term represents the production of heat by intermolecular friction and is called the *viscous heating* term.

In some models that can be found in the literature, the viscous heating term is absent from the temperature equation. Whether the viscous heating term should be taken into account or not depends in fact on the relative size of the fluctuations of velocity field about its average value, and of the fluctuations of temperature field about its average values.

If the fluctuations of velocity field are of a smaller order than the square-root of the temperature fluctuations, then a straightforward scaling argument shows that the viscous heating term can indeed be neglected in (2.18). If however, the fluctuations of velocity field are at least of the same order of magnitude as the square-root of the temperature fluctuations, then the viscous heating term cannot be neglected in (2.18). We shall discuss this alternative further in the description of the incompressible hydrodynamic limits of the Boltzmann equation.

2.7. *Coupling of the velocity and temperature fields by conservative forces*

In our discussion of the incompressible flows as low Mach number limits, we have neglected so far the external force $\mathbf{f}$. Split it as the sum of a gradient field (i.e., of a conservative force) and of a solenoidal field

$$\mathbf{f} = -\nabla_x\phi + \mathbf{f}^{\mathrm{s}}, \qquad \operatorname{div}_x \mathbf{f}^{\mathrm{s}} = 0.$$

Scale ϕ and $\mathbf{f}^{\mathrm{s}}$ as

$$\phi_\varepsilon(t,x) = \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \qquad \mathbf{f}^{\mathrm{s}}_\varepsilon(t,x) = \frac{1}{\varepsilon^3}\mathbf{f}^{\mathrm{s}}\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \tag{2.19}$$

and assume that ρ_ε and θ_ε have fluctuations of order ε about their constant average values

$$\rho_\varepsilon = \bar\rho + \varepsilon\tilde\rho_\varepsilon, \qquad \theta_\varepsilon = \bar\theta + \varepsilon\tilde\theta_\varepsilon.$$

In that case, the leading order in ε of the momentum equation in the Navier–Stokes system (2.8) reduces to

$$\nabla_x\big(\bar\rho\tilde\theta_\varepsilon + \bar\theta\tilde\rho_\varepsilon\big) + \bar\rho\nabla_x\phi_\varepsilon \simeq 0,$$

or in other words,

$$\frac{\tilde{\rho}_\varepsilon}{\bar{\rho}} + \frac{\tilde{\theta}_\varepsilon}{\bar{\theta}} + \frac{\phi_\varepsilon}{\bar{\theta}} \simeq C(t), \tag{2.20}$$

an equality known as *Boussinesq's relation*.[1] In many cases, the boundary conditions (or decay at infinity, or else periodicity conditions) entail that $C(t) = 0$.

The next order in ε of the momentum equation in the Navier–Stokes system is

$$\bar{\rho}\big(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon\big) + \tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon \simeq \lambda \Delta_x u_\varepsilon + \bar{\rho}\mathbf{f}_\varepsilon + \text{gradient field}$$

and one expresses the action of the conservative force $\tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon$ as

$$\tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon \simeq -\frac{\bar{\rho}}{\bar{\theta}} \left(\tilde{\theta}_\varepsilon \nabla_x \phi_\varepsilon + \frac{1}{2} \nabla_x \big(\phi_\varepsilon^2\big) \right)$$

so that the momentum equation reduces to

$$\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon - \frac{\theta_\varepsilon}{\bar{\theta}} \nabla_x \phi_\varepsilon \simeq \nu \Delta_x u_\varepsilon + \mathbf{f}_\varepsilon + \text{gradient field}. \tag{2.21}$$

As for the temperature equation, one should refrain from using directly (2.18). Indeed, this equation has been derived from (2.8) in the purely incompressible case where $\rho = \text{const}$, while in the present case $\rho = \text{const}$ modulo terms of order ε.

In the present case, we must go back to the Navier–Stokes system (2.8) and write the continuity and energy equation in terms of the fluctuations of density and temperature

$$\begin{aligned}
&\varepsilon(\partial_t \tilde{\rho}_\varepsilon + u_\varepsilon \cdot \nabla_x \tilde{\rho}_\varepsilon) + \bar{\rho}\, \mathrm{div}_x\, u_\varepsilon = \mathrm{o}(\varepsilon), \\
&\frac{1}{\gamma - 1} \varepsilon\big(\partial_t \tilde{\theta}_\varepsilon + u_\varepsilon \cdot \nabla_x \tilde{\theta}_\varepsilon\big) + \bar{\theta}\, \mathrm{div}_x\, u_\varepsilon = \varepsilon\, \mathrm{div}_x \left(\frac{\kappa(\bar{\theta})}{\bar{\rho}} \nabla_x \tilde{\theta}_\varepsilon \right) + \mathrm{o}(\varepsilon).
\end{aligned}$$

Next we must eliminate $\mathrm{div}_x\, u_\varepsilon$ between both equations above; indeed, we only know that $\mathrm{div}_x\, u_\varepsilon \simeq 0$ to leading order in ε, and it may not be true that $\mathrm{div}_x\, u_\varepsilon = \mathrm{o}(\varepsilon)$. Dividing the first equation above by $\bar{\rho}$ and the second by $\bar{\theta}$, one arrives at

$$\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x) \left(\frac{1}{\gamma - 1} \frac{\tilde{\theta}_\varepsilon}{\bar{\theta}} - \frac{\tilde{\rho}_\varepsilon}{\bar{\rho}} \right) = \varepsilon \frac{\kappa(\bar{\theta})}{\bar{\rho}} \Delta_x \left(\frac{\tilde{\theta}_\varepsilon}{\bar{\theta}} \right) + \mathrm{o}(\varepsilon).$$

We further eliminate the fluctuation of density by Boussinesq's relation, and eventually

[1] Boussinesq's relation is most often used in a different physical context. It expresses the effect on the buoyancy of the volume expansion of a liquid due to variations of temperature.

arrive at

$$\frac{\gamma}{\gamma-1}\bar{\rho}(\partial_t\theta_\varepsilon + u_\varepsilon\cdot\nabla_x\theta_\varepsilon) + \bar{\rho}u_\varepsilon\cdot\nabla_x\phi_\varepsilon \simeq \kappa\big(\bar{\theta}\big)\Delta_x\theta_\varepsilon. \tag{2.22}$$

Collecting both equations (2.21) and (2.22), we arrive at the coupled system in the small ε limit

$$\begin{aligned} &\partial_t u + u\cdot\nabla_x u + \frac{\theta}{\bar{\theta}}\nabla_x\phi + \nabla_x\pi = \nu\Delta_x u, \qquad \operatorname{div}_x u = 0,\\ &\partial_t\theta + u\cdot\nabla_x\theta + \frac{\gamma-1}{\gamma}u\cdot\nabla_x\phi = \bar{\kappa}\Delta_x\theta, \end{aligned} \tag{2.23}$$

where

$$\bar{\kappa} = \frac{\gamma-1}{\gamma}\frac{\kappa(\bar{\theta})}{\bar{\rho}}.$$

It is interesting to compare the temperature equation in (2.23) with (2.18); notice that the heat conductivity in (2.23) is $1/\gamma$ that in (2.18). Besides there is no viscous heating term in the temperature equation in (2.23), at variance with (2.18). This, however, is a consequence of the scaling considered in the discussion above: indeed, the fluctuations of velocity and temperature fields are of the same order of magnitude, so that viscous heating is a lower-order effect. On the contrary, if one sets the temperature fluctuations to be of the order of the squared fluctuations of velocity field, one recovers a viscous heating term in (2.23).

The material in this section is fairly classical and can be found in most textbooks on fluid mechanics; more information can be gathered from the excellent introductory section of [85]; see also the classical treatise [74].

3. The Boltzmann equation and its formal properties

The Boltzmann equation is the model that governs the evolution of perfect gases in kinetic theory. While fluid dynamics describes the state of a fluid with a few scalar or vector fields defined on the domain filled by the fluid, such as the temperature or velocity fields, kinetic theory describes the state of a gas with the *number density* (also called the *distribution function*) $F \equiv F(t,x,v) \geqslant 0$ that is the density of gas molecules which, at time $t \geqslant 0$, are located at the position $x \in \mathbb{R}^3$ and have velocity $v \in \mathbb{R}^3$. Put in other words, in any infinitesimal volume $\mathrm{d}x\,\mathrm{d}v$ centered at the point $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$ of the single particle phase space, one can find approximately $F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$ like particles at time t. In the classical kinetic theory of gases, the molecular radius is neglected, except in the collision cross-section: this has important consequences, as will be seen later.

The Boltzmann equation takes the form

$$\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), \tag{3.1}$$

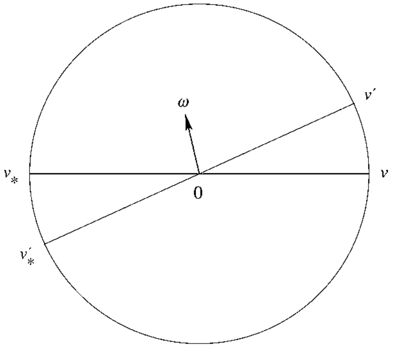

Fig. 1. The pre- and post-collision velocities in the reference frame of the center of mass of the particle pair.

where $\mathcal{B}(F, F)$ is the collision integral. This collision integral $\mathcal{B}(F, F)$ is a quadratic integral operator acting only on the v-argument of the number density F, and takes the form

$$\mathcal{B}(F, F)(t, x, v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} \big(F' F'_* - F F_*\big) b(v - v_*, \omega)\, \mathrm{d}\omega\, \mathrm{d}v_*, \tag{3.2}$$

where the notations F, F_*, F' and F'_* designate respectively the values $F(t, x, v)$, $F(t, x, v_*)$, $F(t, x, v')$ and $F(t, x, v'_*)$, with $v' \equiv v'(v, v_*, \omega)$ and $v'_* \equiv v'_*(v, v_*, \omega)$ given in terms of v, v_* and ω by the formulas

$$v' = v - (v - v_*) \cdot \omega\omega, \qquad v'_* = v_* + (v - v_*) \cdot \omega\omega, \tag{3.3}$$

where $\omega \in \mathbb{S}^2$ is an arbitrary unit vector (see Figure 1). These formulas represent all the solutions $(v', v'_*) \in \mathbb{R}^3 \times \mathbb{R}^3$ of the system of equations

$$v' + v'_* = v + v_*, \qquad \left|v'\right|^2 + \left|v'_*\right|^2 = |v|^2 + |v_*|^2, \tag{3.4}$$

where $(v, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3$ is given. If (v, v_*) are the velocities of a pair of like particles before collision, and (v', v'_*) are the velocities of the same pair of particles after collision – or vice versa, equalities (3.4) express the conservation of momentum and kinetic energy during the collision.

Only the binary collisions are accounted for in Boltzmann's equation. Indeed, since the molecular radius is neglected in the kinetic theory of gases, one can show that collisions involving more than two particles are events that occur with probability zero, and therefore can be neglected for all practical purposes.

Moreover, kinetic energy is the only form of energy conserved during collisions. In fact, the Boltzmann collision integral (3.2) applies only to monatomic gases. Polyatomic gases can also be treated by the methods of kinetic theory; however this require using complicated variants of Boltzmann's original collision integral that involve vibrational and rotational energies in addition to the kinetic energy of the center of mass of each molecule; besides, these additional energy variables are quantized in certain applications. While such considerations are important for understanding some real gas effects, they lead to heavy technicalities which do not belong to an expository article such as the present one. For these reasons, we shall implicitly restrict our attention to monatomic gases and to the collision integral (3.2) in the sequel.

The function $b \equiv b(V, \omega)$ is the *collision kernel*, an a.e. positive function that is of the form

$$b(V, \omega) = |V| \Sigma\big(|V|, \big|\cos\big(\widehat{V, \omega}\big)\big|\big), \tag{3.5}$$

where Σ is the *scattering cross-section* (see Section 3.5 for a precise definition of this notion).

We shall discuss later the physical meaning of the function Σ, together with the usual mathematical assumptions on the collision kernel b. For the moment, assume that b is locally integrable on $\mathbb{R}^3 \times \mathbb{S}^2$, and consider a number density $F \equiv F(t, x, v)$ which, at any arbitrary instant of time t and location x, is continuous with compact support in the velocity variable v. Then, the collision integral $\mathcal{B}(F, F)(t, x, v)$ can be split as

$$\mathcal{B}(F, F)(t, x, v) = \mathcal{B}_+(F, F)(t, x, v) - \mathcal{B}_-(F, F)(t, x, v), \tag{3.6}$$

where

$$\begin{aligned} \mathcal{B}_+(F, F)(t, x, v) &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} F' F'_* b(v - v_*, \omega)\, \mathrm{d}\omega\, \mathrm{d}v_*, \\ \mathcal{B}_-(F, F)(t, x, v) &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} F F_* b(v - v_*, \omega)\, \mathrm{d}\omega\, \mathrm{d}v_* \end{aligned} \tag{3.7}$$

are called respectively the *gain term* and the *loss term* in the collision integral $\mathcal{B}(F, F)$. The physical meaning of both the gain and loss terms – and that of the collision integral itself – can be explained in the following manner:

- $\mathcal{B}_-(F, F)(t, x, v)\, \mathrm{d}v$ is the number of particles located at x at time t that exit the volume element $\mathrm{d}v$ centered at v in the velocity space by colliding with another particle with an arbitrary velocity v_* located at the same position x at the same time t, and
- $\mathcal{B}_+(F, F)(t, x, v)\, \mathrm{d}v$ is the number of particles located at x at time t that enter the volume element $\mathrm{d}v$ centered at v in the velocity space as the result of a collision involving two particles with pre-collisional velocities v' and v'_* at the same time t and the same position x.

Notice that, in this model, collisions are purely local and instantaneous, which is another consequence of having neglected the molecular radius. Moreover, it is assumed that the joint distribution of any pair of particles located at the same position x at the same time t with velocities v and v_* *and that are about to collide* is the product

$$F(t, x, v) F(t, x, v_*).$$

In other words, such particles are assumed to be statistically uncorrelated; however, this assumption, which is crucial in the physical derivation of the Boltzmann equation, is needed only for particle pairs about to collide – and is obviously false for a pair of particles having just collided.

Going back to the Boltzmann equation (3.1) in the form

$$\partial_t F = -v \cdot \nabla_x F + \mathcal{B}(F, F),$$

it follows from the above discussion that the first term on the right-hand side represents the net number of particles entering the infinitesimal phase-space volume $\mathrm{d}x\,\mathrm{d}v$ centered at (x, v) as the result of inertial motion of particles between collisions, while the second term represents the net number of particles entering that same volume as the result of instantaneous and purely local collisions.

3.1. *Conservation laws*

Throughout this subsection, it is assumed that the collision kernel b is locally integrable,

$$b \in L^1_{\mathrm{loc}}\big(\mathbb{R}^3 \times \mathbb{S}^2\big). \tag{3.8}$$

The first major result about the Boltzmann collision integral is the following proposition.

PROPOSITION 3.1. *Let $F \equiv F(v) \in C_{\mathrm{c}}(\mathbb{R}^3)$ and $\phi \in C(\mathbb{R}^3)$. Then*

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{B}(F, F)(v)\phi(v)\,\mathrm{d}v\\&\quad= \frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.\end{aligned}$$

This result is essential to understanding the Boltzmann equation and especially its relations to hydrodynamics. For this reason, we shall give a complete proof of it.

PROOF OF PROPOSITION 3.1. The second relation in (3.4) and the fact that F is compactly supported shows that the support of $(v, v_*, \omega) \mapsto F'F'_* - FF_*$ is compact in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$. Hence both integrals

$$\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{B}(F, F)(v)\phi(v)\,\mathrm{d}v\\&\quad= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\phi b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\end{aligned}$$

are well defined since F and ϕ are continuous and b satisfies (3.8). In the latter integral, apply the change of variables $(v, v_*) \mapsto (v_*, v)$, while keeping ω fixed: (3.3) show that (v', v'_*) is changed into (v'_*, v'), so that the expression $(F'F'_* - FF_*)$ is invariant, while (3.5) shows that the collision kernel satisfies $b(v_* - v, \omega) = b(v - v_*, \omega)$. Hence

$$\begin{aligned}
&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\phi b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\quad = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\phi_* b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\quad = \frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)(\phi + \phi_*) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.
\end{aligned}$$

Now in the latter integral, for a.e. fixed $\omega \in \mathbb{S}^2$, apply the change of variables $(v, v_*) \mapsto (v', v'_*)$ defined by (3.3). It is easily seen that this transformation is an involution of $\mathbb{R}^3 \times \mathbb{R}^3$, so that this change of variables maps (v', v'_*) onto (v, v_*): hence $F'F'_* - FF_*$ is transformed into its opposite $FF_* - F'F'_*$. Formulas (3.3) also show that

$$\big|v' - v'_*\big| = v - v_* \quad \text{and} \quad \big(v' - v'_*\big)\cdot\omega = -(v - v_*)\cdot\omega$$

so that, by (3.5), one has $b(v' - v'_*, \omega) = b(v - v_*, \omega)$. Finally, this change of variables is an isometry of $\mathbb{R}^3 \times \mathbb{R}^3$ by the second relation of (3.4), and therefore preserves the Lebesgue measure. Eventually, we have proved that

$$\begin{aligned}
&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)(\phi + \phi_*) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\quad = -\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi' + \phi'_*\big) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\quad = \frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega
\end{aligned}$$

and this entails the announced formula. □

Notice that it may not be necessary to assume that F has compact support in Proposition 3.1. For instance, the same result holds for all F and $\phi \in C(\mathbb{R}^3)$ if there exists $m > 0$ such that

$$\begin{aligned}
&\big|\phi(v)\big| + \int_{\mathbb{S}^2} b(v, \omega)\,\mathrm{d}\omega = \mathrm{O}\big(|v|^m\big) \quad \text{while } F(v) = \mathrm{O}\big(|v|^{-n}\big) \\
&\quad \text{as } |v| \to +\infty, \text{ with } n > 2m + 3.
\end{aligned} \tag{3.9}$$

An important consequence of this proposition is the following corollary.

COROLLARY 3.2. *Under the same assumptions as in Proposition* 3.1 *or* (3.9), *one has*

$$\int_{\mathbb{R}^3} \mathcal{B}(F,F)(v)\,\mathrm{d}v = 0 \qquad (\textit{conservation of mass}),$$
$$\int_{\mathbb{R}^3} v_k\mathcal{B}(F,F)(v)\,\mathrm{d}v = 0 \qquad (\textit{conservation of momentum}),$$
$$\int_{\mathbb{R}^3} \frac{1}{2}|v|^2\mathcal{B}(F,F)(v)\,\mathrm{d}v = 0 \quad (\textit{conservation of energy}),$$

for $k = 1, 2, 3$.

When applied to a solution of the Boltzmann equation $F \equiv F(t,x,v)$, these five relations are the net conservation of mass – equivalently, of the total number of particles – momentum and energy in each phase-space cylinder $\mathrm{d}x \times \mathbb{R}^3_v$, where $\mathrm{d}x$ is any infinitesimal volume element in the space of positions $\mathbb{R}^3_x$.

PROOF OF COROLLARY 3.2. Assuming that $\phi(v)$ is one of the functions 1, v_k for $k = 1, 2, 3$, and $\frac{1}{2}|v|^2$, one has

$$\phi(v) + \phi(v_*) - \phi\big(v'\big) - \phi\big(v'_*\big) = 0$$

for each $(v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, because of (3.4). Applying Proposition 3.1 shows the five relations stated in Corollary 3.2. □

Let $F \equiv F(t,x,v)$ be a solution of the Boltzmann equation; assume that $F(t,x,\cdot)$ is continuous with compact support on $\mathbb{R}^3_v$ a.e. in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$, or satisfies (3.9). Then Corollary 3.2 implies that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^3} F\,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} vF\,\mathrm{d}v = 0,\\
&\partial_t \int_{\mathbb{R}^3} vF\,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} v\otimes vF\,\mathrm{d}v = 0,\\
&\partial_t \int_{\mathbb{R}^3} \frac{1}{2}|v|^2F\,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} v\frac{1}{2}|v|^2F\,\mathrm{d}v = 0.
\end{aligned} \tag{3.10}$$

These equalities are the local conservation laws of mass, momentum and energy in space–time divergence form.

Assume further, for simplicity, that for a.e. $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v > 0;$$

define then

$$\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v && \text{(macroscopic density)},\\
u(t,x) &= \frac{1}{\rho(t,x)}\int_{\mathbb{R}^3} vF(t,x,v)\,\mathrm{d}v && \text{(bulk velocity)},\\
\theta(t,x) &= \frac{1}{\rho(t,x)}\int_{\mathbb{R}^3} \frac{1}{3}\big|v-u(t,x)\big|^2 F(t,x,v)\,\mathrm{d}v && \text{(temperature)}.
\end{aligned}\tag{3.11}$$

With these definitions, the local conservation laws (3.10) take the form

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0,\\
&\partial_t(\rho u) + \operatorname{div}_x(\rho u\otimes u) + \nabla_x(\rho\theta) = -\operatorname{div}_x \int_{\mathbb{R}^3} A(v-u)F\,\mathrm{d}v,\\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2+\frac{3}{2}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2+\frac{5}{2}\theta\right)\right)\\
&\quad = -\operatorname{div}_x \int_{\mathbb{R}^3} B(v-u)F\,\mathrm{d}v - \operatorname{div}_x \int_{\mathbb{R}^3} A(v-u)\cdot uF\,\mathrm{d}v,
\end{aligned}\tag{3.12}$$

where

$$A(z) = z\otimes z - \frac{1}{3}|z|^2, \qquad B(z) = \frac{1}{2}\big(|z|^2-5\big)z.$$

The left-hand side of the equalities (3.12) coincide with that of the compressible Euler system (2.7) with $\gamma = 5/3$ (the adiabatic exponent for point particles, i.e., for particles with 3 degrees of freedom). The right-hand side, on the contrary, depends on the solution of the Boltzmann equation F and is in general not determined by the macroscopic variables ρ, u and θ.

However, in some limit, it may be possible to approximate the right-hand side of (3.12) by appropriate functions of ρ, u and θ, thereby arriving at a system in closed form with unknown (ρ, u, θ).

For instance, deriving the compressible Euler system (2.7) as some asymptotic limit of the Boltzmann equation would consist in proving that the right-hand side of the second and third equations in (3.12) vanishes in that limit. Deriving the Navier–Stokes system (2.8) from the Boltzmann equation would consist in finding some (other) asymptotic limit such that

$$\int_{\mathbb{R}^3} A(v-u)F\,\mathrm{d}v \simeq -\lambda D(u) \quad \text{and} \quad \int_{\mathbb{R}^3} B(v-u)F\,\mathrm{d}v = -\kappa\nabla_x\theta,$$

and so on.

The problem of finding such closure relations is the key to all the derivations of hydrodynamic models from the Boltzmann equation.

3.2. *Boltzmann's H-theorem*

We have seen in the last subsection how the symmetries of the Boltzmann collision integral entail the local conservation of mass, momentum and energy.

Another important feature of these symmetries is that they also entail a variant of the second principle of thermodynamics, as we shall now explain.

PROPOSITION 3.3 (Boltzmann's H-theorem). *Assume that the collision kernel b satisfies* (3.8)*, that $F \in C(\mathbb{R}^3)$ is positive and rapidly decaying at infinity, and that, for some $m > 0$, one has*

$$\int_{\mathbb{S}^2} b(v,\omega)\,\mathrm{d}\omega + \bigl|\ln F(v)\bigr| = \mathrm{O}\bigl(|v|^m\bigr) \quad \textit{as } |v| \to +\infty.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{B}(F,F)\ln F\,\mathrm{d}v\\&\quad = -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \bigl(F'F'_* - FF_*\bigr)\ln\biggl(\frac{F'F'_*}{FF_*}\biggr) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \leqslant 0.\end{aligned}$$

Moreover, the following conditions are equivalent

(i) $\int_{\mathbb{R}^3} \mathcal{B}(F,F)\ln F\,\mathrm{d}v = 0$,

(ii) $\mathcal{B}(F,F)(v) = 0$ *for all* $v \in \mathbb{R}^3$,

(iii) F *is a Maxwellian distribution, i.e., there exists $\rho,\theta > 0$ and $u \in \mathbb{R}^3$ such that $F = \mathcal{M}_{(\rho,u,\theta)}$, where*

$$\mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} \mathrm{e}^{-|v-u|^2/2\theta} \quad \textit{for each } v \in \mathbb{R}^3. \tag{3.13}$$

As was already the case of Proposition 3.1, Boltzmann's H-theorem is so essential in deriving hydrodynamic equations from the Boltzmann equation that we give a complete proof of it.

PROOF OF PROPOSITION 3.3. The assumptions on F and b are such that F, b and $\phi = \ln F$ satisfy the assumption (3.9). Applying Proposition 3.1 implies that

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{B}(F,F)\ln F\,\mathrm{d}v\\&\quad = -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \bigl(F'F'_* - FF_*\bigr)\ln\biggl(\frac{F'F'_*}{FF_*}\biggr) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.\end{aligned} \tag{3.14}$$

Since the logarithm is an increasing function, one has

$$(f-g)\ln\left(\frac{f}{g}\right)=(f-g)(\ln f-\ln g)\geqslant 0\quad\text{for each } f,g>0,$$

so that the expression on the right-hand side of (3.14) is nonpositive.

If that expression is equal to zero, the integrand must vanish a.e., meaning that

$$F'F'_*=FF_*\quad\text{for a.e. } (v,v_*,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2,$$

since the collision kernel b is a.e. positive. This accounts for the equivalence between conditions (i) and (ii). That (iii) implies (i) is proved by inspection; for instance, one can observe that, if F is a Maxwellian distribution, $\ln F$ is a linear combination of 1, v_1, v_2, v_3 and $|v|^2$, so that

$$\ln F'+\ln F'_*-\ln F-\ln F_*=0\quad\text{for all } (v,v_*,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2,$$

because of the microscopic conservation laws (3.4). Finally, (i) implies (iii), as shown by the next lemma, and this concludes the proof of Boltzmann's H-theorem. □

LEMMA 3.4. *Let* $\phi\geqslant 0$ *a.e. be such that* $(1+|v|^2)\phi\in L^1(\mathbb{R}^3)$. *If*

$$\phi'\phi'_*=\phi\phi_*\quad\textit{for a.e. } (v,v_*,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2,$$

then ϕ *is either a.e.* 0 *or a Maxwellian* (*i.e., is of the form* (3.13)).

The following proof is due to Perthame [104]; Boltzmann's original argument can be found in Section 18 of [16].

PROOF OF LEMMA 3.4. After translation and multiplication by a constant, one can always assume that

$$\int_{\mathbb{R}^3}\phi(v)\,\mathrm{d}v=1,\qquad\int_{\mathbb{R}^3}v\phi(v)\,\mathrm{d}v=0\tag{3.15}$$

unless $\phi=0$ a.e. Denoting by $\hat\phi$ the Fourier transform of ϕ, our assumptions imply that, for a.e. $\omega\in\mathbb{S}^2$, one has

$$\begin{aligned}\hat\phi(\xi)\hat\phi(\xi_*)&=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi(v')\phi(v'_*)\mathrm{e}^{-\mathrm{i}\xi\cdot v-\mathrm{i}\xi_*\cdot v_*}\,\mathrm{d}v\,\mathrm{d}v_*\\&=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi(v)\phi(v_*)\mathrm{e}^{-\mathrm{i}\xi\cdot v'-\mathrm{i}\xi_*\cdot v'_*}\,\mathrm{d}v\,\mathrm{d}v_*\\&=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi(v)\phi(v_*)\mathrm{e}^{-\mathrm{i}\xi\cdot v-\mathrm{i}\xi_*\cdot v_*}\mathrm{e}^{\mathrm{i}((\xi-\xi_*)\cdot\omega)((v-v_*)\cdot\omega)}\,\mathrm{d}v\,\mathrm{d}v_*.\end{aligned}$$

(Notice that the second equality follows from the same change of variables $(v, v_*) \mapsto (v', v'_*)$ as in the proof of Proposition 3.1.) In fact, this relation holds for all $\omega \in \mathbb{S}^2$ since both sides of the equality above are continuous in ω.

Since the left-hand side of the equality above is independent of ω, one can differentiate in ω to obtain that

$$0 = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi(v)\phi(v_*)\mathrm{e}^{-\mathrm{i}\xi\cdot v-\mathrm{i}\xi_*\cdot v_*}(v - v_*) \cdot \omega_0 \,\mathrm{d}v\,\mathrm{d}v_*$$

for any $\xi \neq \xi_* \in \mathbb{R}^3$ and $\omega_0 \in \mathbb{S}^2$ such that $\omega_0 \perp (\xi - \xi_*)$. In other words,

$$\omega_0 \perp (\xi - \xi_*) \quad \Longrightarrow \quad (\nabla_\xi - \nabla_{\xi_*})\hat{\phi}(\xi)\hat{\phi}(\xi_*) \perp \omega_0.$$

This implies that, for all $\xi \neq \xi_* \in \mathbb{R}^3$, one has

$$(\nabla_\xi - \nabla_{\xi_*})\hat{\phi}(\xi)\hat{\phi}(\xi_*) \parallel (\xi - \xi_*). \tag{3.16}$$

Applying this with $\xi_* = 0$ leads to

$$\nabla_\xi\hat{\phi}(\xi) \parallel \xi,$$

on account of the normalization condition (3.15). Hence $\hat{\phi}$ is of the form

$$\hat{\phi}(\xi) = \psi\big(|\xi|^2\big), \quad \xi \in \mathbb{R}^3.$$

Writing (3.16) with this form of $\hat{\phi}$, one finds that

$$\xi\psi'\big(|\xi|^2\big)\psi\big(|\xi_*|^2\big) - \xi_*\psi\big(|\xi|^2\big)\psi'\big(|\xi_*|^2\big) \parallel (\xi - \xi_*).$$

Whenever ξ and ξ_* are not colinear, i.e., for a dense subset of all $\xi, \xi_* \in \mathbb{R}^3$, this implies that

$$\psi'\big(|\xi|^2\big)\psi\big(|\xi_*|^2\big) = \psi\big(|\xi|^2\big)\psi'\big(|\xi_*|^2\big).$$

Since $\phi \in L^1((1 + |v|^2)\,\mathrm{d}v)$, $\hat{\phi} \in C^2(\mathbb{R})$ and the normalization conditions (3.15) imply that $\hat{\phi}(0) = 1$ and $\hat{\phi}'(0) = 0$; hence $\psi \in C^1(\mathbb{R}^2)$ and the relation above holds for each $(\xi, \xi_*) \in \mathbb{R}^3 \times \mathbb{R}^3$. This relation implies in turn that ψ is of the form

$$\psi(r) = \mathrm{e}^{-\theta r/2}.$$

Hence $\hat{\phi}$ is of the form

$$\hat{\phi}(\xi) = \mathrm{e}^{-\theta|\xi|^2/2},$$

so that ϕ is of the form $\phi = \mathcal{M}_{(1,0,\theta)}$. □

At this point, it is natural to introduce the notion of *collision invariant*.

DEFINITION 3.5. A collision invariant is a measurable function ϕ a.e. finite on $\mathbb{R}^3$ that satisfies

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) = 0 \quad \text{a.e. in } (v, v_*, \omega),$$

where $v' \equiv v'(v, v_*, \omega)$ and $v'_* \equiv v'_*(v, v_*, \omega)$ are defined by (3.3).

For instance, in the proof of Boltzmann's H-theorem, Maxwellian densities are characterized as the densities whose logarithms are collision invariants.

A variant of Lemma 3.4 characterizes collision invariants.

PROPOSITION 3.6. *A function ϕ is a collision invariant if and only if there exists five constants $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that*

$$\phi(v) = a_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 |v|^2 \quad \textit{a.e. in } \mathbb{R}^3.$$

See Section 3.1 in [28] for a proof of the proposition above.

3.3. *H-theorem and a priori estimates*

We conclude with the main application of Boltzmann's H-theorem, i.e., getting a priori estimates on the solution of the Boltzmann equation. We shall discuss four different cases.

Case 1: *The periodic box.* Consider the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$
$$F|_{t=0} = F^{\text{in}}.$$

Let F be a solution of the Boltzmann equation such that, for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$, $F(t, x, \cdot)$ satisfies the assumptions of Proposition 3.3. Then the number density F satisfies the local entropy inequality (3.26). Integrating this differential inequality on $[0, t] \times \mathbb{T}^3$, one arrives at

$$\begin{aligned}
&\iint_{\mathbb{T}^3 \times \mathbb{R}^3} F \ln F(t, x, v)\, \mathrm{d}x\, \mathrm{d}v \\
&\qquad + \frac{1}{4} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F' F'_* - F F_*) \ln\left(\frac{F' F'_*}{F F_*}\right) b\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega\, \mathrm{d}x\, \mathrm{d}s \\
&\quad = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} F^{\text{in}} \ln F^{\text{in}}(x, v)\, \mathrm{d}x\, \mathrm{d}v
\end{aligned} \tag{3.17}$$

for each $t \geqslant 0$.

The following definition explains the name "H-theorem".

DEFINITION 3.7. Let $F \geqslant 0$ a.e. be an element of $L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ such that

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left|F \ln F(x,v)\right| \mathrm{d}x\,\mathrm{d}v < +\infty.$$

One denotes by $H(F)$ the quantity

$$H(F) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} F \ln F(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Whenever there is no risk of ambiguity, we use the notation $H(t)$ to designate $H(F(t,\cdot,\cdot))$, when F is a solution of the Boltzmann equation. Equality (3.17) implies that $H(F)$ is a nonincreasing function of time; it was this property that Boltzmann called "the H-theorem". Moreover, $H(F)$ is stationary only if F is a Maxwellian (see Section 3.4.2). Hence, from the physical viewpoint, it is natural to think of $H(F(t,\cdot,\cdot))$ as *minus the entropy* of the system of particles distributed under $F(t,\cdot,\cdot)$.

In order to obtain a bound on the entropy production, it is convenient to introduce another (closely related) concept of entropy.

DEFINITION 3.8. Let $F \geqslant 0$ a.e. and $G > 0$ be two measurable functions on $\mathbb{T}^3 \times \mathbb{R}^3$; the relative entropy of F with respect to G is

$$H(F|G) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(F \ln\left(\frac{F}{G}\right) - F + G\right) \mathrm{d}x\,\mathrm{d}v.$$

Notice that the integrand in the definition of $H(F|G)$ is an a.e. nonnegative measurable function, so that the relative entropy $H(F|G)$ is well defined as an element of $[0,+\infty]$.

Let $\rho, \theta > 0$ and $u \in \mathbb{R}^3$, then

$$\begin{aligned} H(F|\mathcal{M}_{(\rho,u,\theta)}) = H(t) &+ \iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v-u|^2}{2\theta} F\,\mathrm{d}x\,\mathrm{d}v \\ &- \left(1 + \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right) \iint_{\mathbb{T}^3\times\mathbb{R}^3} F\,\mathrm{d}x\,\mathrm{d}v + \rho. \end{aligned} \tag{3.18}$$

Hence, if $F \in L^1(\mathbb{T}^3 \times \mathbb{R}^3; (1+|v|^2)\,\mathrm{d}v\,\mathrm{d}x)$ and if $H(0)$ is finite, then $H(t)$ is finite for each $t \geqslant 0$, and

$$\begin{aligned} &-\left(\left|\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right| + \frac{|u|^2+1}{\theta}\right) \iint_{\mathbb{T}^3\times\mathbb{R}^3} (1+|v|^2) F\,\mathrm{d}v\,\mathrm{d}x - \rho \\ &\quad \leqslant H(t) \leqslant H(0). \end{aligned}$$

On the other hand, F also satisfies the local conservation of mass, momentum, and energy (3.10), so that, integrating these local conservation laws on $[0,t] \times \mathbb{T}^3$, one arrives at the *global* variant of these conservation laws

$$\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{T}^3\times\mathbb{R}^3} F(0,x,v)\,\mathrm{d}v\,\mathrm{d}x,\\
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} vF(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{T}^3\times\mathbb{R}^3} vF(0,x,v)\,\mathrm{d}v\,\mathrm{d}x,\\
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F(0,x,v)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{3.19}$$

Since

$$\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v-u|^2}{2\theta} F\,\mathrm{d}x\,\mathrm{d}v\\
&\quad = \frac{1}{2\theta}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\big(|v|^2+|u|^2\big)F\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{\theta}\iint_{\mathbb{T}^3\times\mathbb{R}^3} u\cdot vF\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

one has

$$H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) = H(t) + \text{globally conserved quantities}$$

so that

$$H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) = H(t) - H(0).$$

Hence, the global entropy relation (3.17) is recast in terms of the relative entropy as

$$\begin{aligned}
&\frac{1}{4}\int_0^t\int_{\mathbb{T}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s\\
&\quad = H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big)
\end{aligned} \tag{3.20}$$

for each $\rho,\theta > 0$ and each $u\in\mathbb{R}^3$. This implies, in particular,

- the relative entropy bound

$$0 \leqslant H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big),\quad t\geqslant 0;$$

- the following entropy control

$$\begin{aligned}
&-\bigg(\bigg|\ln\bigg(\frac{\rho}{(2\pi\theta)^{3/2}}\bigg)\bigg| + \frac{|u|^2+1}{\theta}\bigg)\iint_{\mathbb{T}^3\times\mathbb{R}^3}\big(1+|v|^2\big)F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x - \rho\\
&\quad \leqslant H(F)(t) \leqslant H(F)(0);
\end{aligned}$$

- the entropy production estimate

$$\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{T}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big).$$

Case 2: *A bounded domain with specular reflection on the boundary.* The periodic box is a somewhat academic choice of a spatial domain for studying the Boltzmann equation. The next case that we consider now is very similar but more realistic. Let Ω be a smooth, bounded domain of $\mathbb{R}^3$. Starting from a given number density F^{in} at time $t = 0$, we consider the initial boundary value problem

$$\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), \quad (x,v)\in\Omega\times\mathbb{R}^3,$$
$$F(t,x,v) = F(t,x,\mathcal{R}_x v), \quad (x,v)\in\partial\Omega\times\mathbb{R}^3,$$
$$F|_{t=0} = F^{\text{in}},$$

where $\mathcal{R}_x$ designates the specular reflection defined by the outward unit normal n_x at $x\in\partial\Omega$

$$\mathcal{R}_x v = v - 2(v\cdot n_x)n_x.$$

Assume that the initial boundary value problem above has a solution F satisfying the assumptions of Proposition 3.3. One multiplies the Boltzmann equation above by $\ln F + 1$ and integrates first in v to obtain the identity (3.26), and then integrates in x, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^3} F\ln F\,\mathrm{d}x\,\mathrm{d}v + \iint_{\partial\Omega\times\mathbb{R}^3} F\ln F\,v\cdot n_x\,\mathrm{d}\sigma(x)\,\mathrm{d}v$$
$$= -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Changing the v variable in the boundary term by $v\mapsto w = \mathcal{R}_x v$, one sees that $w\cdot n_x = -v\cdot n_x$, while the specular reflection condition satisfied by F on $\partial\Omega$ implies that $F(t,x,v) = F(t,x,w)$; besides this change of variables preserves the Lebesgue measure $\mathrm{d}v$ since $\mathcal{R}_x$ is an isometry. Hence

$$\iint_{\partial\Omega\times\mathbb{R}^3} F\ln F\,v\cdot n_x\,\mathrm{d}\sigma(x)\,\mathrm{d}v = -\iint_{\partial\Omega\times\mathbb{R}^3} F\ln F\,w\cdot n_x\,\mathrm{d}\sigma(x)\,\mathrm{d}w = 0$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}H(F)(t)$$
$$= -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega. \tag{3.21}$$

Proceeding similarly with the local conservation laws (3.10) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^3} F\,\mathrm{d}x\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^3}\frac{1}{2}|v|^2 F\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{3.22}$$

At this point, we apply the formula (3.18) in the case where $u = 0$ and deduce from (3.22) that

$$H(F|\mathcal{M}_{(\rho,0,\theta)}) = H(F) + \text{globally conserved quantities.}$$

Notice that, at variance with the case of the periodic box, the total momentum is not conserved, so that the formula above only holds with centered Maxwellians (i.e., Maxwellians with zero bulk velocity).

Therefore, as in the case of the periodic box, one has, for each $\rho, \theta > 0$,

$$\begin{aligned}&\frac{1}{4}\int_0^t\int_\Omega\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s\\ &\quad = H\big(F(0)\big|\mathcal{M}_{(\rho,0,\theta)}\big) - H\big(F(t)\big|\mathcal{M}_{(\rho,0,\theta)}\big)\end{aligned} \tag{3.23}$$

for each $t \geqslant 0$. Again we obtain

- the relative entropy bound

$$0 \leqslant H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big), \quad t \geqslant 0;$$

- the following entropy control

$$\begin{aligned}&-\left(\left|\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right| + \frac{1}{\theta}\right)\iint_{\mathbb{T}^3\times\mathbb{R}^3}\big(1+|v|^2\big)F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x - \rho\\ &\quad \leqslant H(F)(t) \leqslant H(F)(0);\end{aligned}$$

- the entropy production estimate

$$\begin{aligned}&\frac{1}{4}\int_0^{+\infty}\int_\Omega\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s\\ &\quad \leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big).\end{aligned}$$

Case 3: *The Euclidean space with Maxwellian equilibrium at infinity.* Next we study cases where the spatial domain is unbounded. The simplest of such cases is that of a cloud of gas which is in Maxwellian equilibrium at infinity. Therefore, we consider the Cauchy problem

$$\begin{aligned}&\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), && (t,x,v)\in\mathbb{R}_+^*\times\mathbb{R}^3\times\mathbb{R}^3,\\ &F(t,x,v)\to\mathcal{M}_{(\rho,u,\theta)}, && |x|\to+\infty,\\ &F|_{t=0} = F^{\mathrm{in}}.\end{aligned}$$

We shall assume that F converges to the Maxwellian state $\mathcal{M}_{(\rho,u,\theta)}$ rapidly enough so that the relative entropy

$$
\begin{aligned}
&H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \\
&\quad = \iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}x\,\mathrm{d}v < +\infty
\end{aligned}
$$

for each $t \geqslant 0$. We claim that the same entropy relation as in the case of the three-torus also holds in the present situation

$$
\begin{aligned}
&\frac{1}{4}\int_0^t\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big)
\end{aligned}
\tag{3.24}
$$

for each $t \geqslant 0$. However, this equality is not obtained in the same way, since neither the globally conserved quantities nor the H-function itself are well-defined objects in this case (these quantities involve divergent integrals because of the Maxwellian condition at infinity).

Observe instead that, by the same argument as in the case of the three-torus, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v \\
&\quad = \int_{\mathbb{R}^3}F\ln F\,\mathrm{d}v + \frac{1}{2\theta}\int_{\mathbb{R}^3}\big(|v|^2+|u|^2\big)F\,\mathrm{d}v \\
&\qquad - \frac{1}{\theta}\int_{\mathbb{R}^3}u\cdot vF\,\mathrm{d}v + \left(1+\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right)\int_{\mathbb{R}^3}F\,\mathrm{d}v + \rho
\end{aligned}
$$

while

$$
\begin{aligned}
&\int_{\mathbb{R}^3}v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v \\
&\quad = \int_{\mathbb{R}^3}vF\ln F\,\mathrm{d}v + \frac{1}{2\theta}\int_{\mathbb{R}^3}v\big(|v|^2+|u|^2\big)F\,\mathrm{d}v \\
&\qquad - \frac{1}{\theta}\int_{\mathbb{R}^3}vu\cdot vF\,\mathrm{d}v + \left(1+\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right)\int_{\mathbb{R}^3}vF\,\mathrm{d}v.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v \\
&\quad = \int_{\mathbb{R}^3}F\ln F\,\mathrm{d}v + \text{locally conserved quantity}
\end{aligned}
$$

while

$$\int_{\mathbb{R}^3} v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v$$
$$=\int_{\mathbb{R}^3} vF\ln F\,\mathrm{d}v+\text{flux of that locally conserved quantity}$$

so that

$$\partial_t\int_{\mathbb{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v$$
$$+\operatorname{div}_x\int_{\mathbb{R}^3} v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v$$
$$=\partial_t\int_{\mathbb{R}^3}F\ln F\,\mathrm{d}v+\operatorname{div}_x\int_{\mathbb{R}^3}vF\ln F\,\mathrm{d}v.$$

Hence

$$\partial_t\int_{\mathbb{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v$$
$$+\operatorname{div}_x\int_{\mathbb{R}^3} v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)\mathrm{d}v$$
$$=-\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\left(F'F'_*-FF_*\right)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Integrating further on $[0,t]\times\mathbb{R}^3$, one arrives at (3.24).

To summarize, we deduce from (3.24)

- the relative entropy bound

$$0\leqslant H\left(F(t)\middle|\mathcal{M}_{(\rho,u,\theta)}\right)\leqslant H\left(F^{\mathrm{in}}\middle|\mathcal{M}_{(\rho,u,\theta)}\right)\quad\text{for each } t\geqslant 0;$$

- the entropy production estimate

$$\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\left(F'F'_*-FF_*\right)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant H\left(F^{\mathrm{in}}\middle|\mathcal{M}_{(\rho,u,\theta)}\right).$$

Case 4: *The Euclidean space with vacuum at infinity.* Finally, we consider the case of a cloud of gas expanding in the vacuum. As we shall see, this case is slightly different from the previous ones. Consider the Cauchy problem

$$
\begin{aligned}
&\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), &&(t,x,v)\in\mathbb{R}_+^*\times\mathbb{R}^3\times\mathbb{R}^3,\\
&F(t,x,v)\to 0, &&|x|,|v|\to+\infty,\\
&F|_{t=0}=F^{\mathrm{in}}.
\end{aligned}
$$

We shall assume that F vanishes rapidly enough at infinity so that the relative entropy

$$H\big(F(t)\big|\mathcal{G}\big)<+\infty \quad \text{for each } t\geqslant 0,$$

where $\mathcal{G}$ is the centered reduced Gaussian

$$\mathcal{G}(x,v)=\frac{1}{(2\pi)^3}\mathrm{e}^{-(|x|^2+|v|^2)/2}.$$

Assume that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} F^{\mathrm{in}}\big(\big|\ln F^{\mathrm{in}}\big|+|x|^2+|v|^2+1\big)\,\mathrm{d}x\,\mathrm{d}v<+\infty.$$

We claim that, for each $t\geqslant 0$,

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v=\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x|^2F(0,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{3.25}$$

Indeed

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_t\big(|x-tv|^2F(t,x,v)\big)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad=\iint_{\mathbb{R}^3\times\mathbb{R}^3}(\partial_t+v\cdot\nabla_x)\big(|x-tv|^2F(t,x,v)\big)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad=\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2(\partial_t+v\cdot\nabla_x)F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad=\int_{\mathbb{R}^3}\bigg(\int_{\mathbb{R}^3}|x-tv|^2\mathcal{B}(F,F)(t,x,v)\,\mathrm{d}v\bigg)\,\mathrm{d}x=0.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
H(F|\mathcal{G})={}&H(F)+\frac12\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big(|x|^2+|v|^2\big)F\,\mathrm{d}x\,\mathrm{d}v\\
&+\big(3\ln(2\pi)-1\big)\iint_{\mathbb{R}^3\times\mathbb{R}^3}F\,\mathrm{d}x\,\mathrm{d}v+1.
\end{aligned}
$$

Because of (3.25), one has

$$
\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^2 F(t)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad\leqslant 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-tv|^2 F(t)\,\mathrm{d}x\,\mathrm{d}v + 2t^2\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 F(t)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad= 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^2 F(0)\,\mathrm{d}x\,\mathrm{d}v + 2t^2\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 F(0)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}
$$

so that

$$
\begin{aligned}
&-\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^2 F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v - \left(\frac{1}{2}+t^2\right)\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad - \bigl(3\ln(2\pi)-1\bigr)\iint_{\mathbb{R}^3\times\mathbb{R}^3} F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v - 1 \leqslant H(t) \leqslant H(0).
\end{aligned}
$$

Integrating on $[0,t]\times\mathbb{R}^3$ the local entropy equality (3.26), one arrives at the equality

$$
\begin{aligned}
&H(0)-H(t) \\
&\quad= \frac{1}{4}\int_0^t\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \bigl(F'F'_* - FF_*\bigr)\ln\left(\frac{F'F'_*}{FF_*}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant C\bigl(1+t^2\bigr)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \bigl(1+|x|^2+|v|^2+\bigl|\ln F^{\mathrm{in}}\bigr|\bigr) F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

As we shall see below, Cases 1–3 are the most useful in the context of hydrodynamic limits. Case 4 is also interesting, although not for hydrodynamic limits: it provides one of the important estimates in the construction of global weak solutions to the Boltzmann equation by R. DiPerna and P.-L. Lions (see further).

Another case, which we did not discuss in spite of its obvious interest for applications, is that of a spatial domain that is the complement in $\mathbb{R}^3$ of a regular compact set, assuming specular reflection of the particles at the boundary of the domain. This case is handled by a straightforward adaptation of the arguments in Cases 2 and 3.

Let us now briefly discuss some of the main consequences of Boltzmann's H-theorem.

3.4. *Further remarks on the H-theorem*

3.4.1. *H-theorem and the second principle of thermodynamics.* To begin with, let F be a solution of the Boltzmann equation such that, for a.e. $(t,x)\in\mathbb{R}_+\times\mathbb{R}^3$, $F(t,x,\cdot)$ satisfies the assumptions of Proposition 3.3. Multiplying both sides of the Boltzmann equation by $\ln F + 1$ and applying Proposition 3.3 and Corollary 3.2 with $\phi\equiv 1$, one arrives at the identity

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^3} F \ln F \, \mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} vF \ln F \, \mathrm{d}v \\
&\quad = -\frac{1}{4} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\quad \leqslant 0.
\end{aligned} \tag{3.26}$$

It is interesting to compare the equality (3.26) with the second principle of thermodynamics applied to any portion of a fluid in a smooth domain Ω. Denoting by n_x the outward unit normal field on $\partial\Omega$, and by s the entropy per unit of mass in the fluid, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \rho s(t,x)\,\mathrm{d}t \geqslant -\int_{\partial\Omega} \rho s u(t,x)\cdot n_x \,\mathrm{d}\sigma(x) - \int_{\partial\Omega} \frac{q(t,x)}{\theta(t,x)} \cdot n_x \,\mathrm{d}\sigma(x),$$

where ρ is the density of the fluid, u the velocity field, θ the temperature, q the heat flux and $\mathrm{d}\sigma(x)$ the surface element on $\partial\Omega$. The infinitesimal version of this inequality is

$$\partial_t(\rho s) + \operatorname{div}_x\left(\rho s u + \frac{q}{\theta}\right) \geqslant 0, \tag{3.27}$$

which is obviously analogous to (3.26). In particular,

$$\text{the quantity } -\int_{\mathbb{R}^3} F \ln F \,\mathrm{d}v \text{ is analogous to } \rho s$$

and

$$\text{the quantity } -\int_{\mathbb{R}^3} vF \ln F \,\mathrm{d}v \text{ is analogous to } \rho s u,$$

while the quantity

$$\frac{1}{4} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$

is the local entropy production. Notice that fluid dynamics does not in general provide any expression of the entropy production in terms of ρ, u, θ, s and q. On the contrary, in the kinetic theory of gases, the entropy production is given in terms of the number density by the integral above.

3.4.2. *Relaxation towards equilibrium.* One application of the second principle of thermodynamics is the relaxation towards equilibrium for closed systems. Assume that a gas described by the Boltzmann equation is enclosed in some container Ω that is a smooth, bounded domain of $\mathbb{R}^3$. At the microscopic level, we assume that the gas molecules are reflected on the surface of the container without exchanging heat. One model for this is the ideal situation where each molecule impinging on the boundary of the container is specularly reflected, this being Case 2 of Section 3.3.

Starting from a given number density F^{in} at time $t = 0$, we consider the initial boundary value problem

$$\begin{aligned}
&\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), && (x, v) \in \Omega \times \mathbb{R}^3,\\
&F(t, x, v) = F(t, x, \mathcal{R}_x v), && (x, v) \in \partial\Omega \times \mathbb{R}^3,\\
&F|_{t=0} = F^{\text{in}},
\end{aligned}$$

where $\mathcal{R}_x$ designates the specular reflection defined by the outward unit normal n_x at $x \in \partial\Omega$

$$\mathcal{R}_x v = v - 2v \cdot n_x n_x.$$

Now pick any sequence $t_n \to +\infty$ such that

$$F_n(t, x, v) := F(t + t_n, x, v) \to E(t, x, v) \tag{3.28}$$

in a weak topology that is compatible with the conservation laws (3.22). Then, by weak convergence and convexity, the bound on entropy production obtained in Case 2 of Section 3.3 implies that $E(t, x, v)$ is a local Maxwellian – meaning that the function $v \mapsto E(t, x, v)$ is a.e. a Maxwellian with parameters ρ, u, θ that are functions of t, x – which satisfies

$$\begin{aligned}
&(\partial_t E + v \cdot \nabla_x E)(t, x, v) = 0, && (x, v) \in \Omega \times \mathbb{R}^3,\\
&E(t, x, v) = E(t, x, \mathcal{R}_x v), && (x, v) \in \partial\Omega \times \mathbb{R}^3.
\end{aligned} \tag{3.29}$$

Whenever Ω is not rotationally invariant with respect to some axis of symmetry, the only local Maxwellians that solve the system of equations (3.29) are the global Maxwellians of the form

$$E(t, x, v) = \mathcal{M}_{(\rho,0,\theta)}(v) \quad \text{for some constant } \rho, \theta > 0. \tag{3.30}$$

Since we assumed that the conservation laws (3.22) are compatible with the topology in which the long time limit holds, the constants ρ and θ are given by

$$\iint_{\Omega\times\mathbb{R}^3} F^{\text{in}}\, \mathrm{d}x\, \mathrm{d}v = \rho|\Omega|, \qquad \iint_{\Omega\times\mathbb{R}^3} \frac{1}{2} F^{\text{in}}\, \mathrm{d}x\, \mathrm{d}v = \frac{3}{2}\rho\theta|\Omega|.$$

However, if Ω is rotationally invariant around some axis of symmetry, the system (3.29) has other solutions than the global Maxwellians (3.30), namely all functions of the form

$$E(t, x, v) = \mathcal{M}_{(\rho,0,\theta)}(v)\mathrm{e}^{\lambda(k\times(x-x_0))\cdot v},$$

where $\lambda \neq 0$ is a constant, while the axis of rotational symmetry for Ω is the line of direction $k \in \mathbb{S}^2$ passing through x_0.

In fact, Lions proved in Section V of [83] that the convergence (3.28) is locally uniform in t with values in the strong topology of $L^1(\Omega \times \mathbb{R}^3)$.

Let us mention that, in spite of his apparent simplicity, the problem of relaxation toward equilibrium for the Boltzmann equation is still open, in spite of recent progress by Desvillettes and Villani [35]. The main issue is the lack of a tightness estimate in the v variable for $|v|^2 F(t,x,v)$ as $t \to +\infty$ that would apply to all initial data of finite mass, energy and entropy. Short of such an estimate, the part of the argument above identifying the temperature in terms of the initial data fails. The only cases where such estimates have been obtained correspond to initial data that are already close enough to some global Maxwellian, or that are independent of the space variable (i.e., the space homogeneous case).

However incomplete, this discussion shows the importance of Boltzmann's H-theorem whenever one seeks to estimate how close to the class of local Maxwellians a given solution of the Boltzmann equation may be. This particular point is of paramount importance for hydrodynamic limits.

3.5. *The collision kernel*

So far, our discussion of the Boltzmann equation – in fact, of the Boltzmann collision integral – did not use much of the properties of the collision kernel b. Indeed, we only took advantage of the symmetries of b in (3.5) and some additional bounds such as (3.8) or (3.9).

However, the derivation of hydrodynamic limits requires further properties of the collision integral, for which a more extensive discussion of the collision kernel becomes necessary.

First we recall some elementary facts concerning the two-body problem. Consider two points of unit mass subject to a repulsive interaction potential $U \equiv U(r)$, where r is the distance between these two points. In other words, assume that U satisfies the properties

$$U \in C^\infty\big(\mathbb{R}_+^*\big) \text{ is decreasing}, \qquad \lim_{r \to 0^+} U(r) = +\infty, \qquad \lim_{r \to +\infty} U(r) = 0^+.$$

It is well known that both points stay in the same plane for all times. Pick a Galilean frame where one of the points is at rest; then, the trajectory of the moving point is easily expressed in polar coordinates (r,θ) with the fixed point as origin. Choosing the origin of polar angles to be the line asymptotic to the trajectory of the moving particle in the past, the trajectory is determined as follows. Let v be the speed of the moving particle at infinity, and let h be the impact parameter defined by

$$vh = r^2(t)\dot{\theta}(t), \quad t \in \mathbb{R},$$

this quantity being a well-known first integral of the motion (see Figure 2). In other words, h is the distance between the line asymptotic to the trajectory of the moving particle in the

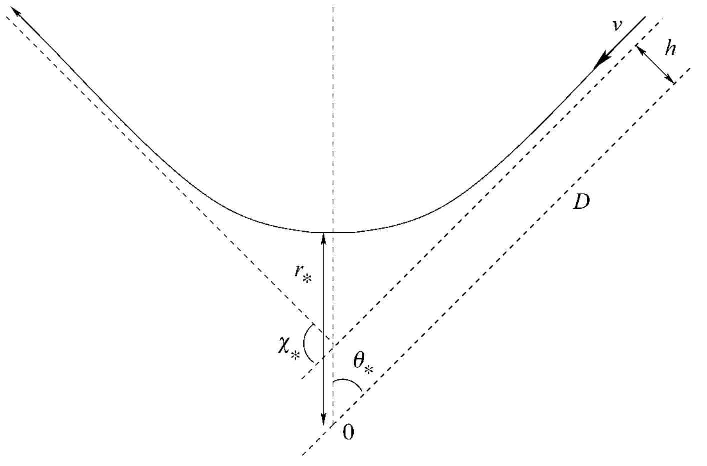

Fig. 2. Deflection of a particle subject to a radial repulsion potential from a particle at rest.

past and the parallel line going through the particle at rest (see Figure 2). Let $z_* > 0$ be the unique solution to

$$1 - z_*^2 - \frac{4}{v^2} U\left(\frac{h}{z_*}\right) = 0,$$

and set

$$r_* = \frac{h}{z_*} \quad \text{and} \quad \theta_* = \int_0^{z_*} \frac{\mathrm{d}z}{\sqrt{1 - z^2 - 4/v^2 U(h/z)}}. \tag{3.31}$$

The point of polar coordinates (r_*, θ_*) is the apse of the trajectory, i.e., the closest to the particle at rest. Then the trajectory of the moving particles is given in polar coordinates by the equation

$$\theta = \begin{cases} \int_0^{h/r} \frac{\mathrm{d}z}{\sqrt{1-z^2-4/v^2U(h/z)}} & \text{for } \theta \in (0, \theta_*] \text{ and } r > r_*, \\ 2\theta_* - \int_0^{h/r} \frac{\mathrm{d}z}{\sqrt{1-z^2-4/v^2U(h/z)}} & \text{for } \theta \in [\theta_*, 2\theta_*) \text{ and } r > r_*. \end{cases}$$

Notice in particular that the moving particle is deflected of an angle $2\theta_*$.

Next we recall the notion of *scattering cross-section*. Pick an arbitrary relative speed v at infinity, and consider the deflection angle $\chi_* = \pi - 2\theta_*$ as a function of the impact parameter h. It is easily seen that

$$\chi_* \text{ is decreasing,} \qquad \lim_{h \to 0^+} \chi_*(h) = \pi^- \quad \text{and} \quad \lim_{h \to +\infty} \chi_*(h) = 0^+.$$

Because the two-body problem is invariant by any rotation around the line D passing through the particle at rest that is parallel to the asymptote in the past to the trajectory

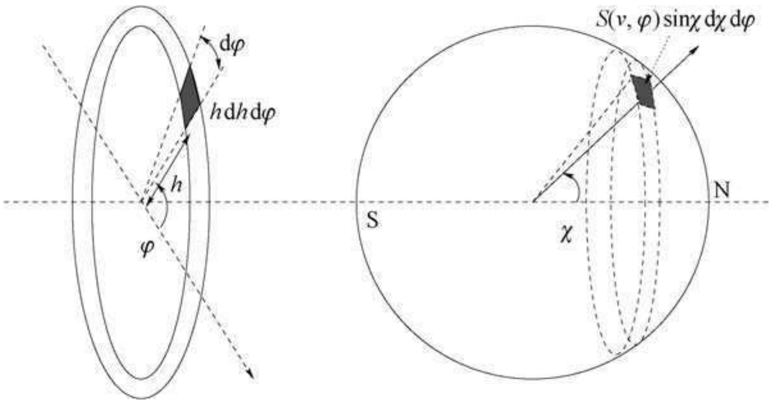

Fig. 3. The scattering cross-section corresponding to the relative speed v and in the direction χ corresponding to the impact parameter h.

of the moving particle, consider the map

$$\begin{aligned}&\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{R}_+^* \times [0, 2\pi) \to (0, \pi) \times [0, 2\pi) \simeq \mathbb{S}^2 \setminus \{N, S\},\\&(h, \phi) \mapsto \big(\chi_*(h), \phi\big),\end{aligned} \tag{3.32}$$

where the first identification is through polar coordinates in the plane orthogonal to D with origin $h = 0$ (the intersection of D with that plane), while the second identification is through spherical coordinates, with D as the polar axis and $\mathbb{S}^2 \cap D = \{N, S\}$, see Figure 3. The image of the Lebesgue measure under this map is a surface measure on $\mathbb{S}^2$ of the form

$$S(v, \chi) \sin \chi \, d\chi \, d\phi,$$

and $S(v, \chi)$ is the scattering cross-section in the direction χ corresponding to the relative speed v. Because S is the density with respect to the Euclidean surface element on the unit sphere (which is dimensionless) of the image of the two-dimensional Lebesgue measure (which has the dimension of a surface) under the map (3.32), it has the dimension of a surface, which justifies the name "cross-section".

The scattering cross-section clearly depends upon the computation of the deflection angle in (3.31)

$$S(v, \chi) = \frac{h}{\sin \chi \, |\chi_*'(h)|}\bigg|_{\chi_*(h) = \chi}.$$

Here are the scattering cross-sections for a few typical interactions:

- in the case of a hard sphere interaction,

$$U(r) = 0 \quad \text{if } r \geqslant d_0 \quad \text{and} \quad U(r) = +\infty \quad \text{if } 0 < r < d_0.$$

In this example, U is not decreasing but only nonincreasing and has finite range. Therefore, the definition of the scattering cross-section must be modified as follows. The map (3.32) is replaced with

$$\begin{aligned} &B(0, d_0) \simeq (0, d_0) \times [0, 2\pi) \to [0, \pi) \times [0, 2\pi) \simeq \mathbb{S}^2 \setminus \{N, S\}, \\ &(h, \phi) \mapsto \big(\chi_*(h), \phi\big), \end{aligned} \tag{3.33}$$

and $S(v, \chi) \sin \chi \, \mathrm{d}\chi \, \mathrm{d}\phi$ is the image under the above map of the restriction to the disk $B(0, d_0) \subset \mathbb{R}^2$ of the Lebesgue measure. With this slightly modified definition, it is found that

$$S(v, \chi) = \frac{1}{4} d_0^2;$$

- if $U(r) = kr^{-s}$ with $s > 0$, set

$$\vartheta(l) = \int_0^{\zeta(l)} \frac{\mathrm{d}z}{\sqrt{1 - z^2 - 2(z/l)^s}},$$

where $l = (v^2/2k)^{1/s} h$, and where $\zeta(l)$ is the only positive root of the denominator of the integrand above; set $\vartheta \mapsto \lambda(\vartheta)$ to be the inverse of $l \mapsto \vartheta(l)$ so defined. Then

$$S(v, \chi) = (2k)^{2/s} v^{-4/s} \frac{\beta((\pi - \chi)/2)}{\sin \chi},$$

where $\beta(\vartheta) = \lambda(\vartheta)\lambda'(\vartheta)$. One finds that β is singular near $\theta = \pi/2$, which corresponds to $\chi = 0$ and $l \to +\infty$, i.e., to collisions with small deflection angles, or equivalently to the case of *grazing collisions*

$$\beta(\theta) \simeq C\left(\frac{\pi}{2} - \theta\right)^{-1-2/s} \quad \text{as } \theta \to \frac{\pi}{2}^-,$$

while $\beta(\theta) = O(\theta)$ for $\theta \to 0$.

Although the usual definition of the scattering cross-section involves the deflection angle χ, one might find it easier to use instead the angle $\theta = (\pi - \chi)/2$ (see Figure 2). It is easily seen that the scattering cross-section S as above can be expressed in terms of a function $\Sigma \equiv \Sigma(v, \mu)$ defined on $\mathbb{R}_+ \times [0, 1)$ by the formula

$$\Sigma\big(v, |\cos\theta|\big) = \frac{1}{2} S(v, \pi - 2\theta) |\cos\theta|.$$

This function Σ has the following geometric interpretation: the mapping

$$\begin{aligned} &\mathbb{S}^2 \setminus \{N, S\} \simeq (0, \pi) \times [0, 2\pi) \to \mathbb{S}^2 \setminus \{N\}, \\ &\omega \simeq (\theta, \phi) \mapsto \big(-\cos(2\theta), \sin(2\theta)\cos\phi, \sin(2\theta)\sin\phi\big) \end{aligned}$$

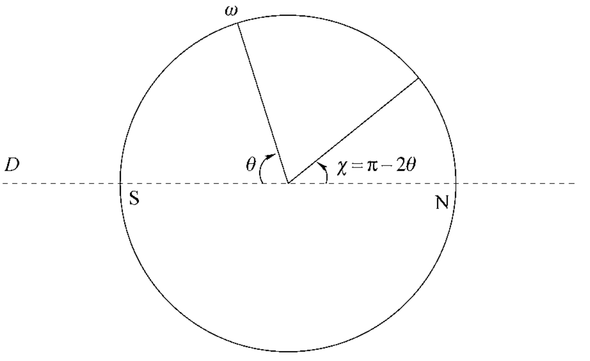

Fig. 4. The double cover $\theta \mapsto \chi$.

is a double cover (see Figure 4) and the image of the surface measure $\Sigma(v, |\cos\theta|) \times \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ under the above mapping is $2S(v, \chi) \sin\chi \, \mathrm{d}\chi \, \mathrm{d}\phi$.

Then, the collision kernel b is given by

$$b(V, \omega) = |V| \Sigma\big(|V|, \big|\cos\big(\widehat{V, \omega}\big)\big|\big).$$

With the formulas for the scattering cross-section given above, one sees that

- in the case of hard-spheres with radius r_0,

$$b(V, \omega) = \frac{1}{2} r_0^2 |V \cdot \omega|; \tag{3.34}$$

- in the case of an interaction potential $U(r) = kr^{-s}$ for $s > 0$,

$$b(V, \omega) = \frac{1}{4}(2k)^{2/s} |V|^{1-4/s} \frac{\beta(\theta)}{\sin\theta} \quad \text{with } \theta = \big(\widehat{V, \omega}\big), \tag{3.35}$$

  where $\beta(\theta) = \mathrm{O}(\theta)$ as $\theta \to 0^+$ while $\beta(\theta) = \mathrm{O}((\pi/2 - \theta)^{-1-2/s})$ as $\theta \to \pi^-/2$;
- whenever $s = 4$, the collision kernel b is independent of $|V|$; such potentials are usually referred to as *Maxwellian* potentials, and considerably facilitate the analysis of the collision integral;
- for $s = 1$, which is the case of a repulsive Coulomb potential, one has

$$b(V, \omega) = k^2 |V|^{-3} \frac{\beta(\theta)}{\sin\theta} \quad \text{with } \beta(\theta) = \mathrm{O}\left(\left(\frac{\pi}{2} - \theta\right)^{-3}\right) \text{ as } \theta \to \frac{\pi}{2}. \tag{3.36}$$

Observe that, for any inverse power-law potential $U(r) = kr^{-s}$,

$$\int_{\mathbb{S}^2} b(V, \omega) \, \mathrm{d}\omega = +\infty \tag{3.37}$$

because of the singularity at $\theta = \pi/2$, so that our earlier assumption (3.8) is violated by such potentials. In particular, whenever the particle interaction is given by an inverse power-law potential, one cannot split the collision integral as

$$\mathcal{B}(F, F) = \mathcal{B}_+(F, F) - \mathcal{B}_-(F, F)$$

as was done earlier in this section. One way around this is to define $\mathcal{B}(F, F)$ in the sense of distributions, as follows

$$\begin{aligned}&\big\langle \mathcal{B}(F, F), \phi\big\rangle \\ &\quad = \frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big) b(v - v_*, \omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\end{aligned}$$

for each $F \in C_c^1(\mathbb{R}^3)$ and $\phi \in C_c^1(\mathbb{R}^3)$. Then

$$\big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big) = \mathrm{O}\Big(\big|(v - v_*)\cdot\omega\big|^2\Big)$$

so that the integrand in the right-hand side above is $\mathrm{O}((\pi/2 - \theta)^{1-2/s})$. This procedure can handle all inverse power-law potentials $U(r) = kr^{-s}$ for $s > 1$; however, the Coulomb case $s = 1$ remains excluded.

Observe that, in addition to the singularity in the deflection angle at $\theta = \pi/2$, the collision kernel of the Coulomb potential also has a $|v - v_*|^{-3}$ singularity in the velocity variable. Physicists deal with the latter singularity by introducing a further truncation near $v = v_*$; the dependence upon this truncation parameter of the collision integral is only logarithmic, so that the result does not depend "too much" on the truncation parameter. In other words, the collision integral is computed modulo a scaling factor known as the *Coulomb logarithm* (see [75], Section 41).

Another way of avoiding the singularity of the collision kernel at $\theta = \pi/2$ consists in assuming that the interaction potential is truncated at large distance, in other words, that

$$U(r) = U(r_{\mathrm{C}}) \quad \text{whenever } r \geqslant r_{\mathrm{C}}. \tag{3.38}$$

In this case (as in the hard sphere case), U is not decreasing but only nonincreasing, and the definition of the scattering cross-section must be modified as in (3.33) with r_{C} in the place of d_0. Call b_{C} the collision kernel corresponding to the potential truncated as in (3.38), and S_{C}, Σ_{C} the associated scattering cross-section. Then

$$\begin{aligned}\int_{\mathbb{S}^2} b_{\mathrm{C}}(V, \omega)\,\mathrm{d}\omega &= |V|\int_0^{2\pi}\!\!\int_0^{\pi} \Sigma_{\mathrm{C}}\big(|V|, |\cos\theta|\big)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= |V|\int_0^{2\pi}\!\!\int_0^{\pi} S_{\mathrm{C}}\big(|V|, \chi\big)\sin\chi\,\mathrm{d}\chi\,\mathrm{d}\phi \\ &= |V|\int_0^{r_c}\!\!\int_0^{2\pi} h\,\mathrm{d}h\,\mathrm{d}\phi = |V|\pi r_{\mathrm{C}}^2\end{aligned}$$

since $S_{\mathrm{C}}(v,\chi)\sin\chi\,\mathrm{d}\chi\,\mathrm{d}\phi$ is the image under the map (3.33) of the Lebesgue measure on the two-dimensional disk of radius r_{C}. Hence the truncation (3.38) leads to a collision kernel whose integral over the angle variables is finite, thereby avoiding the divergence (3.37) that occurs for any infinite range, inverse power law potential.

Yet another way of avoiding the singularity of the collision kernel at $\theta = \pi/2$ was proposed by Grad [63]. He considered molecular force laws for which the collision kernel satisfies the condition

$$\frac{b(V,\omega)}{|\cos(V,\omega)|} \leqslant C\big(|V| + |V|^{1-\varepsilon}\big), \tag{3.39}$$

where $C > 0$ and $\varepsilon \in (0,1)$. Comparing the case of a power law potential $U(r) = kr^{-s}$ with Grad's assumption above, we see that the collision kernel in (3.35) satisfies (3.39) provided that one modifies the function β near $\theta = \pi/2$ so that $\beta(\theta) = \mathrm{O}(\pi/2 - \theta)$. One possibility is to replace β with

$$\tilde{\beta}(\theta) = \beta(\theta)\mathbb{1}_{\theta \leqslant \theta_0}, \tag{3.40}$$

where $\theta_0 \in (0, \frac{\pi}{2})$ is some arbitrary value. Then, the associated truncated collision kernel $\tilde{b}$ satisfies the bounds

$$\tilde{b}(V,\omega) \leqslant C_b\big(1+|V|\big)^{1-4/s} \quad \text{and} \quad \int_{\mathbb{S}^2} b(V,\omega)\,\mathrm{d}\omega \geqslant \frac{1}{C_b}\big(1+|V|\big)^{1-4/s} \tag{3.41}$$

for some positive constant C_b. The potential U is called a *hard cut-off potential* if $s \geqslant 4$, and a *soft cut-off potential* if $s < 4$.

Grad defined more general classes of hard and soft cut-off potentials; specifically, a general hard cut-off potential corresponds to the condition

$$\int_{\mathbb{S}^2} b(V,\omega)\,\mathrm{d}\omega \geqslant \frac{c|V|}{1+|V|}$$

while a soft cut-off potential is defined by the condition

$$\int_{\mathbb{S}^2} b(V,\omega)\,\mathrm{d}\omega \leqslant c\big(1+|V|^{\varepsilon-1}\big)$$

for some $c > 0$ and $\varepsilon \in (0,1)$, in addition to the bound (3.39). In the sequel, we shall mostly restrict our attention to those hard cut-off potentials that satisfy the same bound (3.41) as in the inverse power law case.

The terminology "cut-off potential" attached to Grad's cut-off prescription, is somewhat unfelicitous. Indeed, it is not equivalent to truncating the potential at large intermolecular distances as in (3.38). Indeed, the angular truncation (3.40) prohibits grazing collisions with deflection angle less than a threshold $\pi - 2\theta_0$ that is independent of the relative velocity V. On the contrary, with a potential truncated as in (3.38), there exist grazing collisions with deflection angle arbitrarily small for large enough relative velocity $|V|$. Hence, it would be more appropriate to refer to Grad's procedure as leading to a "cut-off scattering cross-section" rather than a "cut-off potential". Yet the latter terminology is commonly used in the literature, so that changing it would only cause confusion.

Let us conclude this subsection with a few words on the physical relevance of Grad's cut-off assumption. Grad observed that, in gases of neutral particles with short range interactions, grazing collisions are not statistically dominant as in the case of plasmas, where the long range effect of the Coulomb interaction must be accounted for. The latter case requires using a mean-field description of the long-range interaction potential, in addition to the Landau collision integral, an approximation of Boltzmann's collision integral in the regime of essentially grazing collisions.

That the Coulomb potential, probably the best known interaction in physics, is a singular point in the theory of the Boltzmann collision integral may seem highly regrettable. However, in view of the remark above, the reader should bear in mind that the Boltzmann equation is essentially meant to model collisional processes in neutral gases with short-range molecular interactions, and not in plasmas, so that the Coulomb potential is not really relevant in this context.

3.6. *The linearized collision integral*

Let ρ and $\theta > 0$, and pick $u \in \mathbb{R}^3$; the linearization at $\mathcal{M}_{(\rho,u,\theta)}$ of Boltzmann's collision integral is defined as follows

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}} f = -2\mathcal{M}_{(\rho,u,\theta)}^{-1} \mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)} f), \tag{3.42}$$

where $\mathcal{B}$ is the bilinear operator obtained by polarization from the Boltzmann collision integral. In other words,

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}} f = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \big(f + f_* - f' - f'_*\big) b(v - v_*, \omega) \mathcal{M}_{(\rho,u,\theta)}(v_*)\, \mathrm{d}v_*\, \mathrm{d}\omega, \tag{3.43}$$

where f_*, f' and f'_* are the values of f at v_*, v' and v'_*, respectively, and where v' and v'_* are determined in terms of v, v_* and ω by the usual collision relations (3.3).

The dependence on the parameters ρ, u and θ of the linearized collision integral is handled most easily by using the translation and scaling invariance of $\mathcal{L}$.

3.6.1. *Translation and scaling invariance of* $\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}$. We introduce the following notation for the actions of translation and scaling transformations on functions defined on $\mathbb{R}^3$:

$$\tau_u f(v) = f(v - u), \qquad m_\lambda f(v) = \lambda^{-3} f\left(\frac{v}{\lambda}\right). \tag{3.44}$$

For instance, with these notations,

$$\mathcal{M}_{(\rho,u,\theta)} = \rho\tau_u m_{\sqrt{\theta}}\mathcal{M}_{(1,0,1)}. \tag{3.45}$$

Given a collision kernel $b \equiv b(z,\omega)$, we denote by $\mathcal{B}^b$ Boltzmann's collision integral defined by this collision kernel. With the notation so defined, a straightforward change of variables in the collision integral leads to the following relation

$$\begin{aligned} &\tau_u\mathcal{B}^b(\Phi,\Phi) = \mathcal{B}^b(\tau_u\Phi,\tau_u\Phi),\\ &m_\lambda\mathcal{B}^b(\Phi,\Phi) = \lambda^3\mathcal{B}^{m_\lambda b}(m_\lambda\Phi,m_\lambda\Phi) \end{aligned} \tag{3.46}$$

for each continuous, rapidly decaying $\Phi \equiv \Phi(v)$, where, in the expression $m_\lambda b$, it is understood that the scaling transformation acts on the first argument of b, i.e., on the relative velocity. The analogous formula for the linearized collision operator is

$$\rho\tau_u m_{\sqrt{\theta}}\mathcal{L}^b_{\mathcal{M}_{(1,0,1)}}\phi = \theta^{3/2}\mathcal{L}^{m_{\sqrt{\theta}}b}_{\mathcal{M}_{(\rho,u,\theta)}}(\tau_u m_{\sqrt{\theta}}\phi). \tag{3.47}$$

This relation shows that it is enough to study the linearization of the collision integral at the centered reduced Gaussian

$$M = \mathcal{M}_{(1,0,1)}$$

with an arbitrary collision kernel b.

3.6.2. *Rotational invariance of $\mathcal{L}_{\mathcal{M}_{(1,0,1)}}$.* The orthogonal group $O_3(\mathbb{R})$ acts on functions on $\mathbb{R}^3$ by the formula

$$f_R(v) = f\big(R^{\mathrm{T}}v\big), \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3; \tag{3.48}$$

likewise its action on vector fields is defined by

$$U_R(v) = RU\big(R^{\mathrm{T}}v\big), \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3, \tag{3.49}$$

while its action on symmetric matrix fields is given by

$$S_R(v) = RS\big(R^{\mathrm{T}}v\big)R^{\mathrm{T}}, \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3. \tag{3.50}$$

The Boltzmann collision integral is obviously invariant under the action of $O_3(\mathbb{R})$ – indeed, the microscopic collision process is isotropic. In fact, an elementary change of variables in the collision integral shows that

$$\mathcal{B}(\Phi_R,\Phi_R) = \mathcal{B}(\Phi,\Phi)_R \tag{3.51}$$

for each continuous, rapidly decaying Φ. Since the centered unit Gaussian $M = \mathcal{M}_{(1,0,1)}$ is a radial function, this rotation invariance property goes over to $\mathcal{L}_M$,

$$\mathcal{L}_M(\phi_R) = (\mathcal{L}_M\phi)_R. \tag{3.52}$$

Extending $\mathcal{L}_M$ to act componentwise on vector or matrix fields on $\mathbb{R}^3$, one finds that

$$\mathcal{L}_M(U_R) = (\mathcal{L}_M U)_R \tag{3.53}$$

for continuous, rapidly decaying vector fields U and

$$\mathcal{L}_M(S_R) = (\mathcal{L}_M S)_R \tag{3.54}$$

for continuous, rapidly decaying symmetric matrix fields S, where the notations U_R and S_R are as in (3.48)–(3.50).

As we shall see below, this $O_3(\mathbb{R})$-invariance of $\mathcal{L}_M$ has important consequences: it implies in particular that the viscosity and heat conductivity are scalar quantities (and not matrices).

3.6.3. *The Fredholm property.* Henceforth, we assume that the collision kernel b satisfies a hard cut-off assumption in the sense of Grad [63], i.e., there exists $\alpha \in [0,1]$ and $C_b > 0$ such that, for a.e. $z \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$, one has

$$\begin{aligned} &0 < b(z,\omega) \leqslant C_b\big(1+|z|\big)^\alpha \quad \text{and} \\ &\int_{\mathbb{S}^2} b(z,\omega)\,\mathrm{d}\omega \geqslant \frac{1}{C_b}\big(1+|z|\big)^\alpha. \end{aligned} \tag{3.55}$$

Consider $\mathcal{L}$, the linearization of the Boltzmann collision integral at the centered, reduced Gaussian state M above. (Notice that we have discarded the dependence of $\mathcal{L}$ on b and M for notational simplicity.) From (3.43), we infer that $\mathcal{L}$ can be split as the sum of a local (multiplication) operator and of an integral operator, as follows

$$\mathcal{L}\phi(v) = a\big(|v|\big)\phi(v) - \mathcal{K}\phi(v), \tag{3.56}$$

where a is the collision frequency

$$a\big(|v|\big) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} b(v-v_*,\omega)M_*\,\mathrm{d}v_*\,\mathrm{d}\omega. \tag{3.57}$$

The nonlocal operator $\mathcal{K}$ is further split into two parts

$$\mathcal{K}\phi = \mathcal{K}_1\phi - \mathcal{K}_2\phi, \tag{3.58}$$

where

$$\begin{aligned}\mathcal{K}_1\phi &= \iint_{\mathbb{R}^3\times\mathbb{S}^2} \phi_* b(v-v_*,\omega) M_*\,\mathrm{d}v_*\,\mathrm{d}\omega,\\ \mathcal{K}_2\phi &= \iint_{\mathbb{R}^3\times\mathbb{S}^2} \big(\phi' + \phi'_*\big) b(v-v_*,\omega) M_*\,\mathrm{d}v_*\,\mathrm{d}\omega\\ &= 2\iint_{\mathbb{R}^3\times\mathbb{S}^2} \phi' b(v-v_*,\omega) M_*\,\mathrm{d}v_*\,\mathrm{d}\omega.\end{aligned} \tag{3.59}$$

(This last formula is not entirely obvious; it rests on a change of variables that will be explained later.) It is clear that $\mathcal{K}_1$ is a compact operator on $L^2(M\,\mathrm{d}v)$; that $\mathcal{K}_2$ shares the same property is much less obvious, and was proved by Hilbert in the hard sphere case. Fifty years later, Grad introduced the cut-off assumption which now bears his name and used it in particular to extend Hilbert's result to all cut-off potentials.

LEMMA 3.9 (Hilbert [68], Grad [63]). *Assume that b is a collision kernel that satisfies the hard cut-off assumption* (3.55). *Then the operator $\mathcal{K}_2$ is compact on $L^2(M\,\mathrm{d}v)$.*

We shall not give the proof of the Hilbert–Grad lemma here, since it is rather long and technical; the interested reader is referred to the lucid exposition by Glassey [49], or to the treatise by Cercignani, Illner and Pulvirenti [28] for the hard sphere case.

Instead, we shall digress from our discussion of the linearized collision integral and mention a new result by Lions which can be viewed as a nonlinear analogue of the Hilbert–Grad result.

LEMMA 3.10 (Lions [83]). *Assume that b is the collision kernel of a hard sphere gas*: $b(z,\omega) = |z\cdot\omega|$, *and consider the gain term in the Boltzmann collision integral*

$$\mathcal{B}_+(F,F) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} F' F'_* \big|(v-v_*)\cdot\omega\big|\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Then $\mathcal{B}_+$ maps $L^2_{\mathrm{comp}}(\mathbb{R}^3)$ continuously into $H^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Actually, this is not exactly Lions' result; in the reference [83], only compactly supported collision kernels are considered. The proof in [83] is based upon the L^2-continuity of Fourier integral operators of order 0. The statement above was later proved by Bouchut and Desvillettes [17] by a very simple and elegant argument.

Deriving the Hilbert–Grad lemma from the Lions lemma would require additional estimates in order to handle contributions from large $|v|$. Since obtaining these estimates requires essentially as much work as does the proof of the Hilbert–Grad lemma, we do not claim that the Lions lemma leads to a simpler proof of the compactness of $\mathcal{K}$; it is however of great independent interest.

The main properties of $\mathcal{L}$ are now summarized in the following theorem.

THEOREM 3.11. *Assume that the collision kernel b satisfies the hard cut-off assumption* (3.55). *Then $\mathcal{L}$ is an unbounded self-adjoint nonnegative Fredholm operator, with domain $D(\mathcal{L}) = L^2(a^2 M\,\mathrm{d}v)$ (a being the collision frequency defined in* (3.57)). *Its nullspace is the space of collision invariants*

$$\operatorname{Ker}\mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}.$$

Finally, $\mathcal{L}$ satisfies the following relative coercivity property on $(\operatorname{Ker}\mathcal{L})^{\perp}$: there exists $C_0 > 0$ such that, for each $\phi \in L^2(aM\,\mathrm{d}v)$ – the form domain of $\mathcal{L}$ – one has

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,\mathrm{d}v \geqslant C_0 \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 aM\,\mathrm{d}v, \tag{3.60}$$

where Π is the $L^2(M\,\mathrm{d}v)$-orthogonal projection on $\operatorname{Ker}\mathcal{L}$.

SKETCH OF THE PROOF. Let us briefly explain how these various facts are established. It follows from the Hilbert–Grad lemma that $\mathcal{K}$ is a compact operator on $L^2(M\,\mathrm{d}v)$. Hence $\mathcal{L} = a - \mathcal{K}$ is an unbounded Fredholm operator with domain $D(\mathcal{L}) = L^2(a^2 M\,\mathrm{d}v)$. That $\mathcal{L}$ is self-adjoint comes from the symmetries of Boltzmann's collision integral, especially from the computation of

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F)\phi\,\mathrm{d}v$$

as explained in Proposition 3.1. By the same argument, one sees that

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,\mathrm{d}v = \frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \left(\phi + \phi_* - \phi' - \phi'_*\right)^2 bM\,\mathrm{d}vM_*\,\mathrm{d}v_*\,\mathrm{d}\omega \geqslant 0 \tag{3.61}$$

so that $\mathcal{L}$ is a nonnegative operator; moreover, if $\mathcal{L}\phi = 0$, the integral on the right-hand side of the above equality vanishes, so that

$$\phi + \phi' - \phi' - \phi'_* = 0 \quad \text{a.e. in } v, v_*, \omega.$$

Hence ϕ is a collision invariant (see Definition 3.5) and this characterizes $\operatorname{Ker}\mathcal{L}$, see Proposition 3.6. As for the last inequality, the spectral theory of compact operators implies that $\mathcal{L}$ has a spectral gap in $L^2(M\,\mathrm{d}v)$

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,\mathrm{d}v \geqslant C^* \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 M\,\mathrm{d}v. \tag{3.62}$$

On the other hand, by continuity of $\mathcal{K}$ on $L^2(M\,\mathrm{d}v)$, one has

$$\int_{\mathbb{R}^3}\phi\mathcal{L}\phi M\,\mathrm{d}v=\int_{\mathbb{R}^3}(\phi-\Pi\phi)\mathcal{L}(\phi-\Pi\phi)M\,\mathrm{d}v$$
$$\geqslant\int_{\mathbb{R}^3}(\phi-\Pi\phi)^2aM\,\mathrm{d}v-\|\mathcal{K}\|^2\int_{\mathbb{R}^3}(\phi-\Pi\phi)^2M\,\mathrm{d}v.$$

Combining both inequalities leads to the announced estimate. □

The improved spectral gap estimate above is due to Bardos, Caflisch and Nicolaenko [6]. Also, an elegant explicit estimate for the $L^2(M\,\mathrm{d}v)$ spectral gap can be found in [5].

In fact, it is interesting to notice that the weighted spectral gap estimate (3.60) is in some sense more intrinsic than the unweighted one. Indeed, in the case of soft cut-off potentials, (3.62) is false, but (3.60) (which is of course a weaker statement since the collision frequency vanishes for large velocities in the soft potential case) holds true: this was proved in [57], based on decay estimates on the gain term in the linearized collision integral due to Grad and Caflisch [23]. In other words, while the spectral gap estimate (3.62) holds for hard potentials only, the weighted spectral gap estimate (3.60) holds for hard as well as soft potentials, in the cut-off case.

An important consequence of Theorem 3.11 is that the integral equation

$$\mathcal{L}\phi=\psi,\quad \psi\in L^2(M\,\mathrm{d}v),\tag{3.63}$$

satisfies the *Fredholm alternative*:

- either $\psi\perp\operatorname{Ker}\mathcal{L}$, in which case (3.63) has a unique solution

  $$\phi_0\in L^2\big(a^2M\,\mathrm{d}v\big)\cap(\operatorname{Ker}\mathcal{L})^\perp;$$

  then any solution of (3.63) is of the form

  $$\phi=\phi_0+\phi_1,\quad \text{where } \phi_1 \text{ is an arbitrary element of } \operatorname{Ker}\mathcal{L};$$

- or $\psi\notin(\operatorname{Ker}\mathcal{L})^\perp$, in which case (3.63) has no solution.

EXAMPLE. Consider the vector field

$$B(v)=\frac{1}{2}\big(|v|^2-5\big)v$$

and the matrix field

$$A(v)=v\otimes v-\frac{1}{3}|v|^2I.$$

Clearly

$$A_{jk}\perp\operatorname{Ker}\mathcal{L},\qquad B_l\perp\operatorname{Ker}\mathcal{L},\qquad A_{jk}\perp B_l,\quad j,k,l=1,2,3.\tag{3.64}$$

In fact, more is true:

$$\int_{\mathbb{R}^3} A(v) f\big(|v|^2\big) M \,\mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} A(v) v f\big(|v|^2\big) M \,\mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} B(v) f\big(|v|^2\big) M \,\mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} B(v) \cdot v M \,\mathrm{d}v = 0.$$

The second and third formulas are obvious since A is even and B odd. As for the first formula, observe that A is an isotropic matrix, in the sense that $A(Rv) = RA(v)R^{\mathrm{T}}$ for each $R \in O_3(\mathbb{R})$ – with the notation (3.50) for the action of $O_3(\mathbb{R})$ on symmetric matrices, $A_R = A$ for each $R \in O_3(\mathbb{R})$. Hence the matrix

$$\int_{\mathbb{R}^3} A(v) f\big(|v|^2\big) M \,\mathrm{d}v$$

commutes with any $R \in O_3(\mathbb{R})$ – as can be seen by changing v into Rv in the above integral – and is therefore a scalar multiple of the identity matrix. But

$$\operatorname{trace} \int_{\mathbb{R}^3} A(v) f\big(|v|^2\big) M \,\mathrm{d}v = \int_{\mathbb{R}^3} \operatorname{trace} A(v) f\big(|v|^2\big) M \,\mathrm{d}v = 0$$

and hence this scalar multiple of the identity matrix is null. The fourth and last formula is based on the following elementary recursion formula for Gaussian integrals

$$\int_{\mathbb{R}^3} |v|^n M \,\mathrm{d}v = (n+1) \int_{\mathbb{R}^3} |v|^{n-2} M \,\mathrm{d}v, \quad n \geqslant 2 \tag{3.65}$$

(to see this, use spherical coordinates and integrate by parts).

In particular, the Fredholm alternative implies the existence and uniqueness of a matrix field $\widehat{A}$ and of a vector field $\widehat{B}$ such that

$$\begin{aligned} &\mathcal{L}\widehat{A} = A \quad \text{and} \quad \widehat{A} \perp \operatorname{Ker} \mathcal{L}, \\ &\mathcal{L}\widehat{B} = B \quad \text{and} \quad \widehat{B} \perp \operatorname{Ker} \mathcal{L}. \end{aligned} \tag{3.66}$$

Observe that

$$\mathcal{L}\big(\widehat{A}_R\big) = A_R = A, \qquad \widehat{A}_R \perp \operatorname{Ker} \mathcal{L} \quad \text{for all } R \in O_3(\mathbb{R}),$$

$$\mathcal{L}\big(\widehat{B}_R\big) = B_R = B, \qquad \widehat{B}_R \perp \operatorname{Ker} \mathcal{L} \quad \text{for all } R \in O_3(\mathbb{R}),$$

so that, by the uniqueness part in the Fredholm alternative

$$\widehat{A}_R = \widehat{A} \quad \text{and} \quad \widehat{B}_R = \widehat{B} \quad \text{for all } R \in O_3(\mathbb{R}).$$

An elementary geometric argument (see [34]) shows the existence of two scalar functions

$$\mathbf{a}: \mathbb{R}_+ \to \mathbb{R}, \qquad \mathbf{b}: \mathbb{R}_+ \to \mathbb{R}$$

such that

$$\widehat{A}(v) = \mathbf{a}\big(|v|^2\big)A(v) \quad \text{and} \quad \widehat{B}(v) = \mathbf{b}\big(|v|^2\big)B(v). \tag{3.67}$$

As we shall see further, the viscosity and heat conductivity of a gas are expressed as Gaussian integrals of the scalar functions **a** and **b**, and therefore are scalar quantities themselves.

4. Hydrodynamic scalings for the Boltzmann equation

This short section introduces in particular the Knudsen number, *Kn*, a dimensionless parameter of considerable importance for the derivation of hydrodynamic models from the kinetic theory of gases.

4.1. *Notion of a rarefied gas*

Think of a monatomic gas as a cloud of $\mathcal{N}$ hard spheres of radius r confined in a container of volume V; we shall call *excluded volume* the volume V_e that the $\mathcal{N}$ gas molecules would occupy if tightly packed somewhere in the container (as oranges in a grocery store). Clearly

$$\frac{4}{3}\mathcal{N}\pi r^3 \leqslant V_e \leqslant \mathcal{N}(2r)^3,$$

and we shall call a *perfect gas* one for which $V_e \ll V$. It is well known that the equation of state for a perfect gas is given by the Boyle–Mariotte law

$$p = k\rho\theta,$$

where p, ρ and θ are the pressure, the density and the temperature in the gas, while k designates the Boltzmann constant ($k = 1.38 \cdot 10^{-23}$ J K^{-1}).

EXAMPLE. For a monatomic gas at room temperature and atmospheric pressure, about $\mathcal{N} = 10^{20}$ molecules with radius 10^{-8} cm are to be found in a volume V of 1 cm^3. Hence, the excluded volume satisfies $V_e \leqslant 10^{20} \cdot (2 \cdot 10^{-8})^3 = 8 \cdot 10^{-4}$ cm^3, so that $V_e \ll V$.

Another important notion is that of *mean free path*: it is the average distance between two successive collisions for one gas molecule picked at random. There is more than one precise definition of that notion, in particular, because there are several choices of probability measures for computing the mean. One choice could be to use the empirical notion of mean: Pick a typical particle, wait until after this particle has collided $n \gg 1$ times with other particles; the ratio of the distance traveled by the particle between the initial time and the last collision time to the number n of collisions should converge, in the large n limit, to one notion of mean free path. This definition, however, does not provide us with an easy way of estimating the mean free path, because it is fairly difficult to compute the trajectory of one particle in the cloud of the $\mathcal{N} - 1$ other particles.

Intuitively, one expects that the larger the number of particles in the container, the smaller the mean free path; likewise, the bigger the particles, the smaller the mean free path. This suggests using the following formula to estimate the order of magnitude of the mean free path,

$$\text{mean free path} \simeq \frac{1}{(\mathcal{N}/(V - V_\mathrm{e})) \times 4\mathcal{A}},$$

where $\mathcal{A}$ is the surface area of the section of the particles. A mathematical justification of the above formula for the mean free path can be found in [110]; see also [39].

EXAMPLE. For the same monatomic gas as above, $\mathcal{N} = 10^{20}$ molecules, $V - V_\mathrm{e} \simeq V = 1\ \mathrm{cm}^3$, while $\mathcal{A} = \pi \cdot (10^{-8})^2\ \mathrm{cm}^2$. This gives a mean free path of the order of 10^{-5} cm $\ll 1$ cm (the size of the container). Hence, a gas molecule will bump into roughly 10^5 other particles when traveling a distance comparable to the size of the container.

The example above shows that even in the case of a perfect gas (i.e., for an excluded volume negligible when compared to the size of the container) any given particle can collide with a large number of other particles. See, for instance, [2] for more information on the importance of the excluded volume in this context.

EXAMPLE. For the same monatomic gas as above, lower the pressure from $p = 1$ atm to $p = 10^{-4}$ atm. Then, only $\mathcal{N} = 10^{16}$ molecules are to be found in the container, which gives a mean free path of the order of 0.1 cm, comparable to the size of the container.

These examples suggest that the natural way of measuring the degree of rarefaction in the gas is by using the Knudsen number, Kn, defined as the ratio of the mean free path to the size of the container, or more generally, as

$$Kn = \frac{\text{mean free path}}{\text{macroscopic length scale}}.$$

In other words,

- a rarefied gas is a gas for which $Kn \gtrsim 1$, while
- a gas in hydrodynamic regime satisfies $Kn \ll 1$.

4.2. *The dimensionless Boltzmann equation*

In the discussion above, the introduction of the Knudsen number was based on physical arguments. We shall see below that it also appears naturally when writing the Boltzmann equation in dimensionless variables.

Choose a macroscopic length scale L and time scale T, and a reference temperature Θ. This defines *two* velocity scales:

- one is the speed at which some macroscopic portion of the gas is transported over a distance L in time T, i.e.,

$$U = \frac{L}{T};$$

- the other one is the *thermal speed* of the molecules with energy $\frac{3}{2}k\Theta$; in fact, it is more natural to define this velocity scale as

$$c = \sqrt{\frac{5}{3}\frac{k\Theta}{m}},$$

  m being the molecular mass, which is the *speed of sound* in a monatomic gas at the temperature Θ.

Define next the dimensionless variables involved in the Boltzmann equation, i.e., the dimensionless time, space and velocity variables as

$$\hat{t} = \frac{t}{T}, \qquad \hat{x} = \frac{x}{L} \quad \text{and} \quad \hat{v} = \frac{v}{c}.$$

Define also the dimensionless number density

$$\widehat{F}\big(\hat{t}, \hat{x}, \hat{v}\big) = \frac{L^3 c^3}{\mathcal{N}} F(t, x, v),$$

where $\mathcal{N}$ is the total number of gas molecules in a volume L^3. Finally, we must rescale the collision kernel b. As mentioned earlier, $b(z, \omega)$ is the relative velocity multiplied by the scattering cross-section of the gas molecules; define

$$\hat{b}(\hat{z}, \omega) = \frac{1}{c \times 4\pi r^2} b(z, \omega) \quad \text{with } \hat{z} = \frac{z}{c},$$

where r is the molecular radius.

If f satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \big(F' F'_* - F F_*\big) b(v - v_*, \omega)\, \mathrm{d}v_*\, \mathrm{d}\omega,$$

then

$$\frac{L}{cT}\partial_{\hat{t}}\widehat{F}+\hat{v}\cdot\nabla_{\hat{x}}\widehat{F}=\frac{\mathcal{N}\times 4\pi r^2}{L^2}\iint_{\mathbb{R}^3\times\mathbb{S}^2}\big(\widehat{F}'\widehat{F}'_*-\widehat{F}\widehat{F}_*\big)\hat{b}(\hat{v}-\hat{v}_*,\omega)\,\mathrm{d}\hat{v}_*\,\mathrm{d}\omega.$$

The factor multiplying the collision integral is

$$L\times\frac{\mathcal{N}\times 4\pi r^2}{L^3}=\frac{L}{\text{mean free path}}=\frac{1}{Kn},$$

where *Kn* is the Knudsen number defined above. The factor multiplying the time derivative

$$\frac{(1/T)\times L}{c}=:St$$

is called the *kinetic Strouhal number* (by analogy with the notion of Strouhal number used in the dynamics of vortices). Hence the dimensionless form of the Boltzmann equation (see Section 2.9 in [114]) is

$$St\,\partial_{\hat{t}}\widehat{F}+\hat{v}\cdot\nabla_{\hat{x}}\widehat{F}=\frac{1}{Kn}\iint_{\mathbb{R}^3\times\mathbb{S}^2}\big(\widehat{F}'\widehat{F}'_*-\widehat{F}\widehat{F}_*\big)\hat{b}(\hat{v}-\hat{v}_*,\omega)\,\mathrm{d}\hat{v}_*\,\mathrm{d}\omega. \tag{4.1}$$

There is some arbitrariness in the way the length, time and temperature scales L, T, Θ are chosen. For instance, if $F^{\mathrm{in}}\equiv F^{\mathrm{in}}(x,v)$ is the initial number density (at time $t=0$), one can choose

$$\begin{aligned}
\frac{1}{L}&=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big|\nabla_x F^{\mathrm{in}}\big|\,\mathrm{d}x\,\mathrm{d}v\bigg/\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v,\\
U&=\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}vF^{\mathrm{in}}\,\mathrm{d}v\right|\mathrm{d}x\bigg/\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v,\\
\Theta&=\frac{1}{3k}\iint_{\mathbb{R}^3\times\mathbb{R}^3}m|v|^2F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v\bigg/\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

and define $T=L/U$. In addition to Sone's book [114], we also refer to the Introduction of [10] for a presentation of the Boltzmann equation in dimensionless variables.

All hydrodynamic limits of the Boltzmann equation correspond to situations where the Knudsen number, *Kn*, satisfies

$$Kn\ll 1.$$

In other words, the Knudsen number governs the transition from the kinetic theory of gases to hydrodynamic models, just as the Reynolds number in fluid mechanics governs the transition from laminar to turbulent flows, except that the hydrodynamic limit is much better understood than the latter situation.

But there is no universal prescription for the Strouhal number in the context of the hydrodynamic limit; as we shall see below, various hydrodynamic regimes can be derived from the Boltzmann equation by appropriately tuning the Strouhal number.

5. Compressible limits of the Boltzmann equation: Formal results

In this section we study the dimensionless Boltzmann equation (4.1) in the case where

$$Kn = \varepsilon \ll 1, \qquad St = 1.$$

The dimensionless Boltzmann equation takes the form

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \tag{5.1}$$

where

$$\mathcal{B}(F, F) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} \big(F' F'_* - F F_*\big) b(v - v_*, \omega)\, \mathrm{d}v_*\, \mathrm{d}\omega.$$

The collision kernel b is assumed to satisfy Grad's hard cut-off assumption

$$0 < b(z, \omega) \leqslant C_b\big(1 + |z|\big)^\alpha \quad \text{and} \quad \int_{\mathbb{S}^2} b(z, \omega)\, \mathrm{d}\omega \geqslant \frac{1}{C_b}\big(1 + |z|\big)^\alpha \tag{5.2}$$

for a.e. $z \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$, where $\alpha \in [0, 1]$ and $C_b > 0$.

5.1. *The compressible Euler limit: The Hilbert expansion*

In [68], Hilbert proposed to seek the solution of (5.1) as a formal power series in ε,

$$F_\varepsilon(t, x, v) = \sum_{k\geqslant 0} \varepsilon^k F_k(t, x, v) \tag{5.3}$$

with $F_k \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathcal{S}(\mathbb{R}^3))$ for each $k \geqslant 0$.

The equations governing the coefficients F_k are obtained by inserting the right-hand side of (5.3) in (5.1) and equating the coefficients multiplying the successive powers of ε.

Order ε^{-1}. One finds

$$0 = \mathcal{B}(F_0, F_0),$$

hence F_0 is a local Maxwellian, meaning that there exists $\rho \equiv \rho(t, x) > 0$, $\theta \equiv \theta(t, x) > 0$ and $u \equiv u(t, x) \in \mathbb{R}^3$ such that

$$F_0(t, x, v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v) \quad \text{a.e.}$$

Order ε^0. One finds

$$(\partial_t + v \cdot \nabla_x) F_0 = \mathcal{B}(F_1, F_0) + \mathcal{B}(F_0, F_1)$$

which can be recast in terms of the linearization at the local Maxwellian F_0 of the collision integral (see (3.42)) as

$$\mathcal{L}_{F_0}\left(\frac{F_1}{F_0}\right) = -(\partial_t + v \cdot \nabla_x) \ln F_0. \tag{5.4}$$

This is precisely an integral equation for F_1/F_0 of the form (3.63) studied above.

Let us compute the right-hand side of the above equality, i.e., the expression

$$(\partial_t + v \cdot \nabla_x) \ln F_0.$$

For convenience, we shall denote by V the vector

$$V = \frac{1}{\sqrt{\theta}}(v - u).$$

Then

$$\begin{aligned}
&(\partial_t + v \cdot \nabla_x) \ln \mathcal{M}_{(\rho,u,\theta)} \\
&\quad = \frac{1}{\rho}(\partial_t + v \cdot \nabla_x)\rho - \frac{3}{2}\frac{1}{\theta}(\partial_t + v \cdot \nabla_x)\theta \\
&\qquad + \frac{v-u}{\theta}(\partial_t + v \cdot \nabla_x)u + \frac{|v-u|^2}{2\theta^2}(\partial_t + v \cdot \nabla_x)\theta.
\end{aligned}$$

We shall rearrange the right-hand side of the above equality and express it as a linear combination of the functions

$$1, \qquad V_j, \qquad \frac{1}{2}\big(|V|^2 - 3\big), \qquad A(V)_{kl} \quad \text{and} \quad B(V)_j, \qquad j,k,l = 1,2,3.$$

One finds that

$$\begin{aligned}
&(\partial_t + v \cdot \nabla_x) \ln \mathcal{M}_{(\rho,u,\theta)} \\
&\quad = \frac{1}{\rho}(\partial_t \rho + u \cdot \nabla_x \rho + \rho \operatorname{div}_x u) \\
&\qquad + \frac{1}{\sqrt{\theta}} V \cdot \left(\partial_t u + u \cdot \nabla_x u + \nabla_x \theta + \frac{\theta}{\rho}\nabla_x \rho\right) \\
&\qquad + \frac{1}{2}\big(|V|^2 - 3\big)\frac{1}{\theta}\left(\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3}\theta \operatorname{div}_x u\right) \\
&\qquad + A(V) : \nabla_x u + 2B(V) \cdot \nabla_x \sqrt{\theta}.
\end{aligned} \tag{5.5}$$

Because of the orthogonality relations (3.64), the last two terms on the right-hand side of (5.5) belong to $(\operatorname{Ker}\mathcal{L}_{F_0})^{\perp}$, while the first three terms there are in $\operatorname{Ker}\mathcal{L}_{F_0}$. Therefore, the solvability condition for the Fredholm integral equation (5.4) consists in setting to 0 the coefficients of the functions 1, V_j and $\frac{1}{2}(|V|^2-3)$, i.e.,

$$
\begin{aligned}
&\partial_t\rho + u\cdot\nabla_x\rho + \rho\operatorname{div}_x u = 0,\\
&\partial_t u + u\cdot\nabla_x u + \nabla_x\theta + \frac{\theta}{\rho}\nabla_x\rho = 0,\\
&\partial_t\theta + u\cdot\nabla_x\theta + \frac{2}{3}\theta\operatorname{div}_x u = 0,
\end{aligned}
$$

or in other words,

$$
\begin{aligned}
&\partial_t\rho + \operatorname{div}_x(\rho u) = 0,\\
&\partial_t u + u\cdot\nabla_x u + \frac{1}{\rho}\nabla_x(\rho\theta) = 0,\\
&\partial_t\theta + u\cdot\nabla_x\theta + \frac{2}{3}\theta\operatorname{div}_x u = 0.
\end{aligned}
\tag{5.6}
$$

We recognize the system of Euler equations for a compressible fluid (2.7), in the case of a perfect monatomic gas; i.e., for $\gamma = 0$, so that the pressure law and the internal energy are given respectively by

$$
\text{pressure} = \rho\theta \quad \text{and} \quad \text{internal energy} = \frac{3}{2}\theta.
$$

Assuming that ρ, u and θ satisfy these Euler equations, we solve for F_1 the Fredholm integral equation (5.4) to find

$$
\begin{aligned}
F_1 = {} & -\frac{1}{\rho}F_0\big(\mathbf{a}\big(\theta,|V|\big)A(V)\cdot\nabla_x u + 2\mathbf{b}\big(\theta,|V|\big)B(V)\cdot\nabla_x\sqrt{\theta}\big)\\
& + F_0\left(\frac{\rho_1}{\rho} + \frac{1}{\sqrt{\theta}}V\cdot u_1 + \frac{1}{2}\big(|V|^2-3\big)\frac{\theta_1}{\theta}\right).
\end{aligned}
$$

The second term on the right-hand side of the formula giving F_1 represents the arbitrary element of $\operatorname{Ker}\mathcal{L}_{F_0}$ that appears in the general solution of the integral equation (3.63), while the scalar quantities $\mathbf{a}(\theta, V)$ and $\mathbf{b}(\theta, V)$ satisfy (see (3.67))

$$
\mathcal{L}_{F_0}\big(\mathbf{a}\big(\theta,|V|\big)A(V)\big) = A(V) \quad \text{and} \quad \mathcal{L}_{F_0}\big(\mathbf{b}\big(\theta,|V|\big)B(V)\big) = B(V).
$$

More precisely, let b be the collision kernel satisfying (5.2) considered in this chapter. Applying the notation in Section 3.6.1, and especially (3.44), we define scalar functions $\mathbf{a}(\theta,\cdot)$ and $\mathbf{b}(\theta,\cdot)$ as in (3.67) with a collision kernel $m_{1/\sqrt{\theta}}b$.

Order ε^1. One finds that

$$\partial_t F_1 + v \cdot \nabla_x F_1 - \mathcal{B}(F_1, F_1) = 2\mathcal{B}(F_0, F_2)$$

which can be put in the form

$$\mathcal{L}_{F_0}\left(\frac{F_2}{F_0}\right) = \mathcal{Q}_{F_0}\left(\frac{F_1}{F_0}, \frac{F_1}{F_0}\right) - \frac{1}{F_0}(\partial_t + v \cdot \nabla_x)F_1$$

and this is therefore a Fredholm integral equation of the type (3.63). Here $\mathcal{Q}_{F_0}$ is the Boltzmann collision integral intertwined with the multiplication by the local Maxwellian F_0, i.e.,

$$\mathcal{Q}_{F_0}(\phi, \phi) = F_0^{-1}\mathcal{B}(F_0\phi, F_0\phi).$$

For this equation to have a solution, one must verify the compatibility conditions

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 \,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 \,\mathrm{d}v = 0.$$

These five compatibility conditions are five PDEs for the five unknown functions ρ_1, u_1 and θ_1.

Order ε^n. One finds

$$\partial_t F_n + v \cdot \nabla_x F_n - \sum_{k+l=n, 1\leqslant k,l,\leqslant n} \mathcal{B}(F_k, F_l) = 2\mathcal{B}(F_0, F_{n+1})$$

which is a Fredholm equation of the same type as above.

Here again, the compatibility condition reduces to

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n \,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n \,\mathrm{d}v = 0.$$

More generally, the compatibility condition at order $n + 1$ (to guarantee the existence of F_{n+1}) provides the system of five equations satisfied by that part of F_n which belongs to the nullspace of $\mathcal{L}_{F_0}$. See [114] or Chapter V, Section 2 in [27] for more on Hilbert's expansion.

Conclusion. The Hilbert expansion method shows that the leading order in ε of Hilbert's formal solution (5.3) of the scaled Boltzmann equation (5.1) with Strouhal number $St = 1$ and Knudsen number $Kn = \varepsilon \ll 1$ is a local Maxwellian state whose parameters are governed by the Euler equations of gas dynamics (for a perfect monatomic gas).

In general, one cannot hope that Hilbert's formal power series (5.3) has a positive radius of convergence. Yet, Hilbert's expansion method can be used to obtain a rigorous derivation of the Euler equations of gas dynamics from the Boltzmann equation, as we shall see in Section 9.

5.2. *The compressible Navier–Stokes limit: The Chapman–Enskog expansion*

In this subsection, we shall seek higher-order (in ε) corrections to the compressible Euler system. The Hilbert expansion presented above is not well suited for this purpose, because linear combinations of collision invariants (i.e., hydrodynamic modes) appear at each order in ε instead of being all concentrated in the leading order term.

For that reason, we shall use a slightly different expansion method, the Chapman–Enskog expansion. Thus, we seek a solution of the scaled Boltzmann equation (5.1) as a Chapman–Enskog formal power series,

$$F_\varepsilon(t,x,v) = \sum_{n\geqslant 0} \varepsilon^n F^{(n)}\big[\vec{P}(t,x)\big](v), \tag{5.7}$$

parametrized by the vector $\vec{P}$ of conserved densities of F_ε.

NOTATION. $F^n[\vec{P}(t,x)](v)$ designates a quantity that depends smoothly on $\vec{P}$ and any finite number of its derivatives with respect to the x-variable at the same point (t,x), and on the v-variable.

In particular, $F^n[\vec{P}(t,x)](v)$ does not contain time-derivatives of $\vec{P}$: the Chapman–Enskog method is based on eliminating $\partial_t \vec{P}$ in favor of x-derivatives via conservation laws satisfied by $\vec{P}$.

That $\vec{P}$ is the vector of conserved densities of F_ε means that

$$\begin{aligned}
&\int_{\mathbb{R}^3} F^{(0)}\big[\vec{P}\big](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = \vec{P}, \\
&\int_{\mathbb{R}^3} F^{(n)}\big[\vec{P}\big](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = \vec{0}, \quad n \geqslant 1.
\end{aligned} \tag{5.8}$$

These conserved densities satisfy a formal system of conservation laws of the form

$$\partial_t \vec{P} = \sum_{n\geqslant 0} \varepsilon^n \operatorname{div}_x \Phi^{(n)}\big[\vec{P}\big], \tag{5.9}$$

where the formal fluxes $\Phi^{(n)}$ are obtained from the local conservation laws associated to the Boltzmann equation by the formulas

$$\Phi^{(n)}\big[\vec{P}\big] = -\int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F^{(n)}\big[\vec{P}\big](v)\,\mathrm{d}v \tag{5.10}$$

for $n \geqslant 0$.

Let us analyze the first orders in ε of the Chapman–Enskog expansion.

Order 0. One has

$$\mathcal{B}\big(F^{(0)}\big[\vec{P}\big], F^{(0)}\big[\vec{P}\big]\big) = 0 \quad \text{and thus} \quad F^{(0)}\big[\vec{P}\big] = \mathcal{M}_{(\rho,u,\theta)},$$

here

$$\vec{P} = \begin{pmatrix} \rho \\ \rho u \\ \rho(\frac{1}{2}|u|^2 + \frac{3}{2}\theta) \end{pmatrix}, \qquad \Phi^{(0)}\big[\vec{P}\big] = -\begin{pmatrix} \rho u \\ \rho u^{\otimes 2} + \rho\theta I \\ \rho u(\frac{1}{2}|u|^2 + \frac{5}{2}\theta) \end{pmatrix}.$$

Hence the formal conservation law at order 0 is

$$\partial_t \vec{P} = \operatorname{div}_x \Phi^{(0)}\big[\vec{P}\big] \quad \mathrm{mod}\ \mathrm{O}(\varepsilon)$$

whose explicit form is

$$\begin{aligned} &\partial_t \rho + u \cdot \nabla_x \rho + \rho \operatorname{div}_x u = 0, \\ &\partial_t u + (u \cdot \nabla_x) u + \frac{1}{\rho} \nabla_x(\rho\theta) = 0 \quad \mathrm{mod}\ \mathrm{O}(\varepsilon), \\ &\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3}\theta \operatorname{div}_x u = 0. \end{aligned} \tag{5.11}$$

In other words, the 0th order of the Chapman–Enskog expansion gives the compressible Euler system, as does the Hilbert expansion.

The Hilbert and Chapman–Enskog methods differ at order 1 in ε, as we shall see below.

Order 1. One has

$$(\partial_t + v \cdot \nabla_x) F^{(0)}\big[\vec{P}\big] = 2\mathcal{B}\big(F^{(0)}\big[\vec{P}\big], F^{(1)}\big[\vec{P}\big]\big). \tag{5.12}$$

Using the formal conservation law at order 0, we eliminate $\partial_t F^{(0)}[\vec{P}]$ and replace it with x-derivatives of $F^{(0)}[\vec{P}]$, by using (5.5) and the conservation laws at order 0 (5.11) (i.e., Euler's system) as follows

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x)\mathcal{M}_{(\rho,u,\theta)} \\ &\quad = \mathcal{M}_{(\rho,u,\theta)}\big(A(V) : D(u) + 2B(V) \cdot \nabla_x \sqrt{\theta}\,\big) + \mathrm{O}(\varepsilon) \end{aligned} \tag{5.13}$$

with the notation

$$V = \frac{v-u}{\sqrt{\theta}}, \qquad A(V) = V \otimes V - \frac{1}{3}|V|^2 I, \qquad B(V) = \frac{1}{2} V\big(|V|^2 - 5\big),$$

and where $D(u)$ is the traceless part of the deformation tensor of u

$$D(u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{3}\operatorname{div}_x u I\right).$$

In view of (5.12) and (5.13), $F^{(1)}[\vec{P}]$ is determined by the conditions

$$A(V) : D(u) + 2B(V) \cdot \nabla_x \sqrt{\theta} = -\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}\left(\frac{F^{(1)}[\vec{P}]}{\mathcal{M}_{(\rho,u,\theta)}}\right),$$

$$\int F^{(1)}\big[\vec{P}\big](v)\begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = 0.$$

By Hilbert's lemma, $\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}$ is a Fredholm operator on $L^2(M\,\mathrm{d}v)$; thus

$$F^{(1)}\big[\vec{P}\big](v) = -\mathcal{M}_{(1,u,\theta)}\big(\mathbf{a}\big(\theta, |V|\big)A(V) : D(u) + 2\mathbf{b}\big(\theta, |V|\big)B(V) \cdot \nabla_x \sqrt{\theta}\,\big),$$

where we recall that the scalar functions **a** and **b** are defined by

$$\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}\big(\mathbf{a}\big(\theta, |V|\big)A(V)\big) = A(V) \quad \text{and} \quad \mathbf{a}\big(\theta, |V|\big)A(V) \perp \operatorname{Ker}\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}$$

while

$$\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}\big(\mathbf{b}\big(\theta, |V|\big)B(V)\big) = B(V) \quad \text{and} \quad \mathbf{b}\big(\theta, |V|\big)B(V) \perp \operatorname{Ker}\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}.$$

Hence the first-order correction to the fluxes in the formal conservation law is

$$\Phi^{(1)}\big[\vec{P}\big] = \begin{pmatrix} 0 \\ \mu(\theta)D(u) \\ \mu(\theta)D(u) \cdot u + \kappa(\theta)\nabla_x \theta \end{pmatrix}.$$

Therefore, the formal conservation law at first order is

$$\partial_t \vec{P} = \operatorname{div}_x \Phi^{(0)}\big[\vec{P}\big] + \varepsilon \operatorname{div}_x \Phi^{(1)}\big[\vec{P}\big] \quad \operatorname{mod} \mathrm{O}\big(\varepsilon^2\big),$$

i.e., the compressible Navier–Stokes system with $\mathrm{O}(\varepsilon)$ dissipation terms

$$
\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0,\\
&\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = \varepsilon \operatorname{div}_x\big(\mu D(u)\big) \quad \operatorname{mod} \mathrm{O}\big(\varepsilon^2\big),\\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right)\\
&\quad = \varepsilon \operatorname{div}_x(\kappa \nabla_x \theta) + \varepsilon \operatorname{div}_x\big(\mu D(u)\cdot u\big).
\end{aligned}
\tag{5.14}
$$

The viscosity μ and heat conduction κ are computed as follows

$$
\begin{aligned}
&\theta \int_{\mathbb{R}^3} \mathbf{a}\big(\theta, |V|\big) A_{ij}(V) A_{kl}(V) \mathcal{M}_{(1,u,\theta)}\,\mathrm{d}v = \mu(\theta)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right),\\
&\theta \int_{\mathbb{R}^3} \mathbf{b}\big(\theta, |V|\big) B_i(V) B_j(V) \mathcal{M}_{(1,u,\theta)}\,\mathrm{d}v = \kappa(\theta)\delta_{ij},
\end{aligned}
$$

or in other words,

$$
\begin{aligned}
&\mu(\theta) = \frac{2}{15}\theta \int_0^{+\infty} \mathbf{a}(\theta, r) r^6 \mathrm{e}^{-r^2/2}\frac{\mathrm{d}r}{\sqrt{2\pi}},\\
&\kappa(\theta) = \frac{1}{6}\theta \int_0^{+\infty} \mathbf{b}(\theta, r) r^4 \big(r^2 - 5\big)^2 \mathrm{e}^{-r^2/2}\frac{\mathrm{d}r}{\sqrt{2\pi}}.
\end{aligned}
\tag{5.15}
$$

In the hard sphere case, one finds that the viscosity and heat conduction are of the form

$$
\mu(\theta) = \mu_0\sqrt{\theta}, \qquad \kappa(\theta) = \kappa_0\sqrt{\theta},
$$

where μ_0 and ν_0 are positive constants.

Conclusion. The Chapman–Enskog expansion method shows that the first-order (in ε) correction to the compressible Euler system in the limit of (5.1) – with $St = 1$ and $Kn = \varepsilon \to 0$ – is the compressible Navier–Stokes system (5.14).

Notice that the class of Navier–Stokes systems obtained in this way is by no means the most general: only the equation of state of a perfect monatomic gas can be obtained in this way, as in the case of the compressible Euler limit of the Boltzmann equation. In addition, the viscous dissipation tensor obtained in this way involves *only one* viscosity coefficient instead of two.

Finally, the Chapman–Enskog expansion can be pushed further to obtain higher-order corrections to the compressible Euler equations. For instance, the second-order correction to the compressible Euler equations is a system known as *the Burnett equations*; further corrections have also been computed and are known as *the super-Burnett equations*. See Chapter 5, Section 3 in [27] and Section 25 in [62] for more material on the Chapman–Enskog expansion, as well as for a comparison with Hilbert's expansion.

It should be noted however that these further corrections to the Euler system, beyond the Navier–Stokes system, i.e., the Burnett and super-Burnett equations, are in general not well posed, so that their practical interest in fluid mechanics is unclear. Therefore, we shall not pursue this line of investigation. However, Levermore recently proposed a subtle modification of the Chapman–Enskog expansion which leads to well-posed variants of the Burnett systems [81].

5.3. *The compressible Euler limit: The moment method*

We shall now present a method for deriving hydrodynamic equations from the Boltzmann equation that differs from either the Hilbert or Chapman–Enskog expansions. It consists of passing to the limit as the Knudsen number vanishes in the local conservation laws of mass, momentum and energy that are satisfied by "well-behaved" solutions of the Boltzmann equation. We describe this method on the derivation of the compressible Euler equations from the scaled Boltzmann equation (5.1).

Start from the Cauchy problem for the Boltzmann equation in the periodic box

$$\begin{aligned} &\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3, \\ &F_\varepsilon|_{t=0} = \mathcal{M}_{(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})}. \end{aligned} \tag{5.16}$$

As before, the collision kernel b in the Boltzmann collision integral satisfies the cut-off assumption (5.2).

THEOREM 5.1. *Let* $\rho^{\mathrm{in}} \geqslant 0$ *a.e.*, u^{in} *and* $\theta^{\mathrm{in}} > 0$ *a.e. be such that*

$$\int_{\mathbb{T}^3} \rho^{\mathrm{in}}\big(1 + |u^{\mathrm{in}}|\big)\big(|u^{\mathrm{in}}|^2 + \theta^{\mathrm{in}} + |\ln \rho^{\mathrm{in}}| + |\ln \theta^{\mathrm{in}}|\big)\,\mathrm{d}x < +\infty.$$

For each $\varepsilon > 0$, *let* F_ε *be a solution of* (5.16) *that satisfies the local conservation laws of mass, momentum, and energy, as well as the local entropy relation. Assume that*

$$F_\varepsilon \to F \quad \textit{a.e.}$$

as well as

$$\begin{aligned} &\int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} F\,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big), \\ &\int_{\mathbb{R}^3} vF_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} vF\,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big), \\ &\int_{\mathbb{R}^3} |v|^2F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} |v|^2F\,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big), \end{aligned}$$

while

$$\begin{aligned}
&\int_{\mathbb{R}^3} v \otimes v F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} v \otimes v F \, \mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),\\
&\int_{\mathbb{R}^3} v|v|^2 F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} v|v|^2 F \, \mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),\\
&\int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} F \ln F \, \mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),\\
&\int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} v F \ln F \, \mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),
\end{aligned}$$

as $\varepsilon \to 0$. *Then*

$$F = \mathcal{M}_{(\rho,u,\theta)},$$

where (ρ, u, θ) *is an entropic solution of the compressible Euler system*

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0,\\
&\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = 0,\\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) = 0,
\end{aligned} \tag{5.17}$$

that satisfies the initial condition

$$(\rho, u, \theta)|_{t=0} = \big(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\big). \tag{5.18}$$

PROOF. The moment method involves three steps.

Step 1 (Entropy production bound implies convergence to local equilibrium). The entropy relation in the 3-torus implies the entropy production bound

$$\begin{aligned}
&\frac{1}{4}\int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'_\varepsilon F'_{\varepsilon*} - F_\varepsilon F_{\varepsilon*}\big) \ln\left(\frac{F'_\varepsilon F'_{\varepsilon*}}{F_\varepsilon F_{\varepsilon*}}\right) b \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega\\
&\quad \leqslant \varepsilon H\big(\mathcal{M}_{(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})} \big| \bar{\mathcal{M}}\big),
\end{aligned} \tag{5.19}$$

where $\bar{\mathcal{M}}$ is any global Maxwellian state, for instance, one could choose

$$\bar{\mathcal{M}} = \mathcal{M}_{(\overline{\rho^{\mathrm{in}}}, \overline{u^{\mathrm{in}}}, \overline{\theta^{\mathrm{in}}})},$$

where

$$\overline{\rho^{\mathrm{in}}} = \int_{\mathbb{T}^3} \rho^{\mathrm{in}}(x) \, \mathrm{d}x,$$

$$\overline{u^{\mathrm{in}}} = \frac{1}{\overline{\rho^{\mathrm{in}}}} \int_{\mathbb{T}^3} \rho^{\mathrm{in}} u^{\mathrm{in}}(x)\,\mathrm{d}x,$$

$$\overline{\theta^{\mathrm{in}}} = \frac{1}{\overline{\rho^{\mathrm{in}}}} \int_{\mathbb{T}^3} \rho^{\mathrm{in}} \left(\frac{1}{3} \left| u^{\mathrm{in}} - \overline{u^{\mathrm{in}}} \right|^2 + \theta^{\mathrm{in}} \right)(x)\,\mathrm{d}x.$$

Next we apply Fatou's lemma: assuming that $F_\varepsilon \to F$ a.e. on $\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$, one has

$$\begin{aligned} 0 &\leqslant \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \left(F' F'_* - F F_*\right) \ln\left(\frac{F' F'_*}{F F_*}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \varliminf_{\varepsilon\to 0} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \left(F'_\varepsilon F'_{\varepsilon *} - F_\varepsilon F_{\varepsilon *}\right) \ln\left(\frac{F'_\varepsilon F'_{\varepsilon *}}{F_\varepsilon F_{\varepsilon *}}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\ &= 0. \end{aligned}$$

Hence F is a local Maxwellian, i.e.,

$$F(t,x,v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v)$$

for some $\rho(t,x) \geqslant 0$, $\theta(t,x) > 0$ and $u(t,x) \in \mathbb{R}^3$.

Step 2 (Passing to the limit in the local conservation laws). For each positive ε, the number density F_ε satisfies the local conservation laws recalled below

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v = 0, \\ &\partial_t \int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v\otimes v F_\varepsilon\,\mathrm{d}v = 0, \\ &\partial_t \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \frac{1}{2}|v|^2 F_\varepsilon\,\mathrm{d}v = 0. \end{aligned} \tag{5.20}$$

It follows from our assumptions and Step 1 that

$$\begin{aligned} &\int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} \mathcal{M}_{(\rho,u,\theta)}\,\mathrm{d}v = \rho, \\ &\int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v \mathcal{M}_{(\rho,u,\theta)}\,\mathrm{d}v = \rho u, \\ &\int_{\mathbb{R}^3} v\otimes v F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v\otimes v \mathcal{M}_{(\rho,u,\theta)}\,\mathrm{d}v = \rho(u\otimes u + \theta I), \\ &\int_{\mathbb{R}^3} v|v|^2 F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v|v|^2 \mathcal{M}_{(\rho,u,\theta)}\,\mathrm{d}v = \rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right), \end{aligned}$$

in the limit as $\varepsilon \to 0$. Hence the functions ρ and θ, and the vector field u satisfy the system of PDEs (5.17). It also satisfies the initial condition (5.18) because the convergence of the conserved densities is locally uniform in t.

Step 3 (Passing to the limit in the local entropy relation). Finally, we recall the local entropy relation

$$\partial_t \int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \,\mathrm{d}v = -\text{local entropy production rate} \leqslant 0.$$

It follows from our assumptions that

$$\begin{aligned}\int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \,\mathrm{d}v &\to \int_{\mathbb{R}^3} \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} \,\mathrm{d}v \\ &= \rho \ln\left(\frac{\rho}{\theta^{3/2}}\right) - \frac{3}{2}\big(1+\ln(2\pi)\big)\rho, \\ \int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \,\mathrm{d}v &\to \int_{\mathbb{R}^3} v\mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} \,\mathrm{d}v \\ &= \rho u \ln\left(\frac{\rho}{\theta^{3/2}}\right) - \frac{3}{2}\big(1+\ln(2\pi)\big)\rho u\end{aligned}$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$ as $\varepsilon \to 0^+$, so that, by passing to the limit in the local entropy relation, on account of the continuity equation in (5.17), one arrives at the differential inequality

$$\partial_t\left(\rho \ln\left(\frac{\rho}{\theta^{3/2}}\right)\right) + \operatorname{div}_x\left(\rho u \ln\left(\frac{\rho}{\theta^{3/2}}\right)\right) \leqslant 0. \tag{5.21}$$

In other words, (ρ, u, θ) is a solution of the compressible Euler equations that satisfies the Lax–Friedrichs entropy condition. □

Theorem 5.1 and its proof are taken from [7].

5.4. *The acoustic limit*

Start from the Boltzmann equation with the same scaling as before, but with initial data that are small perturbations of a uniform Maxwellian equilibrium. By Galilean invariance, one can assume without loss of generality that this uniform equilibrium is the centered reduced Gaussian

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2}.$$

Thus, the problem (5.16) reduces to

$$
\begin{aligned}
&\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3, \\
&F_\varepsilon|_{t=0} = \mathcal{M}_{(1+\eta_\varepsilon \rho^{\mathrm{in}}, \eta_\varepsilon u^{\mathrm{in}}, 1+\eta_\varepsilon \theta^{\mathrm{in}})},
\end{aligned}
\tag{5.22}
$$

where $0 < \varepsilon \ll 1$ and $0 < \eta_\varepsilon \ll 1$. The same moment method as above shows that the limiting behavior of the solution to (5.22) under these assumptions is governed by the acoustic system.

THEOREM 5.2. *Assume that ρ^{in}, u^{in} and θ^{in} belong to $L^2(\mathbb{T}^3)$ and that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. For each $\varepsilon > 0$, let F_ε be a solution of* (5.22) *that satisfies the local conservation laws of mass momentum and energy, as well as the local entropy relation.*

Assume that

$$
g_\varepsilon = \frac{F_\varepsilon - M}{\eta_\varepsilon M} \to g \quad \textit{in the sense of distributions},
$$

while

$$
\eta_\varepsilon \mathcal{B}(F_\varepsilon - M, F_\varepsilon - M) \to 0 \quad \textit{in the sense of distributions},
$$

as well as

$$
\begin{aligned}
&\int_{\mathbb{R}^3} g_\varepsilon M \,\mathrm{d}v \to \int_{\mathbb{R}^3} g M \,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'\big(\mathbb{T}^3\big)\big), \\
&\int_{\mathbb{R}^3} v g_\varepsilon M \,\mathrm{d}v \to \int_{\mathbb{R}^3} v g M \,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'\big(\mathbb{T}^3\big)\big), \\
&\int_{\mathbb{R}^3} |v|^2 g_\varepsilon M \,\mathrm{d}v \to \int_{\mathbb{R}^3} |v|^2 g M \,\mathrm{d}v && \textit{in } C\big(\mathbb{R}_+; \mathcal{D}'\big(\mathbb{T}^3\big)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} v \otimes v g_\varepsilon M \,\mathrm{d}v \to \int_{\mathbb{R}^3} v \otimes v g M \,\mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big), \\
&\int_{\mathbb{R}^3} v|v|^2 g_\varepsilon M \,\mathrm{d}v \to \int_{\mathbb{R}^3} v|v|^2 g M \,\mathrm{d}v && \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big)
\end{aligned}
$$

as $\varepsilon \to 0$. Then

$$
g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\big(|v|^2 - 3\big),
$$

where (ρ, u, θ) *is the solution of the acoustic system*

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x u = 0,\\
&\partial_t u + \nabla_x(\rho + \theta) = 0, \qquad (5.23)\\
&\frac{3}{2}\partial_t \theta + \operatorname{div}_x u = 0,
\end{aligned}$$

that satisfies the initial condition

$$(\rho, u, \theta)|_{t=0} = \left(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\right). \qquad (5.24)$$

The proof of this theorem is an easy variant of the formal compressible Euler limit, and is left to the reader. See [52] for the missing details of this formal proof.

6. Incompressible limits of the Boltzmann equation: Formal results

So far, we have discussed various limits of the kinetic theory of gases leading to hydrodynamic models for compressible fluids that satisfy the equation of state of perfect gases. As we shall see in this section, incompressible hydrodynamic models describing *incompressible flows* of perfect gases can also be derived from the Boltzmann equation.

6.1. *The incompressible Navier–Stokes limit*

The scaling on the Boltzmann equation that leads to the incompressible Navier–Stokes equations in the hydrodynamic limit is defined by

$$Kn = St = \varepsilon \ll 1.$$

However, this scaling is not sufficient by itself: as in all long time scalings, one should assume that the length and time scales L and T that enter the definition of St capture the speed of the fluid motion. In other words, situations where

$$\left|u(t,x)\right| \gg \frac{L}{T} \quad \text{must be excluded,}$$

where u is the bulk velocity of the gas, i.e.,

$$u(t,x) = \frac{\int_{\mathbb{R}^3} vF\,\mathrm{d}v}{\int_{\mathbb{R}^3} F\,\mathrm{d}v}.$$

In other words, since $St = (L/T)/\text{speed of sound}$, one must take

$$Ma = \mathrm{O}(St).$$

The case $Ma \sim St$ corresponds to the largest possible velocity field compatible with the above condition, and therefore leads to the Navier–Stokes equation, while the case $Ma = o(\varepsilon)$ leads to the linearized version of the Navier–Stokes equations, i.e., the Stokes equations.

Hence the complete Navier–Stokes scaling is

$$Kn = St = Ma = \varepsilon \ll 1. \tag{6.1}$$

We consider therefore the scaled Boltzmann equation posed on the spatial domain $\mathbb{R}^3$ with uniform Maxwellian equilibrium at infinity – without loss of generality, this Maxwellian equilibrium is assumed to be the centered reduced Gaussian distribution

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

The problem to be studied is therefore

$$\begin{aligned} &\varepsilon\, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3, \\ &F_\varepsilon(t, x, v) \to M \quad \text{as } |x| \to +\infty. \end{aligned} \tag{6.2}$$

That $Ma = O(\varepsilon)$ is seen on the number density F_ε, and not on the Boltzmann equation itself. Here is an example of number density with $Ma = O(\varepsilon)$.

EXAMPLE 1. Take F_ε of the form

$$F_\varepsilon(t, x, v) = \mathcal{M}_{(1,\varepsilon u(t,x),1)}(v).$$

Indeed,

$$\frac{\int_{\mathbb{R}^3} v F_\varepsilon \, dv}{\int_{\mathbb{R}^3} F_\varepsilon \, dv} := \varepsilon u.$$

The speed of sound for the state of the gas described by F_ε is $\sqrt{\frac{5}{3}\theta}$ where

$$\theta(t, x) :\equiv \frac{\int_{\mathbb{R}^3} (1/3)|v - \varepsilon u|^2 F_\varepsilon \, dv}{\int_{\mathbb{R}^3} F_\varepsilon \, dv} = 1.$$

Hence the Mach number for the state of the gas associated to F_ε is

$$\frac{\varepsilon |u(t, x)|}{\sqrt{(5/3)\theta(t, x)}} = O(\varepsilon).$$

Here is another example that also involves fluctuations of temperature.

EXAMPLE 2. Take F_ε of the form

$$F_\varepsilon(t,x,v) = \mathcal{M}_{(1-\varepsilon\theta(t,x),\frac{\varepsilon u(t,x)}{1-\varepsilon\theta(t,x)},\frac{1}{1-\varepsilon\theta(t,x)})}(v)$$

with $\theta \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and $\varepsilon\|\theta\|_{L^\infty} < 1$. One easily checks that

$$\frac{\int_{\mathbb{R}^3} vF_\varepsilon \, \mathrm{d}v}{\int_{\mathbb{R}^3} F_\varepsilon \, \mathrm{d}v} = \varepsilon u + \mathrm{O}\big(\varepsilon^2\big),$$

while

$$\frac{\int_{\mathbb{R}^3}(1/3)|v - \varepsilon u(t,x)/(1-\varepsilon\theta(t,x))|^2 F_\varepsilon \, \mathrm{d}v}{\int_{\mathbb{R}^3} F_\varepsilon \, \mathrm{d}v} = 1 + \varepsilon\theta + \mathrm{O}\big(\varepsilon^2\big).$$

Hence the Mach number is

$$\frac{\varepsilon u + \mathrm{O}(\varepsilon^2)}{\sqrt{(5/3)(1+\varepsilon\theta+\mathrm{O}(\varepsilon^2))}} = \mathrm{O}(\varepsilon).$$

More generally, if F_ε is a number density of the form

$$F_\varepsilon = M(1+\varepsilon g_\varepsilon) \quad \text{such that} \quad g_\varepsilon \geqslant -\frac{1}{\varepsilon} \quad \text{a.e.,} \tag{6.3}$$

one can check that, provided that $\|g_\varepsilon\|_{L^\infty} = \mathrm{O}(1)$, the Mach number for the state of the gas defined by F_ε is $\mathrm{O}(\varepsilon)$.

Hence, we shall supplement the scaled Boltzmann equation (6.2) with the initial condition

$$F_\varepsilon|_{t=0} \equiv \mathcal{M}_{(1-\varepsilon\theta^{\mathrm{in}}(x),\frac{\varepsilon u^{\mathrm{in}}(x)}{1-\varepsilon\theta^{\mathrm{in}}(x)},\frac{1}{1-\varepsilon\theta^{\mathrm{in}}(x)})}(v), \tag{6.4}$$

where

$$\operatorname{div}_x u^{\mathrm{in}} = 0.$$

For each $\varepsilon > 0$, define the number density fluctuation

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$

In terms of the number density fluctuation g_ε, the Boltzmann equation (6.2) takes the form

$$\varepsilon\,\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon = \mathcal{Q}_M(g_\varepsilon, g_\varepsilon), \quad t>0, x, v \in \mathbb{R}^3, \tag{6.5}$$

where $\mathcal{Q}_M$ is the collision integral intertwined with the multiplication by M

$$\mathcal{Q}_M(\phi,\phi) = M^{-1}\mathcal{B}(M\phi, M\phi). \tag{6.6}$$

We shall also need the following notation for moments

$$\langle\phi\rangle = \int_{\mathbb{R}^3} \phi(v)M(v)\,\mathrm{d}v.$$

Hence the local conservation laws of mass momentum and energy satisfied by F_ε take the form

$$\begin{aligned}
&\varepsilon\,\partial_t\langle g_\varepsilon\rangle + \mathrm{div}_x\langle v g_\varepsilon\rangle = 0 && \text{(mass)},\\
&\varepsilon\,\partial_t\langle v g_\varepsilon\rangle + \mathrm{div}_x\langle v\otimes v g_\varepsilon\rangle = 0 && \text{(momentum)},\\
&\varepsilon\,\partial_t\left\langle \frac{1}{2}|v|^2 g_\varepsilon\right\rangle + \mathrm{div}_x\left\langle v\frac{1}{2}|v|^2 g_\varepsilon\right\rangle = 0 && \text{(energy)}.
\end{aligned} \tag{6.7}$$

THEOREM 6.1 (Bardos, Golse and Levermore [8,9]). *For each $\varepsilon > 0$, let F_ε be a solution of* (6.2)–(6.4). *Assume that*

$$\frac{F_\varepsilon - M}{\varepsilon M} \to g \quad \textit{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle v g_\varepsilon\rangle \to \langle v g\rangle \quad \textit{and} \quad \big\langle\big(|v|^2-5\big)g_\varepsilon\big\rangle \to \big\langle\big(|v|^2-5\big)g\big\rangle \quad \textit{in } C\big(\mathbb{R}_+, \mathcal{D}'\big(\mathbb{R}^3\big)\big)$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\begin{aligned}
&\langle v\otimes v g_\varepsilon\rangle \to \langle v\otimes v g\rangle, \qquad \langle B g_\varepsilon\rangle \to \langle B g\rangle,\\
&\big\langle\widehat{A}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon)\big\rangle \to \big\langle\widehat{A}\mathcal{Q}_M(g,g)\big\rangle \quad \textit{and} \quad \big\langle\widehat{A}\otimes v g_\varepsilon\big\rangle \to \big\langle\widehat{A}\otimes v g\big\rangle,\\
&\big\langle\widehat{B}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon)\big\rangle \to \big\langle\widehat{B}\mathcal{Q}_M(g,g)\big\rangle \quad \textit{and} \quad \big\langle\widehat{B}\otimes v g_\varepsilon\big\rangle \to \big\langle\widehat{B}\otimes v g\big\rangle
\end{aligned}$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

Then g is of the form

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 5\big),$$

where (u, θ) *satisfy the incompressible Navier–Stokes–Fourier system*

$$\begin{aligned} &\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \\ &\partial_t \theta + \operatorname{div}_x(u\theta) = \kappa \Delta_x \theta, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \nu &= \frac{1}{10}\langle \widehat{A} : A\rangle = \frac{2}{15}\int_0^{+\infty} \mathbf{a}(1, r) r^6 \mathrm{e}^{-r^2/2} \frac{\mathrm{d}r}{\sqrt{2\pi}}, \\ \kappa &= \frac{2}{15}\langle \widehat{B} \cdot B\rangle = \frac{1}{15}\int_0^{+\infty} \mathbf{b}(1, r) r^2 \big(r^2 - 5\big)^2 \mathrm{e}^{-r^2/2} \frac{\mathrm{d}r}{\sqrt{2\pi}}. \end{aligned} \tag{6.9}$$

We recall that

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I, \qquad B(v) = \frac{1}{2}\big(|v|^2 - 5\big) v, \tag{6.10}$$

and that there exists $\widehat{A}$ and $\widehat{B} \in L^2(\mathbb{R}^3, M\,\mathrm{d}v)$ uniquely determined by

$$\begin{aligned} &\mathcal{L}_M \widehat{A} = A, \qquad \widehat{A} \perp \operatorname{Ker} \mathcal{L}_M, \\ &\mathcal{L}_M \widehat{B} = B, \qquad \widehat{B} \perp \operatorname{Ker} \mathcal{L}_M. \end{aligned} \tag{6.11}$$

Furthermore, there exists two scalar functions **a** and **b** such that

$$\widehat{A}(v) = \mathbf{a}\big(1, |v|\big) A(v) \quad \text{and} \quad \widehat{B}(v) = \mathbf{b}\big(1, |v|\big) B(v). \tag{6.12}$$

PROOF OF THEOREM 6.1. We shall use the moment method, although either the Hilbert or the Chapman–Enskog expansions would also lead to the incompressible Navier–Stokes limit. However, the moment method is the closest to a complete (instead of formal) convergence proof for global solutions without restriction on the size of the initial data.

Step 1 (Asymptotic form of the fluctuations). Multiply the Boltzmann equation (6.5) by ε, so that

$$\varepsilon^2 \partial_t g_\varepsilon + \varepsilon v \cdot \nabla_x g_\varepsilon + \mathcal{L}_M g_\varepsilon = \varepsilon \mathcal{Q}_M(g_\varepsilon, g_\varepsilon).$$

Passing to the limit as $\varepsilon \to 0$ in the above equation, we arrive at

$$\mathcal{L}_M g = 0.$$

Hence, for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $g(t, x, \cdot) \in \operatorname{Ker} \mathcal{L}_M$, which means that g is of the form

$$g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\big(|v|^2 - 3\big). \tag{6.13}$$

Step 2 (Passing to the limit in the local conservation laws). Passing to the limit in (6.7) we arrive at

$$\begin{aligned}&\operatorname{div}_x \langle vg\rangle = 0,\\ &\operatorname{div}_x \langle v\otimes vg\rangle = 0,\\ &\operatorname{div}_x \big\langle v|v|^2 g\big\rangle = 0.\end{aligned}$$

Together with the relation (6.13), the first and the last relation reduce to the incompressibility condition for u

$$\operatorname{div}_x u = 0. \tag{6.14}$$

Together with the relation (6.13), the second relation gives

$$\nabla_x(\rho+\theta) = 0.$$

Since $g \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3, M\,\mathrm{d}v))$, this implies the Boussinesq relation

$$\rho + \theta = 0. \tag{6.15}$$

With this last relation, the asymptotic form of g becomes

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 5\big). \tag{6.16}$$

It remains to derive the motion and heat equations from the local conservation laws. To do this, we recast the local conservation law of momentum as

$$\partial_t\langle vg_\varepsilon\rangle + \operatorname{div}_x \frac{1}{\varepsilon}\langle Ag_\varepsilon\rangle + \nabla_x \frac{1}{\varepsilon}\left\langle \frac{1}{3}|v|^2 g_\varepsilon\right\rangle = 0,$$

and we combine the local conservation laws of mass and energy into

$$\partial_t\left\langle \frac{1}{2}\big(|v|^2-5\big)g_\varepsilon\right\rangle + \operatorname{div}_x \frac{1}{\varepsilon}\langle Bg_\varepsilon\rangle = 0.$$

One easily checks with (6.16) that

$$\begin{aligned}&\langle vg_\varepsilon\rangle \to \langle vg\rangle = u,\\ &\left\langle \frac{1}{2}\big(|v|^2-5\big)g_\varepsilon\right\rangle \to \left\langle \frac{1}{2}\big(|v|^2-5\big)g\right\rangle = \frac{5}{2}\theta\end{aligned} \tag{6.17}$$

in $C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^3))$ as $\varepsilon \to 0$.

Next we pass to the limit in the flux terms. Since $\mathcal{L}_M$ is self-adjoint, one has

$$\frac{1}{\varepsilon}\langle A g_\varepsilon\rangle = \frac{1}{\varepsilon}\big\langle\big(\mathcal{L}_M \widehat{A}\big) g_\varepsilon\big\rangle = \left\langle \widehat{A}\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon\right\rangle.$$

Then we eliminate the term $\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon$ with (6.5)

$$\left\langle \widehat{A}\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon\right\rangle = \big\langle \widehat{A}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon)\big\rangle - \big\langle \widehat{A}(\varepsilon\partial_t + v\cdot\nabla_x) g_\varepsilon\big\rangle$$

so that, passing to the limit as $\varepsilon \to 0$ leads to

$$\frac{1}{\varepsilon}\langle A g_\varepsilon\rangle \to \big\langle \widehat{A}\mathcal{Q}_M(g, g)\big\rangle - \big\langle \widehat{A} v\cdot\nabla_x g\big\rangle \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big). \tag{6.18}$$

Likewise

$$\frac{1}{\varepsilon}\langle B g_\varepsilon\rangle \to \big\langle \widehat{B}\mathcal{Q}_M(g, g)\big\rangle - \big\langle \widehat{B} v\cdot\nabla_x g\big\rangle \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big), \tag{6.19}$$

as $\varepsilon \to 0$.

With (6.16), one easily finds that

$$\begin{aligned} \big\langle \widehat{A} v\cdot\nabla_x g\big\rangle &= \nu\left(\nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{3}(\mathrm{div}_x\, u) I\right), \\ \big\langle \widehat{B} v\cdot\nabla_x g\big\rangle &= \frac{5}{2}\kappa\nabla_x\theta, \end{aligned} \tag{6.20}$$

where ν and κ are given by (6.9).

The nonlinear term is slightly more difficult. Its computation involves in particular the following classical lemma.

LEMMA 6.2. *Let $\phi \in \operatorname{Ker}\mathcal{L}_M$, then*

$$\mathcal{Q}_M(\phi, \phi) = \frac{1}{2}\mathcal{L}_M\big(\phi^2\big).$$

PROOF. Differentiate twice the relation

$$\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}) = 0$$

with respect to ρ, u and θ. See, for instance, [25] or [9] for the missing details. $\square$

Then

$$\big\langle \widehat{A}\mathcal{Q}_M(g, g)\big\rangle = \frac{1}{2}\big\langle \widehat{A}\mathcal{L}_M\big(g^2\big)\big\rangle = \frac{1}{2}\big\langle\big(\mathcal{L}_M \widehat{A}\big) g^2\big\rangle = \frac{1}{2}\big\langle A g^2\big\rangle = u\otimes u - \frac{1}{3}|u|^2 I, \tag{6.21}$$

and likewise

$$\left\langle \widehat{B}\mathcal{Q}_M(g,g)\right\rangle = \frac{1}{2}\left\langle \widehat{B}\mathcal{L}_M\left(g^2\right)\right\rangle = \frac{1}{2}\left\langle \left(\mathcal{L}_M\widehat{B}\right)g^2\right\rangle = \frac{1}{2}\left\langle Bg^2\right\rangle = \frac{5}{2}u\theta. \tag{6.22}$$

Observe that

$$\mathrm{div}_x\left(u\otimes u - \frac{1}{3}|u|^2 I\right) = \mathrm{div}_x(u\otimes u) - \frac{1}{3}\nabla_x|u|^2,$$

while

$$\begin{aligned}\mathrm{div}_x\left(\nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{3}(\mathrm{div}_x\, u)I\right) &= \Delta_x u + \nabla_x(\mathrm{div}_x\, u) - \frac{2}{3}\nabla_x(\mathrm{div}_x\, u)\\ &= \Delta_x u,\end{aligned}$$

because of the divergence-free condition on u.

Gathering (6.17)–(6.22), we arrive at

$$\begin{aligned}&\partial_t u + \mathrm{div}_x(u\otimes u) - \nu\Delta_x u = 0 \quad \text{modulo gradients},\\ &\partial_t\theta + \mathrm{div}_x(u\theta) - \kappa\Delta_x\theta = 0.\end{aligned}$$

□

6.2. *The incompressible Stokes and Euler limits*

By the same moment method, one can derive other incompressible models from the Boltzmann equation. We just state the results below without giving the proofs (which are anyway simpler than that of the Navier–Stokes limit).

6.2.1. *The Stokes limit.* The Stokes–Fourier system is the linearization about $u = 0$ and $\theta = 0$ of the Navier–Stokes–Fourier system. Thus, in order to derive the Stokes–Fourier system from the Boltzmann equation, one keeps the same scaling as for the incompressible Navier–Stokes limit on the Boltzmann equation, i.e.,

$$Kn = St = \varepsilon \ll 1$$

and one scales the Mach number as

$$Ma = \eta_\varepsilon = \mathrm{o}(\varepsilon).$$

In other words, we start from the following Cauchy problem

$$\begin{aligned}&\varepsilon\,\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t,x,v)\in\mathbb{R}_+^*\times\mathbb{R}^3\times\mathbb{R}^3,\\ &F_\varepsilon(t,x,v)\to M \quad \text{as } |x|\to+\infty,\end{aligned} \tag{6.23}$$

with initial condition

$$F_\varepsilon|_{t=0} = \mathcal{M}_{(1-\eta_\varepsilon\theta^{\mathrm{in}}, \frac{\eta_\varepsilon u^{\mathrm{in}}}{1-\eta_\varepsilon\theta^{\mathrm{in}}}, \frac{1}{1-\eta_\varepsilon\theta^{\mathrm{in}}})}. \tag{6.24}$$

In this subsection, the number density fluctuation g_ε is defined to be

$$g_\varepsilon = \frac{F_\varepsilon - M}{\eta_\varepsilon M}.$$

THEOREM 6.3 (Bardos, Golse and Levermore [9]). *For each $\varepsilon > 0$, let F_ε be a solution of* (6.23)–(6.24). *Assume that*

$$\frac{F_\varepsilon - M}{\eta_\varepsilon M} \to g \quad \textit{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle v g_\varepsilon\rangle \to \langle v g\rangle \quad \textit{and} \quad \big\langle\big(|v|^2 - 5\big)g_\varepsilon\big\rangle \to \big\langle\big(|v|^2 - 5\big)g\big\rangle \quad \textit{in } C\big(\mathbb{R}_+, \mathcal{D}'\big(\mathbb{R}^3\big)\big)$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\langle v\otimes v g_\varepsilon\rangle \to \langle v\otimes v g\rangle, \qquad \langle B g_\varepsilon\rangle \to \langle B g\rangle,$$
$$\big\langle \widehat{A}\otimes v g_\varepsilon\big\rangle \to \big\langle \widehat{A}\otimes v g\big\rangle \quad \textit{and} \quad \big\langle \widehat{B}\otimes v g_\varepsilon\big\rangle \to \big\langle \widehat{B}\otimes v g\big\rangle$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

Then g is of the form

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 5\big),$$

where (u,θ) satisfy the incompressible Stokes–Fourier system

$$\begin{aligned} &\partial_t u + \nabla_x p = \nu\Delta_x u, \qquad \operatorname{div}_x u = 0,\\ &\partial_t\theta = \kappa\Delta_x\theta, \end{aligned} \tag{6.25}$$

with ν and κ given by formula (6.9).

6.2.2. *The incompressible Euler limit.* The incompressible Euler equations are formally the inviscid limit of the Navier–Stokes equations. Thus, in order to derive the incompressible Euler equations from the Boltzmann equation, one chooses a scaling that increases the strength of the nonlinear term. In other words, one takes

$$St = Ma = \varepsilon \quad \text{while} \quad Kn = \eta_\varepsilon = \mathrm{o}(\varepsilon).$$

In fact, this is consistent with the von Karman relation, which relates the Mach, Knudsen and Reynolds numbers as follows

$$Kn = \frac{Ma}{Re}. \tag{6.26}$$

Indeed, one gets $Re = \varepsilon/\eta_\varepsilon \to +\infty$ as $\varepsilon \to 0$: therefore, the limiting equation obtained in this way is incompressible (since $Ma \to 0$) and inviscid (since $Re \to +\infty$).

In other words, we start from the following Cauchy problem

$$\begin{aligned} &\varepsilon\, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\eta_\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3, \\ &F_\varepsilon(t, x, v) \to M \quad \text{as } |x| \to +\infty, \end{aligned} \tag{6.27}$$

with initial condition

$$F_\varepsilon|_{t=0} = \mathcal{M}_{(1-\varepsilon\theta^{\mathrm{in}}, \frac{\varepsilon u^{\mathrm{in}}}{1-\varepsilon\theta^{\mathrm{in}}}, \frac{1}{1-\varepsilon\theta^{\mathrm{in}}})}. \tag{6.28}$$

In this subsection, the number density fluctuation g_ε is defined to be

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$

THEOREM 6.4 (Bardos, Golse and Levermore [9]). *For each $\varepsilon > 0$, let F_ε be a solution of* (6.27)–(6.28). *Assume that*

$$\frac{F_\varepsilon - M}{\varepsilon M} \to g \quad \textit{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle v g_\varepsilon \rangle \to \langle v g \rangle \quad \textit{and} \quad \big\langle \big(|v|^2 - 5\big) g_\varepsilon \big\rangle \to \big\langle \big(|v|^2 - 5\big) g \big\rangle \quad \textit{in } C\big(\mathbb{R}_+, \mathcal{D}'\big(\mathbb{R}^3\big)\big)$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \textit{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\langle v \otimes v g_\varepsilon \rangle \to \langle v \otimes v g \rangle, \qquad \langle B g_\varepsilon \rangle \to \langle B g \rangle,$$
$$\big\langle \widehat{A} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \big\rangle \to \big\langle \widehat{A} \mathcal{Q}_M(g, g) \big\rangle \quad \text{and} \quad \big\langle \widehat{B} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \big\rangle \to \big\langle \widehat{B} \mathcal{Q}_M(g, g) \big\rangle$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

Then g is of the form

$$g(t, x, v) = u(t, x) \cdot v + \theta(t, x) \frac{1}{2} \big(|v|^2 - 5\big),$$

where (u, θ) satisfy the system

$$\begin{aligned} &\partial_t u + \operatorname{div}_x (u \otimes u) + \nabla_x p = 0, \qquad \operatorname{div}_x u = 0, \\ &\partial_t \theta + \operatorname{div}_x (u\theta) = 0. \end{aligned} \tag{6.29}$$

6.3. *Other incompressible models*

There are many possible variants of the incompressible Navier–Stokes–Fourier limit described above. To begin with, it is possible to include a conservative force in the Boltzmann equation. The scaling is as follows. Start from equation

$$\varepsilon\, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon - \varepsilon \nabla_x \phi(x) \cdot \nabla_v F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon),$$

where $\phi \equiv \phi(x)$ is a given (smooth) potential. Writing the (x, v)-derivative in the Boltzmann equation above as a Poisson bracket, i.e.,

$$v \cdot \nabla_x F_\varepsilon - \varepsilon \nabla_x \phi(x) \cdot \nabla_v F_\varepsilon = \left\{ \frac{1}{2} |v|^2 + \varepsilon \phi(x);\, F_\varepsilon \right\}$$

suggests to seek the solution F_ε in the form

$$F_\varepsilon(t, x, v) = \mathrm{e}^{-\varepsilon \phi(x)} M(v) \big(1 + \varepsilon g_\varepsilon(t, x, v)\big).$$

Indeed,

$$\mathrm{e}^{-\varepsilon \phi(x)} M(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2 - \varepsilon \phi(x)}$$

is both an element of the nullspace of the Poisson bracket $\{\frac{1}{2}|v|^2 + \varepsilon \phi(x); \cdot\}$ and a Maxwellian.

The same formal argument as above shows that

$$g_\varepsilon(t,x,v) \to u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 5\big),$$

where u and θ satisfy

$$\begin{aligned}
&\partial_t u + \mathrm{div}_x(u\otimes u) + \nabla_x p - \theta\nabla_x\phi = \nu\Delta_x u,\\
&\partial_t\theta + \mathrm{div}_x(u\theta) + \frac{2}{5}u\cdot\nabla_x\phi = \kappa\Delta_x\theta,
\end{aligned}$$

with

$$\nu = \frac{1}{10}\big\langle \widehat{A} : A\big\rangle \quad \text{and} \quad \kappa = \frac{2}{15}\big\langle \widehat{B}\cdot B\big\rangle$$

as in the incompressible Navier–Stokes–Fourier limit theorem above. Of course, this is in agreement with the discussion in Section 2.7. We refer the interested reader to [8] for more details on the formal derivation.

In still another variant of the Navier–Stokes–Fourier limit presented above, it is possible to recover viscous heating terms as in Section 2.6. As explained in that subsection, viscous heating terms should appear when the fluctuations of velocity field are of the order of the square root of temperature fluctuations. In [13], Bardos and Levermore used the following very elegant approach: start from the Boltzmann equation in the Navier–Stokes scaling (6.2) and seek the number density F_ε as

$$F_\varepsilon(t,x,v) = M(v)\big(1 + \varepsilon g_\varepsilon^-(t,x,v) + \varepsilon^2 g_\varepsilon^+(t,x,v)\big),$$

where g_ε^- is odd in v while g_ε^+ is even in v. Because the Boltzmann collision integral is rotation invariant (see Section 3.6.2 and especially (3.51)),

$\mathcal{B}(\Phi^+,\Phi^+)$ and $\mathcal{B}(\Phi^-,\Phi^-)$ are even in v, while
$\mathcal{B}(\Phi^-,\Phi^+)$ and $\mathcal{B}(\Phi^+,\Phi^-)$ are odd in v.

Bardos and Levermore gave a formal argument showing that

$$g_\varepsilon^-(t,x,v) \to u(t,x)\cdot v,$$

while

$$\begin{aligned}
g_\varepsilon^+(t,x,v) \to{}& \rho(t,x) + \frac{1}{2}\big(\big|u(t,x)\big|^2 + 3\theta(t,x)\big)\left(\frac{1}{3}|v|^2 - 1\right)\\
&+ \frac{1}{2}A : u(t,x)\otimes u(t,x) - \widehat{A} : \nabla_x u(t,x),
\end{aligned}$$

where ρ, u and θ satisfy the relations

$$\operatorname{div}_x u = 0, \qquad p = \rho + \theta,$$
$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u,$$
$$\partial_t \left(\frac{5}{2}\theta + \frac{1}{2}|u|^2 - p \right) + \operatorname{div}_x \left(u \left(\frac{5}{2}\theta + \frac{1}{2}|u|^2 \right) \right)$$
$$= \frac{5}{2}\kappa \Delta_x \theta + \mu \operatorname{div}_x \big(\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}} \big) \cdot u \big).$$

In the system above, ν and κ are given by the same formulas as in the Navier–Stokes–Fourier limit theorem, i.e., (6.9).

7. Mathematical theory of the Cauchy problem for hydrodynamic models

In this section we have gathered a few mathematical results bearing on the various hydrodynamic models that appear as limits of the Boltzmann equation. We shall leave aside the compressible Navier–Stokes system, since its derivation from the Boltzmann equation leads to dissipation terms that are of the order of the Knudsen number, and therefore vanish in the hydrodynamic limit. Put in other words, the compressible Navier–Stokes system is an asymptotic expansion of the Boltzmann equation in the Knudsen number, and not a limit thereof. Hence, a mathematical treatment of this limit would involve existence results on the compressible Navier–Stokes system that are uniform in the Reynolds and Péclet numbers, which is beyond current knowledge on this model at the time of this writing.

7.1. *The Stokes and acoustic systems*

We begin with the simplest hydrodynamic models

- the Stokes–Fourier system, and
- the acoustic system,

which are variants of the heat and the wave equations.

7.1.1. *The Stokes–Fourier system.* Consider the Stokes–Fourier system

$$\partial_t u + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^3,$$
$$\partial_t \theta = \kappa \Delta_x \theta,$$
$$(u, \theta)|_{t=0} = \big(u^{\mathrm{in}}, \theta^{\mathrm{in}}\big),$$

where ν and $\kappa > 0$.

THEOREM 7.1. *For each* $(u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathbb{R}^3)$ *such that* $\operatorname{div}_x u^{\mathrm{in}} = 0$, *there exists a unique solution* (u, θ) *of the Stokes–Fourier system such that*

$$(u, \theta) \in C\big(\mathbb{R}_+; L^2\big(\mathbb{R}^3\big)\big) \quad \textit{and} \quad p \in C\big(\mathbb{R}_+^*; L^2\big(\mathbb{R}^3\big)/\mathbb{R}\big).$$

In addition, the pressure p is constant and

$$(u,\theta) \in C^\infty\big(\mathbb{R}_+^* \times \mathbb{R}^3\big).$$

PROOF. Applying the divergence to both sides of the motion equation gives

$$\Delta_x p = 0;$$

since for each $t > 0$ the function $p(t,\cdot) \in L^2(\mathbb{R}^3)/\mathbb{R}$ is harmonic, it is a constant in variable x. Since the divergence operator commutes with the heat operator on $\mathbb{R}^3$, the Stokes–Fourier system above reduces to a system of uncoupled heat equations, whence the announced result. □

7.1.2. *The acoustic system.* Consider the acoustic system

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x u = 0,\\
&\partial_t u + \nabla_x(\rho+\theta) = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3,\\
&\frac{3}{2}\,\partial_t\theta + \operatorname{div}_x u = 0,\\
&(\rho,u,\theta)|_{t=0} = \big(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\big).
\end{aligned}$$

THEOREM 7.2. *For each $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathbb{R}^3)$, there exists a unique solution (ρ,u,θ) of the acoustic system such that*

$$(\rho,u,\theta) \in C\big(\mathbb{R}_+; L^2\big(\mathbb{R}^3\big)\big).$$

PROOF. Applying the Helmholtz decomposition to $u(t,\cdot)$ gives

$$u(t,\cdot) = u_s(t,\cdot) - \nabla_x\phi(t,\cdot), \qquad \operatorname{div}_x u_s = 0.$$

Hence the acoustic system becomes

$$\begin{aligned}
&\partial_t\rho - \Delta_x\phi = 0, \qquad && \frac{3}{2}\,\partial_t\theta - \Delta_x\phi = 0,\\
&\partial_t\phi - \rho - \theta = 0, \qquad && \partial_t u_s = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\partial_t\phi - \rho - \theta = 0, \qquad && \partial_t(\rho+\theta) - \frac{5}{3}\Delta_x\phi = 0,\\
&\partial_t\left(\rho - \frac{3}{2}\theta\right) = 0.
\end{aligned}$$

Therefore, letting $\psi = \rho + \theta$, we arrive at

$$\partial_{tt}\phi - \frac{5}{3}\Delta_x \phi = 0,$$
$$\partial_{tt}\psi - \frac{5}{3}\Delta_x \psi = 0.$$

This is a system of two uncoupled wave equations; then, (ρ, u, θ) is reconstructed from ψ and $\mu \equiv \mu(x) = \rho(t,x) - \frac{3}{2}\theta(t,x)$ by the formulas

$$\rho(t,x) = \frac{3}{5}\psi(t,x) + \frac{2}{5}\mu(x),$$
$$\theta(t,x) = \frac{2}{5}\big(\psi(t,x) - \mu(x)\big),$$

while

$$u(t,x) = u_{\mathrm{s}}(0,x) - \nabla_x \phi(t,x).$$

Applying the classical theory of the Cauchy problem for the wave equations satisfied by ϕ and ψ leads to the announced result. $\square$

7.2. *The incompressible Navier–Stokes equations*

Next, we consider the incompressible Navier–Stokes equations. The mathematical theory of the Cauchy problem for these equations was developed by J. Leray in the early 1930s. Here is a quick summary of his results, see for instance [30,45,85] for more information on this subject.

Consider the Navier–Stokes equations

$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^D,$$
$$u|_{t=0} = u^{\mathrm{in}},$$

where $\nu > 0$.

We begin with the three-dimensional case.

THEOREM 7.3 (Leray [79]). *For each $u^{\mathrm{in}} \in L^2(\mathbb{R}^3)$ such that $\operatorname{div}_x u^{\mathrm{in}} = 0$, there exists a solution in the sense of distributions to the Cauchy problem for the Navier–Stokes equations such that*

$$u \in C\big(\mathbb{R}_+; w - L^2(\mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}_+; H^1(\mathbb{R}^3)\big).$$

Moreover, the function $t \mapsto \|u(t,\cdot)\|_{L^2}$ *is nonincreasing on* $\mathbb{R}_+$ *and satisfies, for each* $t > 0$, *the Leray energy inequality*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu \left|\nabla_x u(s,x)\right|^2 \mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{2}\|u^{\mathrm{in}}\|_{L^2}^2.$$

A solution of the Navier–Stokes equations in the sense of distributions that belongs to $C(\mathbb{R}_+; w - L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{R}^3))$ and satisfies the Leray energy inequality is called a "Leray solution".

In fact, a modification of Leray's original argument allows constructing weak solutions that satisfy the *local* energy inequality

$$\partial_t \frac{1}{2}|u|^2 + \mathrm{div}_x\left(u\left(\frac{1}{2}|u|^2 + p\right)\right) + \nu|\nabla_x u|^2 \leqslant \nu \Delta_x\left(\frac{1}{2}|u|^2\right) \tag{7.1}$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

It is not known whether Leray solutions are uniquely determined by their initial data; however, Leray was able to prove that regular solutions of the Navier–Stokes equations are unique within the class of Leray solutions.

THEOREM 7.4 (Leray [79]). *Let* $u^{\mathrm{in}} \in L^2(\mathbb{R}^3)$ *such that* $\mathrm{div}_x\, u^{\mathrm{in}} = 0$. *Assume that there exists a classical solution,*

$$v \in C^1\left(\mathbb{R}_+; H^1\left(\mathbb{R}^3\right)\right) \cap C\left(\mathbb{R}_+; H^2\left(\mathbb{R}^3\right)\right) \quad \textit{such that} \quad \nabla_x v \in L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^3\right),$$

of the Navier–Stokes equations with initial condition

$$v|_{t=0} = u^{\mathrm{in}}.$$

Then, any Leray solution u *of the Navier–Stokes equations with initial data*

$$u|_{t=0} = u^{\mathrm{in}}$$

coincides with v *a.e.*

Whether the space dimension is $D = 2$ or $D = 3$ leads to fundamental differences in the regularity theory for the Navier–Stokes equations.

THEOREM 7.5 (Leray [78]). *Let* $u^{\mathrm{in}} \in L^2(\mathbb{R}^2)$ *such that* $\mathrm{div}_x\, u^{\mathrm{in}} = 0$. *Then there exists a unique weak solution* u *to the Navier–Stokes equations with initial data*

$$u|_{t=0} = u^{\mathrm{in}}$$

such that

$$u \in C\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right) \cap L^2\left(\mathbb{R}_+; H^1\left(\mathbb{R}^2\right)\right).$$

Furthermore, this solution is smooth for $t > 0$

$$u \in C^\infty\big(\mathbb{R}_+^* \times \mathbb{R}^2\big)$$

and satisfies the energy equality

$$\frac{1}{2}\big\|u(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu\big|\nabla_x u(s,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\big\|u^{\mathrm{in}}\big\|_{L^2}^2$$

for each $t \geqslant 0$.

In space dimension 3, it is an outstanding open problem to determine whether Leray solutions with C^∞ initial data remain C^∞ for all times.

What is known to this date is the following partial regularity theorem which improves on an earlier result by Scheffer [111].

THEOREM 7.6 (Caffarelli, Kohn and Nirenberg [21]). *In space dimension* 3, *let* u *be a Leray solution of the Navier–Stokes equations that satisfies the local variant of Leray's energy inequality* (7.1). *Let the singular set of* u *be*

$$S(u) = \big\{(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3 \,\big|\, u \text{ is not bounded in a neighborhood of } (t,x)\big\}.$$

Then, $S(u)$ *has parabolic Hausdorff dimension less than* 1.

This definition of the singular set $S(u)$ comes from a bootstrap argument due to Serrin [112] showing that, if a Leray solution u of the Navier–Stokes equations is bounded in $B((t,x), R)$, then u is C^∞ in $B((t,x), R/2)$.

The parabolic Hausdorff dimension is defined through coverings with translates of $(-r^2, r^2) \times B(0,r)$ in $\mathbb{R} \times \mathbb{R}^3$ (the usual Hausdorff dimension being defined with balls for the Euclidean metric of $\mathbb{R}^4$).

This result implies that the singular set $S(u)$ must be smaller than a curve in space–time: in other words singularities of solutions to the Navier–Stokes equations in space dimension 3 are rare (if they exist at all).

7.3. *The incompressible Navier–Stokes–Fourier system*

By mimicking the compactness method in the proof of the Leray existence theorem, we can also treat the case of the Navier–Stokes–Fourier system

$$\begin{aligned}
&\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu\Delta_x u, \qquad \mathrm{div}_x\, u = 0, \quad && (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3,\\
&\partial_t \theta + \mathrm{div}_x(u\theta) = \kappa\Delta_x\theta, && (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3,\\
&(u,\theta)|_{t=0} = \big(u^{\mathrm{in}}, \theta^{\mathrm{in}}\big),
\end{aligned}$$

where $\kappa > 0$ and $\nu > 0$.

The analogue of Leray's theorem for the Navier–Stokes–Fourier system is the following one.

THEOREM 7.7. *For each* $(u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathbb{R}^3)$ *such that* $\mathrm{div}_x\, u^{\mathrm{in}} = 0$, *there exists a solution in the sense of distributions to the Cauchy problem for the Navier–Stokes–Fourier system such that*

$$(u, \theta) \in C\big(\mathbb{R}_+; w - L^2\big(\mathbb{R}^3\big)\big) \cap L^2\big(\mathbb{R}_+; H^1\big(\mathbb{R}^3\big)\big).$$

This solution satisfies, for each $t > 0$

$$\frac{1}{2}\big\|u(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu \big|\nabla_x u(s,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{2}\big\|u^{\mathrm{in}}\big\|_{L^2}^2,$$
$$\frac{1}{2}\big\|\theta(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \kappa \big|\nabla_x \theta(s,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{2}\big\|\theta^{\mathrm{in}}\big\|_{L^2}^2.$$

7.4. *The compressible Euler system*

The compressible Euler system is a quasilinear hyperbolic system. The existence and uniqueness theory for this system is not entirely satisfying in its present state, especially in space dimension greater than or equal to 2. More information on the theory of hyperbolic system of conservation laws can be found for instance in [32,77] and [19].

Consider the Cauchy problem for the compressible Euler system (with perfect monatomic gas equation of state)

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}_x(\rho u) = 0,\\
&\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = 0,\\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) = 0.
\end{aligned} \tag{7.2}$$

We begin with a local existence and uniqueness result which is a particular case of a general theorem on quasilinear *symmetrizable systems*. The theory of symmetric hyperbolic system was developed very early by Friedrichs; the importance of the notion of symmetrizable systems was recognized by Godunov [51], and then by Friedrichs and Lax [43] – see also [80] for more information on the theory of hyperbolic systems. The result below comes from [90]; the case of general systems is studied in [77] and [71].

THEOREM 7.8. *Let* $D \geqslant 1$ *and* $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in H^m(\mathbb{R}^D)$ *with* $m > D/2 + 1$. *There exists* $T > 0$ *and a unique solution* $(\rho, u, \theta) \in C([0,T); H^m(\mathbb{R}^D)) \cap C^1([0,T); H^{m-1}(\mathbb{R}^D))$ *of the compressible Euler system in the sense of distributions on* $(0,T) \times \mathbb{R}^D$.

Such solutions are regular (since $H^m(\mathbb{R}^D) \subset C^{[m-D/2]}(\mathbb{R}^D)$ by Sobolev embedding). In general, one does not expect that regular solutions to (7.2) may exist for all times. Singularities – such as shock waves – are expected to appear in finite time for a large class of smooth initial data. Here is an interesting result in this direction, due to Sideris [113].

THEOREM 7.9. *Let $D = 3$, and let $R > 0$. Pick the initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in C^\infty(\mathbb{R}^3)$ to be such that $\rho^{\text{in}}, \theta^{\text{in}} > 0$ on $\mathbb{R}^3$ with*

$$\operatorname{supp}\big(\rho^{\text{in}} - 1\big) \subset B(0, R), \qquad \operatorname{supp}\big(u^{\text{in}}\big) \subset B(0, R),$$
$$\operatorname{supp}\big(\theta^{\text{in}} - 1\big) \subset B(0, R) \quad \textit{and} \quad \rho^{\text{in}} \geqslant \big(\theta^{\text{in}}\big)^{3/2} \quad \textit{on } \mathbb{R}^3.$$

Assume further the existence of $R_0 \in (0, R)$ such that

$$\int_{|x|>r} \frac{(|x| - r)^2}{|x|} \big(\rho^{\text{in}}(x) - 1\big)\,\mathrm{d}x > 0$$

and

$$\int_{|x|>r} \frac{(|x|^2 - r^2)}{|x|^3} \rho^{\text{in}}(x) x \cdot u^{\text{in}}(x)\,\mathrm{d}x \geqslant 0$$

for each $r \in (R_0, R)$. Then the life-span T of the C^1 solution to (7.2) with such initial data is finite.

In dimension greater than or equal to 2, there is no satisfying theory of weak solutions that could extend classical solutions after blow-up time.

At variance, in the one-dimensional case, there is a rather complete theory of weak solutions, that are obtained as superpositions of interacting Riemann problems (i.e., a Cauchy problem for (5.6) whose initial data is a step function with only one jump). Liu studied the compressible Euler system in space dimension 1 written in Lagrangian coordinates. Denoting by a the Lagrangian particle label, this system reads

$$\begin{aligned}
&\partial_t V - \partial_a U = 0,\\
&\partial_t U + \partial_a \left(\frac{\Theta}{V}\right) = 0,\\
&\partial_t \left(\frac{1}{2}U^2 + \frac{3}{2}\Theta\right) + \partial_a \left(\frac{\Theta U}{V}\right) = 0,\\
&(V, U, \Theta)|_{t=0} = \big(V^{\text{in}}, U^{\text{in}}, \Theta^{\text{in}}\big),
\end{aligned} \tag{7.3}$$

with the notation

$$V(t, a) = \frac{1}{\rho(t, X(t, 0, a))},$$

$$U(t,a) = u\big(t, X(t,0,a)\big),$$
$$\Theta(t,a) = \theta\big(t, X(t,0,a)\big),$$

where $t \mapsto X(t,0,a)$ is the path of the particle which is at the position a at time $t = 0$.

The following existence result is based on Glimm's algorithm [50] for constructing BV solutions to hyperbolic systems of conservation laws in the one-dimensional case that are global in time for initial data of small total variation.

THEOREM 7.10 (Liu [88]). *Assume that* $(V^{\text{in}}, U^{\text{in}}, \Theta^{\text{in}}) \in BV(\mathbb{R})$ *with* $\Theta^{\text{in}} \geqslant \theta_* > 0$ *while* $V^{\text{in}} \leqslant V^*$ *on* $\mathbb{R}$. *There exists* $\eta_0 > 0$ *such that the Cauchy problem* (7.3) *has a global weak solution provided that* $TV(V^{\text{in}}, U^{\text{in}}, \Theta^{\text{in}}) \leqslant \eta_0$. *Moreover, this solution is "entropic", i.e.,*

$$\partial_t \ln\big(V^{2/3}\Theta\big) \geqslant 0.$$

(*In other words, the entropy density cannot decrease along particle paths.*)

7.5. *The incompressible Euler equations*

There are essentially two main directions in the mathematical theory of the incompressible Euler equations:

- the PDE viewpoint, and
- the geometric viewpoint.

In the geometric viewpoint, Euler's equations for incompressible fluids in the periodic box $\mathbb{T}^D$ are the equations of geodesics on the group of volume preserving diffeomorphisms of $\mathbb{T}^D$, endowed with the metric defined by the kinetic energy. We shall say nothing of this part of the theory, for which we refer the reader to the beautiful book by Arnold and Khesin [4].

Instead, we shall just recall a few results on the Euler equations as nonlinear PDEs on $\mathbb{T}^D$. The reader is advised to read [91] and [85] for more information on that topic.

The incompressible Euler equations are

$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = 0, \qquad \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{T}^D,$$
$$u|_{t=0} = u^{\text{in}}.$$

We begin with a local existence result for classical solutions in the three-dimensional case, due to Kato.

THEOREM 7.11 (Kato [70]). *Let* $u^{\text{in}} \in H^m(\mathbb{T}^3)$, $m \in \mathbb{N}$, $m \geqslant 3$, *such that* $\operatorname{div}_x u^{\text{in}} = 0$. *Then, there exists* $T > 0$ *and a unique local solution* u *of the incompressible Euler equations with initial data* u^{in}, *such that*

$$u \in C\big([0,T); H^m\big(\mathbb{T}^3\big)\big) \cap AC_{\text{loc}}\big([0,T); H^{m-1}\big(\mathbb{T}^3\big)\big).$$

Whether the life-span of such solutions is finite is an outstanding open problem in the theory of nonlinear PDEs. What is known is that the blow-up time, if finite, does not depend on the regularity index m, as shown by the following remarkable result.

THEOREM 7.12 (Beale, Kato and Majda [15]). *Under the same assumptions as in the previous theorem, if T is finite, then*

- *either*

$$\int_0^T \left\| \operatorname{curl}_x u(t,\cdot) \right\|_{L^\infty} \mathrm{d}t = +\infty,$$

- *or $u \in C([0,T]; H^m(\mathbb{T}^3))$.*

(*In the second case, the solution u can be extended to an interval of time $[0,T')$ with $T' > T$.*)

In the two-dimensional case, there is global existence and uniqueness of a classical solution to the Cauchy problem for the incompressible Euler equations.

THEOREM 7.13 (Yudovich [122]). *Let $u^{\mathrm{in}} \in H^m(\mathbb{T}^2)$, $m \in \mathbb{N}$, $m \geqslant 3$, such that $\operatorname{div}_x u^{\mathrm{in}} = 0$. Then there exists a unique global solution u of the incompressible Euler equations with initial data u^{in} such that*

$$u \in C\big(\mathbb{R}_+; H^m\big(\mathbb{T}^2\big)\big) \cap C^1\big(\mathbb{R}_+; H^{m-1}\big(\mathbb{T}^2\big)\big).$$

A good reference on the two-dimensional case of the incompressible Euler equations is [29].

In view of the importance of the vorticity field $\operatorname{curl}_x u$ for the regularity of the solution to the incompressible Euler equations, the main difference between the two- and the three-dimensional cases comes from the following observation.

If one considers the two-dimensional flow as given by a three-dimensional velocity field u of the form

$$u(t,x) = \begin{pmatrix} u_1(t,x_1,x_2) \\ u_2(t,x_1,x_2) \\ 0 \end{pmatrix},$$

the vorticity field is

$$\operatorname{curl}_x u(t,x) = \begin{pmatrix} 0 \\ 0 \\ \omega(t,x_1,x_2) \end{pmatrix},$$

where

$$\omega(t,x_1,x_2) = \partial_{x_1} u_2(t,x_1,x_2) - \partial_{x_2} u_1(t,x_1,x_2).$$

The scalar quantity ω satisfies the transport equation

$$\partial_t \omega + u \cdot \nabla_x \omega = 0.$$

The maximum principle holds for the transport equation above, so that

$$\|\omega\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^2)} = \left\|\omega(0, \cdot)\right\|_{L^\infty(\mathbb{T}^2)}.$$

Hence, in the two-dimensional case, if u^{in} is sufficiently regular, the vorticity is globally bounded and therefore, by the Beale–Kato–Majda criterion, the regularity of the initial data is propagated for all times.

In the three-dimensional case, the vorticity $\mathrm{curl}_x\, u$ is a *vector* field that satisfies the analogue of the scalar transport equation above for vectors

$$\partial_t \,\mathrm{curl}_x\, u + (u \cdot \nabla_x)\,\mathrm{curl}_x\, u = \nabla_x u \cdot \mathrm{curl}_x\, u.$$

The length of the vector $\mathrm{curl}_x\, u$ can be amplified, or damped, by the matrix $\nabla_x u$: this mechanism is called “vortex stretching” and so far, there is no satisfying method for controlling it. Therefore, there is no a priori bound on the vorticity as in the two-dimensional case, and this is why the question of global existence or finite-time blow-up for classical solutions to the incompressible Euler equations in the three-dimensional case remains very much open.

Finally, we conclude this section with an important class of global solutions on the periodic box to the incompressible Euler equations. Choose

$$U(t, x) = \begin{pmatrix} u_1(t, x_1, x_2) \\ u_2(t, x_1, x_2) \\ w(t, x_1, x_2) \end{pmatrix}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^3,$$

where

$$u(t, x_1, x_2) = \begin{pmatrix} u_1(t, x_1, x_2) \\ u_2(t, x_1, x_2) \end{pmatrix}$$

is a C^1-solution of the two-dimensional incompressible Euler equations on $\mathbb{R}_+ \times \mathbb{T}^2$. If w satisfies the transport equation

$$\partial_t w + \mathrm{div}_x(wu) = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^2,$$

the vector field U solves the three-dimensional incompressible Euler equation on $\mathbb{R}_+ \times \mathbb{T}^3$. Such solutions are referred to as *2D–3C solutions of the incompressible Euler equations* – see, in particular, section 4.3, pp. 150–153 of [85] for an interesting application of 2D–3C solutions to the problem of a priori estimates on the Euler equations.

8. Mathematical theory of the Cauchy problem for the Boltzmann equation

8.1. *Global classical solutions for "small" data*

In this subsection we briefly review two early existence theories for the Boltzmann equation:

- Ukai's theory for perturbations of uniform Maxwellian states; and
- the Illner–Shinbrot theory for small perturbations of the vacuum state.

Consider the Cauchy problem for the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$
$$F|_{t=0} = F^{\mathrm{in}}.$$

8.1.1. *Small perturbations of the vacuum.* The collision kernel b in the collision integral is supposed to satisfy

$$0 \leqslant b(z, \omega) \leqslant C\big(1 + |z|\big) \quad \text{for a.e. } (z, \omega) \in \mathbb{R}^3 \times \mathbb{S}^2.$$

THEOREM 8.1 (Illner and Shinbrot [69]). *Pick $c > 0$; there exists $\eta > 0$ such that for each $C \in (0, \eta)$ and each initial data F^{in} satisfying*

$$0 \leqslant F^{\mathrm{in}}(x, v) \leqslant C \mathrm{e}^{-c(|x|^2+|v|^2)}, \quad x, v \in \mathbb{R}^3,$$

then the Cauchy problem for the Boltzmann equation with initial data F^{in} has a unique global (*mild*) *solution*.

The key to this result is to dominate the solution F of the Boltzmann equation with a Maxwellian traveling wave $\mathrm{e}^{-c|x-tv|^2}$.

Obviously this result is not useful in the context of incompressible hydrodynamic limits: it bears on rarefied clouds of gas that expand in the vacuum, and therefore never approach any global Maxwellian equilibrium.

8.1.2. *Small perturbations of a global Maxwellian state.* The following result, due to Ukai, is the first global existence and uniqueness result proved on the (space inhomogeneous) Boltzmann equation. It bears on the dynamics of a gas whose state is a perturbation of a global Maxwellian equilibrium. It uses a detailed spectral analysis of the linearization of the collision integral at the background Maxwellian state, see [40].

THEOREM 8.2 (Ukai [115]). *Assume that $b(z, \omega) = |z \cdot \omega|$ (hard sphere case). Let $s > 3/2$, and $\beta > 3$; define*

$$\|f\|_{s,\beta} = \sup_{v \in \mathbb{R}^3} \big(1 + |v|\big)^{\beta} \big\| f(\cdot, v)\big\|_{H^s(\mathbb{R}^3)}$$

and let

$$H^s_\beta = \left\{ f \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^3_v; H^s_x\right) \,\middle|\, \|f\|_{s,\beta} < +\infty \right\}.$$

There exists $\eta > 0$ *such that, for any* $f^{\mathrm{in}} \equiv f^{\mathrm{in}}(x, v)$ *satisfying*

$$\left\| f^{\mathrm{in}} \right\|_{s,\beta} < \eta \quad \textit{and} \quad f^{\mathrm{in}} \geqslant -\mathcal{M}^{1/2}_{(1,0,1)},$$

the Cauchy problem for the Boltzmann equation with initial data

$$F^{\mathrm{in}} = \mathcal{M}_{(1,0,1)} + \mathcal{M}^{1/2}_{(1,0,1)} f^{\mathrm{in}}$$

has a unique solution F *such that*

$$F \in L^\infty\left(\mathbb{R}_{+;} H^s_\beta\right) \cap C\left(\mathbb{R}_{+;} H^{s-\varepsilon}_{\beta-\varepsilon}\right) \cap C^1\left(\mathbb{R}_{+;} H^{s-1-\varepsilon}_{\beta-1-\varepsilon}\right)$$

for each $\varepsilon > 0$.

Ukai's theory describes the evolution of number densities that are close to a uniform Maxwellian state, and therefore, one could think of using it in the context of incompressible hydrodynamic limits. However, one should bear in mind that the parameter η that monitors the size of the initial number density fluctuation is not uniform in the Knudsen number, so that applying Ukai's ideas to derive, say, the incompressible Navier–Stokes equations from the Boltzmann equation requires nontrivial modifications, due to Bardos and Ukai [14].

Hence, for the purpose of deriving hydrodynamic models from the Boltzmann equation, it is desirable to have at one's disposal a global existence theory based on a priori estimates that are uniform in the Knudsen number. The only such existence theory so far is the DiPerna–Lions theory of weak solutions of the Boltzmann equation that is described below.

8.2. *The DiPerna–Lions theory*

As already mentioned in our presentation of the Boltzmann equation, the collision integral is local in t and x and an integral operator in v. In other words, the collision integral acts as a multiplication operator in the variables (t, x), and as kind of convolution in the variable v.

On the other hand, the natural a priori bound for the number density is

$$F \in L^\infty_t(L \ln L_x).$$

For such an F, expressions like

$$F^2 \quad \text{or} \quad F \int_{\mathbb{R}^3} F \,\mathrm{d}v$$

are defined only as measurable functions, and not as distributions. Hence the collision integral does not define a distribution for all number densities that satisfy the natural a priori bounds on solutions of the Boltzmann equation.

To get around this, DiPerna and Lions proposed to use the following notion of solution.

DEFINITION 8.3. A nonnegative function $F \in C(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is a renormalized solution of the Boltzmann equation if

$$\frac{\mathcal{B}(F,F)}{\sqrt{1+F}} \in L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v),$$

and if, for each $\Gamma \in C^1(\mathbb{R}_+)$ such that

$$\Gamma'(Z) \leqslant \frac{C}{\sqrt{1+Z}} \quad \text{for all } Z \geqslant 0,$$

one has

$$(\partial_t + v \cdot \nabla_x)\Gamma(F) = \Gamma'(F)\mathcal{B}(F,F)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3$.

In this subsection, we shall consider collision kernels that satisfy the following weak cut-off assumption

$$\lim_{|v|\to+\infty} \frac{1}{1+|v|^2} \int_{|v_*|<R} \int_{\mathbb{S}^2} b(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_* = 0 \quad \text{for each } R > 0. \tag{8.1}$$

The following result was proved by DiPerna and Lions in [37].

THEOREM 8.4. *Let* $F^{\mathrm{in}} \equiv F^{\mathrm{in}}(x,v) \geqslant 0$ *a.e. on* $\mathbb{R}^3 \times \mathbb{R}^3$ *satisfy*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(1+|x|^2+|v|^2+\left|\ln F^{\mathrm{in}}\right|\right) F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v < +\infty.$$

Then, there exists a renormalized solution of the Boltzmann equation such that $F|_{t=0} = F^{\mathrm{in}}$. *This renormalized solution satisfies*

- *the continuity equation* (*local conservation of mass*)

$$\partial_t \int_{\mathbb{R}^3} F\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} vF\,\mathrm{d}v = 0;$$

- *the global conservation of momentum*: *for each* $t \geqslant 0$,

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} vF(t)\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^3\times\mathbb{R}^3} vF^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x;$$

- *the energy inequality*: *for each* $t \geqslant 0$,

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F(t)\,\mathrm{d}v\,\mathrm{d}x \leqslant \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x;$$

- *and the H inequality*: *for each* $t > 0$, *one has*

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3} F \ln F(t)\,\mathrm{d}v\,\mathrm{d}x\\&\qquad + \frac{1}{4}\int_0^t\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s\\&\quad \leqslant \iint_{\mathbb{R}^3\times\mathbb{R}^3} F^{\mathrm{in}}\ln F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

A complete description of the proof of the DiPerna–Lions theorem is beyond the scope of the present work. We shall just explain the main ideas in it.

8.2.1. *The role of the normalizing nonlinearity.* Assume that $F \equiv F(t,x,v) \geqslant 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfies

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \big(1+|v|^2\big) F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leqslant C \quad \text{for each } t \geqslant 0, \tag{8.2}$$

and

$$\int_0^T\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \leqslant C_T \tag{8.3}$$

for each $T > 0$. A solution to the Boltzmann equation satisfies the first inequality (by the global conservation law of energy) and the second estimate by the entropy production bound deduced from Boltzmann's H-theorem.

Then

$$\frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \in L^1_{\mathrm{loc}}\big(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\big).$$

Indeed, we first recall the elementary inequality

$$\big(\sqrt{X} - \sqrt{Y}\big)^2 \leqslant \frac{1}{4}(X-Y)(\ln X - \ln Y) \quad \text{for each } X, Y > 0.$$

Then

$$\begin{aligned}\big|F'F'_* - FF_*\big| &= \Big(\sqrt{F'F'_*} + \sqrt{FF_*}\Big)\Big|\sqrt{F'F'_*} - \sqrt{FF_*}\Big|\\&\leqslant \Big(\sqrt{F'F'_*} - \sqrt{FF_*}\Big)^2 + 2\sqrt{FF_*}\Big|\sqrt{F'F'_*} - \sqrt{FF_*}\Big|.\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(\sqrt{F'F'_*} - \sqrt{FF_*}\big)^2 b \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + 2\bigg(\int_0^T \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(\sqrt{F'F'_*} - \sqrt{FF_*}\big)^2 b \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2} \\
&\qquad \times \bigg(\int_0^T \iint_{\mathbb{R}^3\times\mathbb{R}^3} F_* \bigg(\iint_{|v|<R,|\omega|=1} b(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v\bigg)\mathrm{d}v_*\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2}.
\end{aligned}
$$

By using the entropy production bound (8.3) and the weak cut-off assumption above on the collision kernel b, this inequality becomes

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant C_T + 2\sqrt{C_T} \\
&\qquad \times \bigg(\int_0^T \iint_{\mathbb{R}^3\times\mathbb{R}^3} F_* \bigg(\iint_{|v|<R,|\omega|=1} b(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v\bigg)\mathrm{d}v_*\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2} \\
&\quad \leqslant C_T + 2\sqrt{C_T}\bigg(\int_0^T \iint_{\mathbb{R}^3\times\mathbb{R}^3} F_* \mathrm{O}\big(1+|v_*|^2\big)\,\mathrm{d}v_*\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2} \\
&\quad \leqslant C_T + 2\sqrt{C_T \mathrm{O}(C)}.
\end{aligned}
$$

In the sequel, we restrict our attention for simplicity to the case of bounded collision kernels

$$
0 < b(z,\omega) \leqslant C_b, \quad (z,\omega) \in \mathbb{R}^3 \times \mathbb{S}^2. \tag{8.4}
$$

8.2.2. *Compactness by velocity averaging.* A renormalized solution of the Boltzmann equation is obtained as a limit point of the sequence F_n of solutions to the regularized Boltzmann equation

$$
\begin{aligned}
&\partial_t F_n + v \cdot \nabla_x F_n = \frac{\mathcal{B}(F_n,F_n)}{1 + (1/n)\int_{\mathbb{R}^3} F_n \,\mathrm{d}v}, \\
&F_n|_{t=0} = F^{\mathrm{in}}
\end{aligned}
\tag{8.5}
$$

in the weak topology of $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$. As in the case of all nonlinear PDEs, the main difficulty is to pass to the limit in the nonlinear term – here, in the collision integral. This requires a class of compactness results on the Boltzmann equation that are somehow adapted to the collision integral. In particular, since the collision integral acts as a convolution operator in the variable v and a multiplication operator in the variables t and x, one should seek compactness "with respect to the t and x variables only" – a notion which, of course, remains to be defined.

The appropriate class of compactness theorems was discovered a few years before being applied to the Boltzmann equation. They are known as "velocity averaging results", and were first introduced by Golse, Perthame and Sentis in [55] within the context of the diffusion approximation of the neutron transport equation. (Independently, analogous regularity results for the coefficients of the spherical harmonic expansion of the solution to the free transport equation were announced in [1].)

Obviously, whatever compactness in the strong topology of L^1_{loc} is to be found on a sequence of solutions to the Boltzmann equation (regularized or not) has to come from the streaming (free transport) operator $\partial_t + v \cdot \nabla_x$.

Being hyperbolic, the transport operator $v \cdot \nabla_x$ propagates singularities along characteristics. Therefore, at first sight it seems hopeless that one might obtain any regularizing effect from the free streaming part of the Boltzmann equation, or of any other similar kinetic model. One can think of the following elementary example.

EXAMPLE. Let $f \in L^2(\mathbb{R})$; for a.e. $x, v \in \mathbb{R}^2$, define $F(x, v) = f(v_2 x_1 - v_1 x_2)$. Clearly, $F \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$ and $v \cdot \nabla_x F = 0$. However, since f can be any function in $L^2(\mathbb{R})$, $F \notin H^s_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$ for any $s > 0$, although $v \cdot \nabla_x F \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$.

The key to obtaining regularizing effects from the transport operator $v \cdot \nabla_x$ is to seek those effects not on the number density itself, but on velocity averages thereof, in other words, on the macroscopic densities. Here is the prototype of all velocity averaging results.

THEOREM 8.5 (Golse, Perthame and Sentis [55]). *Let F_ε be a bounded family of functions in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$. Assume that the family $v \cdot \nabla_x F_\varepsilon$ is also bounded in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$. Then, for each $\phi \in L^2(\mathbb{R}^D)$, the family of moments ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v) \phi(v)\, \mathrm{d}v$$

is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Set $G_\varepsilon = F_\varepsilon + v \cdot \nabla_x F_\varepsilon$, and let $\widehat{F}_\varepsilon$ and $\widehat{G}_\varepsilon$ denote respectively the Fourier transforms of F_ε and G_ε in the x variable only. The assumptions on F_ε imply that the family G_ε is bounded in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$, and that

$$\hat{\rho}_\varepsilon(\xi) = \int_{\mathbb{R}^D} \frac{\widehat{G}_\varepsilon(\xi, v) \phi(v)\, \mathrm{d}v}{1 + \mathrm{i} v \cdot \xi}.$$

We need to study how $\hat{\rho}_\varepsilon(\xi)$ decays for $|\xi|$ large. By the Cauchy–Schwarz inequality

$$\left|\hat{\rho}_\varepsilon(\xi)\right|^2 \leqslant \frac{1}{m(\xi)} \int_{\mathbb{R}^D} \left|\widehat{G}_\varepsilon(\xi, v)\right|^2 \mathrm{d}v,$$

with

$$\frac{1}{m(\xi)} := \int_{\mathbb{R}^D} \frac{|\phi(v)|^2 \, \mathrm{d}v}{1 + |v \cdot \xi|^2} = \int_{\mathbb{R}^D} \frac{|\phi(v)|^2 \, \mathrm{d}v}{1 + |v \cdot \omega|^2 |\xi|^2},$$

where $\omega = \xi/|\xi| \in \mathbb{S}^{D-1}$ for all $\xi \in \mathbb{R}^D \setminus \{0\}$. The latter integral is a decreasing family indexed by $|\xi|$ of continuous functions of ω; this family vanishes pointwise in ω as $|\xi| \to +\infty$ by dominated convergence. By Dini's theorem, it vanishes uniformly in $\omega \in \mathbb{S}^{D-1}$, and therefore $m(\xi) \to +\infty$ as $|\xi| \to +\infty$. Since the family

$$\int_{\mathbb{R}^D} \left|\hat{\rho}_\varepsilon(\xi)\right|^2 m(\xi) \, \mathrm{d}\xi \leqslant \iint_{\mathbb{R}^D \times \mathbb{R}^D} \left|\widehat{G}_\varepsilon(\xi, v)\right|^2 \mathrm{d}\xi \, \mathrm{d}v$$

is bounded by Plancherel's theorem, ρ_ε is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^D)$ (by a variant of Rellich's compactness theorem). □

Since the operator $(I + v \cdot \nabla_x)^{-1}$ (which maps G on the solution F of $F + v \cdot \nabla_x F = G$) is a contraction mapping on both $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ and $L^\infty(\mathbb{R}^D \times \mathbb{R}^D)$, the velocity averaging result above also holds in L^p for all $p \in (1, +\infty)$ by interpolation. However, it fails in L^1, as the following example shows. (It also fails in L^∞, see [54], p. 124.)

EXAMPLE ([54], pp. 123–124). Consider G_ε, a bounded family of $L^1(\mathbb{R}^D \times \mathbb{R}^D)$, and for each ε, let F_ε be the solution to $F_\varepsilon + v \cdot \nabla_x F_\varepsilon = G_\varepsilon$. Assume that $G_\varepsilon \to \delta_0 \otimes \delta_{v^*}$ weakly, where $|v^*| = 1$. Then both F_ε and $v \cdot \nabla_x F_\varepsilon$ are bounded in $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ and

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v) \, \mathrm{d}v = \int_0^{+\infty} \int_{\mathbb{R}^D} \mathrm{e}^{-t} G_\varepsilon(x - tv, v) \, \mathrm{d}v \, \mathrm{d}t$$

so that, for any test function $\phi \in C_{\mathrm{c}}(\mathbb{R}^D)$,

$$\int_{\mathbb{R}^D} \rho_\varepsilon(x) \phi(x) \, \mathrm{d}x \to \int_0^{+\infty} \mathrm{e}^{-t} \phi\big(tv^*\big) \, \mathrm{d}t$$

as $\varepsilon \to 0$. Hence ρ_ε converges weakly to a density carried by the half-line $\mathbb{R}_+ v^*$, so that in particular the family ρ_ε is not relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

This example rests on the possible build-up of concentrations in F_ε and $v \cdot \nabla_x F_\varepsilon$. If such concentrations are ruled out, the same interpolation argument as above entails the following L^1 variant of velocity averaging.

PROPOSITION 8.6 (Golse, Lions, Perthame and Sentis [54]). *Let F_ε be a family of measurable functions on $\mathbb{R}^D \times \mathbb{R}^D$ such that, for each compact subset K of $\mathbb{R}^D$, both families F_ε and $v \cdot \nabla_x F_\varepsilon$ are uniformly integrable and tight*[2] *on $K \times \mathbb{R}^D$. Then the family ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v)\,\mathrm{d}v$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Let $\chi \equiv \chi(x)$ belongs to $C^\infty_{\mathrm{c}}(\mathbb{R}^D)$. Set

$$G_\varepsilon(x, v) = \chi(x) F_\varepsilon(x, v) + v \cdot \nabla_x \big(\chi(x) F_\varepsilon(x, v)\big)$$

and, for each $\lambda > 0$, decompose $\chi\rho_\varepsilon$ as follows

$$\chi\rho_\varepsilon = \chi\rho^{>}_{\lambda,\varepsilon} + \chi\rho^{<}_{\lambda,\varepsilon}$$

with

$$\chi\rho^{>}_{\lambda,\varepsilon} = \int_{|v|\leqslant\lambda} (I + v \cdot \nabla_x)^{-1} (G_\varepsilon \mathbb{1}_{|G_\varepsilon|>\lambda})\,\mathrm{d}v + \int_{|v|>\lambda} \chi F_\varepsilon\,\mathrm{d}v,$$

$$\chi\rho^{<}_{R,\lambda,\varepsilon} = \int_{|v|\leqslant\lambda} (I + v \cdot \nabla_x)^{-1} (G_\varepsilon \mathbb{1}_{|G_\varepsilon|\leqslant\lambda})\,\mathrm{d}v.$$

The assumptions on F_ε imply that G_ε is uniformly integrable and tight, so that $G_\varepsilon \mathbb{1}_{|G_\varepsilon|>\lambda} \to 0$ and $\chi F_\varepsilon \mathbb{1}_{|v|>\lambda} \to 0$ in $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ uniformly in ε as $\lambda \to +\infty$; thus $\chi\rho^{>}_{l,\varepsilon} \to 0$ in $L^1(\mathbb{R}^D)$ uniformly in ε as $\lambda \to +\infty$.

On the other hand, for each λ, the family $G_\varepsilon \mathbb{1}_{|G_\varepsilon|\leqslant\lambda}$ indexed by ε is bounded in $L^2(\mathbb{R}^D)$; thus, by velocity averaging in L^2, $\chi\rho^{<}_{\lambda,\varepsilon}$ is relatively compact in $L^2(\mathbb{R}^D)$ – and therefore in $L^1(\mathbb{R}^D)$, since it has support in $\operatorname{supp}\chi$ which is compact.

Hence $\chi\rho_\varepsilon$ is relatively compact in $L^1(\mathbb{R}^D)$. Since χ is arbitrary, this eventually implies that ρ_ε is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$. □

In fact, by a further interpolation argument, one can get rid of the assumption of uniform integrability on derivatives.

THEOREM 8.7 (Golse and Saint-Raymond [59]). *Let F_ε be a family of measurable functions on $\mathbb{R}^D \times \mathbb{R}^D$ such that, for each compact subset K of $\mathbb{R}^D$, the family F_ε is uniformly integrable and tight on $K \times \mathbb{R}^D$, while $v \cdot \nabla_x F_\varepsilon$ is bounded in $L^1(K \times \mathbb{R}^D)$. Then the family ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v)\,\mathrm{d}v$$

[2] A bounded family f_ε of $L^1(X)$ is uniformly integrable if $\int_A |f_\varepsilon|\,\mathrm{d}x \to 0$ as $|A| \to 0$ uniformly in ε; it is tight if $\int_X |f_\varepsilon| \mathbb{1}_{|x|>R}\,\mathrm{d}x \to 0$ as $R \to 0$ uniformly in ε.

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Without loss of generality, assume that all the F_ε are supported in $K \times \mathbb{R}^D$. Write

$$\rho_\varepsilon(x) = \lambda \int_{\mathbb{R}^D} (\lambda I + v \cdot \nabla_x)^{-1} F_\varepsilon(x, v)\,\mathrm{d}v + \int_{\mathbb{R}^D} (\lambda I + v \cdot \nabla_x)^{-1} (v \cdot \nabla_x F_\varepsilon)(x, v)\,\mathrm{d}v.$$

Since

$$\left\| (\lambda I + v \cdot \nabla_x)^{-1} \right\|_{\mathcal{L}(L^1_{x,v})} \leqslant \frac{1}{\lambda},$$

the second term on the right-hand side of the equality above is $O(1/\lambda)$ in L^1_x uniformly in ε, while the first term is strongly relatively compact in L^1_x for each $\lambda > 0$ by the previous proposition. Hence the family ρ_ε is strongly relatively compact in L^1_x. □

There are several extensions and variants of the velocity averaging results recalled here, see for instance [36,38,46–48,105]. Except for the extension of the above results to the evolution problem, which is needed in the construction of renormalized solutions to the Boltzmann equation, we shall not discuss these extensions in the present notes, but refer the interested reader to Chapter 1 of [18] for a survey of that theory as of 2000.

Here is the analogue of the L^1-variant of velocity averaging for evolution problems.

THEOREM 8.8. *Consider* $F_\varepsilon \equiv F_\varepsilon(t, x, v)$, *a family of measurable functions defined a.e. on* $\mathbb{R}_+ \times \mathbb{R}^D \times \mathbb{R}^D$ *such that, for each* $T > 0$ *and each compact* $K \subset \mathbb{R}^D$, F_ε *is uniformly integrable and tight on* $[0, T] \times K \times \mathbb{R}^D$ *while the family* $(\partial_t + v \cdot \nabla_x) F_\varepsilon$ *is bounded in* $L^1([0, T] \times K \times \mathbb{R}^D)$. *Then the family* ρ_ε *defined by*

$$\rho_\varepsilon(t, x) = \int_{\mathbb{R}^D} F_\varepsilon(t, x, v)\,\mathrm{d}v$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^D)$.

The proof is a straightforward variant of the arguments for the steady transport operator $v \cdot \nabla_x$ given above.

8.2.3. *Conclusion.* Let us briefly explain how the renormalization procedure is combined with compactness by velocity averaging in the proof of the DiPerna–Lions theorem.

Choose as normalizing nonlinearity the function

$$\Gamma_\delta(Z) = \frac{1}{\delta} \ln(1 + \delta Z), \quad \delta > 0,$$

and consider the truncated Boltzmann equation (8.5). We leave it to the reader to verify that, under the condition (8.4), the map

$$F \mapsto \frac{\mathcal{B}(F, F)}{1 + (1/n)\int_{\mathbb{R}^3} F\,\mathrm{d}v}$$

is Lipschitz continuous on $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, so that the truncated Boltzmann equation (8.5) has a global solution. Because the truncation factor

$$\frac{1}{1 + (1/n)\int_{\mathbb{R}^3} F\,\mathrm{d}v}$$

is independent of v, the symmetries of the collision integral that imply the local conservation laws of mass, momentum and energy, and the H-theorem hold also for the truncated collision integral

$$\frac{\mathcal{B}(F, F)}{1 + (1/n)\int_{\mathbb{R}^3} F\,\mathrm{d}v}$$

so that, in particular,

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \bigl(1 + |x - tv|^2 + |v|^2\bigr) F_n(t, x, v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \bigl(1 + |x|^2 + |v|^2\bigr) F^{\mathrm{in}}(x, v)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

for all $t \geqslant 0$, while

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} F_n \ln F_n(t, x, v)\,\mathrm{d}x\,\mathrm{d}v \leqslant \iint_{\mathbb{R}^3\times\mathbb{R}^3} F^{\mathrm{in}} \ln F^{\mathrm{in}}(x, v)\,\mathrm{d}x\,\mathrm{d}v.$$

As explained in Case 4 of Section 3.3, this implies the existence of a positive constant C such that

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \bigl(1 + |x|^2 + |v|^2\bigr) F_n(t, x, v)\,\mathrm{d}x\,\mathrm{d}v \leqslant C\bigl(1 + t^2\bigr) \quad \text{and} \\
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} F_n |\ln F_n|(t, x, v)\,\mathrm{d}x\,\mathrm{d}v \leqslant C\bigl(1 + t^2\bigr) \quad \text{for all } t \geqslant 0.
\end{aligned} \tag{8.6}$$

Hence F_n is weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ by the Dunford–Pettis theorem. Since, for each $\delta > 0$, one has

$$\Gamma_\delta(Z) \leqslant Z \quad \text{for each } Z \geqslant 0,$$

the sequence $\Gamma_\delta(F_n)$ is also weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$. On the other hand, the discussion in Section 8.2.1 shows that

$$(\partial_t + v \cdot \nabla_x)\Gamma_\delta(F_n) = \frac{\mathcal{B}(F_n, F_n)}{(1+\delta F_n)(1+(1/n)\int_{\mathbb{R}^3} F_n \,\mathrm{d}v)}$$

is bounded in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. By velocity averaging (Theorem 8.8), this implies that the sequence

$$\int_{\mathbb{R}^3} \Gamma_\delta(F_n)\,\mathrm{d}v$$

is strongly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$. On the other hand, because of the bound (8.6),

$$\int_{\mathbb{R}^3} F_n \,\mathrm{d}v - \int_{\mathbb{R}^3} \Gamma_\delta(F_n)\,\mathrm{d}v \to 0$$

in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$ as $\delta \to 0$, uniformly in $n \geqslant 1$. Hence we conclude that the sequence

$$\int_{\mathbb{R}^3} F_n \,\mathrm{d}v$$

is strongly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$.

Hence, modulo extraction of a subsequence

$$F_n \to F \quad \text{weakly in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+; L^1\big(\mathbb{R}^3 \times \mathbb{R}^3\big)\big)$$

while

$$\int_{\mathbb{R}^3} F_n \,\mathrm{d}v \to \int_{\mathbb{R}^3} F \,\mathrm{d}v \quad \text{strongly in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+; L^1\big(\mathbb{R}^3\big)\big).$$

Since the collision integral acts as a convolution in the v-variable and a multiplication operator in the t and x variables, this compactness theorem implies that, for each $\phi \in C_{\mathrm{c}}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \mathcal{B}(F_n, F_n)\phi \,\mathrm{d}v \to \int_{\mathbb{R}^3} \mathcal{B}(F, F)\phi \,\mathrm{d}v \quad \text{in measure on } [0, T] \times K$$

for each $T > 0$ and each compact $K \subset \mathbb{R}^3$. At first sight, this is not enough to pass to the limit in the sense of distributions in both sides of the truncated Boltzmann equation (8.5).

Instead, one integrates the truncated Boltzmann equation along characteristics by treating the gain term in the truncated collision integral as a source term. One easily sees that the limit F is a supersolution of the limiting Boltzmann equation integrated along characteristics; notice that this does not make use of the renormalization procedure. That F is a subsolution is more involved, we refer to [37] for a complete proof.

8.3. *Variants of the DiPerna–Lions theory*

The original DiPerna–Lions theorem considers a cloud of gas that expands in the vacuum, without any restriction on its degree of rarefaction, i.e., for an initial number density of arbitrary size that has finite mass, energy, second moment in x and entropy. As explained in Section 8.1.1, this kind of situation is not compatible with incompressible hydrodynamic limits.

For that purpose, we describe below two variants of the DiPerna–Lions theory that are particularly relevant in the context of incompressible hydrodynamic limits.

8.3.1. *The periodic box.* The first variant of the DiPerna–Lions theory that we discuss here is the case of the spatial domain $\mathbb{T}^3$ (the periodic box). The collision kernel b satisfies the weak cut-off condition (8.1). Let M be the centered reduced Gaussian

$$M(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2}.$$

We shall say that $F \in C(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is a *renormalized solution* of the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$
$$F|_{t=0} = F^{\mathrm{in}},$$

relative to M if and only if, for each normalizing nonlinearity $\Gamma \in C^1(\mathbb{R}_+)$ such that

$$\Gamma'(Z) \leqslant \frac{C}{\sqrt{1+Z}}, \quad Z \geqslant 0,$$

one has

$$M(\partial_t + v \cdot \nabla_x)\Gamma\left(\frac{F}{M}\right) = \Gamma'\left(\frac{F}{M}\right)\mathcal{B}(F, F)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3$.

THEOREM 8.9. *Let $F^{\mathrm{in}} \geqslant 0$ a.e. be a measurable function such that $H(F^{\mathrm{in}}|M) < +\infty$. There exists a renormalized solution F relative to M of the Cauchy problem for the Boltzmann equation with initial data F^{in}. This solution satisfies*

- *the continuity equation*

$$\partial_t \int_{\mathbb{R}^3} F \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} vF \,\mathrm{d}v = 0;$$

- *the following variant of the local conservation of momentum*

$$\partial_t \int_{\mathbb{R}^3} vF \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \otimes vF \,\mathrm{d}v + \mathrm{div}_x m = 0,$$

where $m \in L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{T}^3, M_3(\mathbb{R})))$ *with values in nonnegative symmetric matrices*;

- *the following energy relation*

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\iint_{\mathbb{T}^3\times\mathbb{R}^3} \mathrm{trace}\big(m(t)\big)$$
$$= \iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F^{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

for each $t > 0$;

- *and the* H *inequality*: *for each* $t > 0$, *one has*

$$\frac{1}{4}\int_0^t \int_{\mathbb{T}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \big(F'F'_* - FF_*\big)\ln\left(\frac{F'F'_*}{FF_*}\right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant H\big(F^{\mathrm{in}}\big|M\big) - H\big(F(t)\big|M\big) - \frac{1}{2}\int \mathrm{trace}\big(m(t)\big).$$

The above result – especially the existence of the defect measure m – is due to Lions and Masmoudi [87].

8.3.2. *The Euclidean space with uniform Maxwellian state at infinity.* The next variant of the DiPerna–Lions theory that we consider is the case of a spatial domain that is the Euclidean space with uniform Maxwellian equilibrium at infinity. Consider the Cauchy problem

$$\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), \quad (t,x,v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$
$$F(t,x,v) \to M \quad \text{as } |x| \to +\infty,$$
$$F|_{t=0} = F^{\mathrm{in}},$$

where M is the centered reduced Gaussian

$$M(v) = \frac{1}{(2\pi)^{3/2}}\mathrm{e}^{-|v|^2/2}.$$

Here again, the collision kernel b satisfies the weak cut-off assumption (8.1). The notion of "renormalized solution relative to M" of the Cauchy problem above is the same as in the case of the periodic box.

THEOREM 8.10 (Lions [82]). *Let* $F^{\mathrm{in}} \geqslant 0$ *a.e. be a measurable function such that* $H(F^{\mathrm{in}}|M) < +\infty$. *There exists a renormalized solution* F *relative to* M *of the Cauchy problem for the Boltzmann equation with initial data* F^{in}. *This solution satisfies*

- *the continuity equation*

$$\partial_t \int_{\mathbb{R}^3} F\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} vF\,\mathrm{d}v = 0;$$

- *and the H-inequality*: *for each* $t > 0$, *one has*

$$\frac{1}{4}\int_0^t\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\bigl(F'F'_* - FF_*\bigr)\ln\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant H\bigl(F^{\mathrm{in}}\big|M\bigr) - H\bigl(F(t)\big|M\bigr).$$

9. The Hilbert expansion method: Application to the compressible Euler limit

In this section we describe a first method for deriving hydrodynamic models from the Boltzmann equation. In spite of its numerous shortcomings (which we shall discuss at the end of the present section), this method is extremely robust, and can be applied to various kinetic models other than the Boltzmann equation.

Start from the Boltzmann equation in the compressible Euler scaling

$$\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t,x,v)\in\mathbb{R}_+^*\times\mathbb{T}^3\times\mathbb{R}^3. \tag{9.1}$$

For simplicity, we only consider in this section the case of a hard sphere gas, so that the collision kernel b in Boltzmann's collision integral $\mathcal{B}$ is given by the expression

$$b(z,\omega) = |z\cdot\omega|, \quad (z,\omega)\in\mathbb{R}^3\times\mathbb{S}^2.$$

Consider next the compressible Euler system for a perfect monatomic gas

$$\begin{aligned}
&\partial_t\rho + \mathrm{div}_x(\rho u) = 0,\\
&\partial_t(\rho u) + \mathrm{div}_x(\rho u\otimes u) + \nabla_x(\rho\theta) = 0,\\
&\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) = 0,\\
&(\rho,u,\theta)|_{t=0} = \bigl(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\bigr),
\end{aligned} \tag{9.2}$$

where

$$\rho^{\mathrm{in}}, u^{\mathrm{in}} \text{ and } \theta^{\mathrm{in}}\in H^5\bigl(\mathbb{T}^3\bigr), \qquad \rho^{\mathrm{in}}\geqslant 0 \quad\text{and}\quad \theta^{\mathrm{in}} > 0 \quad\text{on } \mathbb{T}^3. \tag{9.3}$$

Let (ρ,u,θ) be the solution to (9.2) predicted by Theorem 7.8 under the assumption (9.3), and call $T > 0$ its lifespan. Finally, define the local Maxwellian

$$E(t,x,v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}.$$

THEOREM 9.1 (Caflisch [22]). *There exists* ε_0 *such that, for each* $\varepsilon\in(0,\varepsilon_0)$, *there is a unique solution* F_ε *of the Boltzmann equation* (9.1) *on* $[0,T)\times\mathbb{T}^3\times\mathbb{R}^3$ *satisfying the estimate*

$$\sup_{0\leqslant t\leqslant T'}\bigl\|F(t,\cdot,\cdot) - E(t,\cdot,\cdot)\bigr\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} = \mathrm{O}(\varepsilon) \quad \text{as } \varepsilon\to 0^+$$

for each $T' < T$.

We shall not give a complete proof of this result, which is fairly technical. Instead, we simply discuss below the main ideas in it.

First, the solution F_ε is sought as a truncated Hilbert expansion, plus a remainder term

$$F_\varepsilon(t,x,v) = \sum_{k=0}^{6} \varepsilon^k F_k(t,x,v) + \varepsilon^3 R_\varepsilon(t,x,v), \tag{9.4}$$

such that the last term in the truncated expansion satisfies

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_6 \, \mathrm{d}v = 0. \tag{9.5}$$

We recall from our discussion in Section 5.1 that the projection of a term F_k in Hilbert's expansion on the space of collision invariants is determined by postulating the existence of the next term in that expansion, i.e., F_{k+1}. In the truncated expansion above, we obviously do not postulate the existence of a F_7, so that there is a certain amount of arbitrariness in the choice of F_6, which is resolved by condition (9.5). The other terms F_k, $k = 0, \ldots, 5$, are computed as in Section 5.1.

Next we write an equation for the remainder R_ε: inserting the right-hand side of (9.4) in the scaled Boltzmann equation (9.1) we arrive at

$$\begin{aligned}(\partial_t + v \cdot \nabla_x) R_\varepsilon &= \frac{2}{\varepsilon} \mathcal{B}(F_0, R_\varepsilon) + 2\mathcal{B}\left(\sum_{k=1}^{6} \varepsilon^{k-1} F_k, R_\varepsilon \right) + \varepsilon^2 \mathcal{B}(R_\varepsilon, R_\varepsilon) \\ &\quad + \sum_{k+l \geqslant 7} \varepsilon^{k+l-4} \mathcal{B}(F_k, F_l) - \varepsilon^3 (\partial_t + v \cdot \nabla_x) F_6. \end{aligned} \tag{9.6}$$

Indeed, for $k = 0, \ldots, 5$, the terms F_k are chosen as explained in Section 5.1, so that

$$(\partial_t + v \cdot \nabla_x) \sum_{k=0}^{5} \varepsilon^k F_k = \sum_{l+m \leqslant 6} \varepsilon^{l+m-1} \mathcal{B}(F_l, F_m).$$

Notice that, in (9.6), the nonlinear term is multiplied by an ε^2 factor: hence (9.6) is a weakly nonlinear equation. However, the linear term in (9.6) is

$$\frac{2}{\varepsilon} \mathcal{B}(F_0, R_\varepsilon)$$

at leading order, which defines a nonpositive operator in a weighted L^2 space in the v-variable. Specifically, the operator $R \mapsto \mathcal{B}(F_0, R)$ is self-adjoint in $L^2(\mathbb{R}^3; F_0^{-1} \, \mathrm{d}v)$. A slight difficulty in this setting is that the weight F_0^{-1} depends on (t, x).

To avoid this, we use a slightly different definition of the linearized collision integral than in Section 3.6. Define

$$\mathbf{L}_M\phi = -2M^{-1/2}\mathcal{B}\big(M, M^{1/2}\phi\big),$$

where M is a Maxwellian density. As in Section 3.6, the operator $\mathbf{L}_M$ is split into

$$\mathbf{L}_M\phi(v) = a\big(|v|\big)\phi(v) - \mathbf{K}_M\phi(v),$$

where

$$a_M\big(|v|\big) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} \big|(v-v_*)\cdot\omega\big| M_* \,\mathrm{d}v_*\,\mathrm{d}\omega$$

and

$$\mathbf{K}_M\phi(v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} \big(\sqrt{M}\phi_* - \sqrt{M'_*}\phi' - \sqrt{M'}\phi'_*\big)\big|(v-v_*)\cdot\omega\big|\sqrt{M_*}\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

The properties of $\mathbf{L}_M$ are summarized in the following theorem.

THEOREM 9.2. *The operator $\mathbf{K}_M$ is compact on $L^2(\mathbb{R}^3)$, while $a_M(|v|) \sim a^*|v|$ as $|v| \to +\infty$. Hence the operator $\mathbf{L}_M$ is an unbounded self-adjoint Fredholm operator on $L^2(\mathbb{R}^3)$ with domain $L^2(\mathbb{R}^3; (1+|v|)\,\mathrm{d}v)$ and nullspace*

$$\operatorname{Ker}\mathbf{L}_M = \operatorname{span}\big\{\sqrt{M}, \sqrt{M}v_1, \sqrt{M}v_2, \sqrt{M}v_3, \sqrt{M}|v|^2\big\}.$$

The advantage in using $\mathbf{L}_M$ instead of the operator $R \mapsto \mathcal{B}(F_0, R)$ is that the former operator is self-adjoint on the unweighted space $L^2(\mathbb{R}^3)$.

Formulating (9.6) in terms of the operator $\mathbf{L}_M$ leads to an equation of the form

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x)\big(E^{-1/2}R_\varepsilon\big) + \frac{1}{\varepsilon}\mathbf{L}_E\big(E^{-1/2}R_\varepsilon\big)\\
&\quad = \varepsilon^2\mathbf{Q}_E\big(E^{-1/2}R_\varepsilon, E^{-1/2}R_\varepsilon\big) + 2E^{-1/2}\mathcal{B}\left(\sum_{k=1}^{6}\varepsilon^{k-1}F_k, R_\varepsilon\right)\\
&\qquad + \sum_{k+l\geqslant 7}\varepsilon^{k+l-4}E^{-1/2}\mathcal{B}(F_k, F_l) - \varepsilon^3 E^{-1/2}(\partial_t + v\cdot\nabla_x)F_6\\
&\qquad + \frac{1}{2}R_\varepsilon(\partial_t + v\cdot\nabla_x)\ln E,
\end{aligned}\tag{9.7}$$

where E is the local Maxwellian whose parameters solve (9.2) and the quadratic operator $\mathbf{Q}_M$ is defined by

$$\mathbf{Q}_M(\phi,\phi) = M^{-1/2}\mathcal{B}\big(\sqrt{M}\phi, \sqrt{M}\phi\big).$$

The last term on the right-hand side of (9.7) is the most annoying one, since it grows like

$$|v|^3 \frac{|\nabla_x \theta|}{\theta^2} R_\varepsilon;$$

in particular, it cannot be controlled by the damping part of the linearized collision integral, since $a_M(|v|) = \mathrm{O}(|v|)$ as $|v| \to +\infty$.

To overcome this difficulty, R. Caflisch introduced a new Maxwellian state $\mathbf{M}$ defined by

$$\mathbf{M}(v) = \mathcal{M}_{(1,0,\hat{\theta})}(v) = \frac{1}{(2\pi\hat{\theta})^{3/2}} \mathrm{e}^{-|v|^2/2\hat{\theta}},$$

where

$$\hat{\theta} = 2\|\theta\|_{L^\infty}.$$

Hence, there exists a constant C, that depends on $\|\rho\|_{L^\infty}$, $\|u\|_{L^\infty}$ and $\|1/\theta\|_{L^\infty}$ such that

$$E \leqslant C\mathbf{M}.$$

Next, one decomposes R_ε as

$$R_\varepsilon = \sqrt{E} r_\varepsilon + \sqrt{\mathbf{M}} q_\varepsilon,$$

where the new unknowns r_ε and q_ε are governed by the coupled system

$$\begin{aligned}
(\partial_t + v \cdot \nabla_x) r_\varepsilon &= -\frac{1}{\varepsilon} \mathbf{L}_E r_\varepsilon + \mathbb{1}_{|v| \leqslant c} \frac{1}{\varepsilon} \sigma \mathbf{K}_{\mathbf{M}} q_\varepsilon, \\
(\partial_t + v \cdot \nabla_x) q_\varepsilon &= -\mathbf{M}^{-1/2} r_\varepsilon (\partial_t + v \cdot \nabla_x) E^{1/2} - \frac{1}{\varepsilon} (a_{\mathbf{M}} - \mathbb{1}_{|v|>c} \mathbf{K}_{\mathbf{M}}) q_\varepsilon \\
&\quad + 2\mathbf{M}^{-1/2} \mathcal{B}\left(\sum_{k=1}^{6} \varepsilon^{k-1} F_k, \mathbf{M}^{1/2} (\sigma r_\varepsilon + q_\varepsilon) \right) \\
&\quad + 2\mathbf{Q}_{\mathbf{M}} (\sigma r_\varepsilon + q_\varepsilon, \sigma r_\varepsilon + q_\varepsilon) + \varepsilon^2 s,
\end{aligned} \tag{9.8}$$

where

$$\sigma = \sqrt{\frac{E}{\mathbf{M}}},$$

where c is a truncation parameter to be chosen below, and the source term s is

$$s = \mathbf{M}^{-1/2} \sum_{k+l \geqslant 7} \varepsilon^{k+l-6} \mathcal{B}(F_k, F_l) - \varepsilon \mathbf{M}^{-1/2} (\partial_t + v \cdot \nabla_x) F_6.$$

This system is solved for the initial condition

$$r_\varepsilon|_{t=0} = q_\varepsilon|_{t=0} = 0$$

by a fixed point argument in the norm

$$\|\phi\|_{\alpha,\beta} = \sup_{\xi\in\mathbb{R}^3} \big(1+|\xi|\big)^\alpha \big\|f(\cdot,\xi)\big\|_{H^\beta(\mathbb{R}^3)}$$

for $\beta > 3/2$. We refer to [22] for the complete proof.

REMARKS. Several remarks are in order.

1. The construction in Caflisch's theorem leads to a solution of the Boltzmann equation that exists and approximates the Maxwellian built on the solution of the compressible Euler system for as long as the solution of that system exists and remains smooth. This is very satisfying: should there be a blow-up in finite time in the solution of the Boltzmann equation, it cannot happen before the onset of singularities in the Euler system.
2. There is however a rather unpleasant feature in Caflisch's construction: the solution of the Boltzmann equation so constructed is, in general, not everywhere nonnegative, and therefore loses physical meaning. This is most easily seen on the initial data for F_ε in the form (9.4): in Caflisch's paper, $R_\varepsilon|_{t=0} = 0$, so that, in the particular case of Maxwell's molecules, $F_\varepsilon|_{t=0}$ is a polynomial in v that is not (at least in general) everywhere nonnegative. It could be that an improvement of Caflisch's ansatz with initial layers as in [72] helps avoiding this; however, there is no mention of this difficulty in either [22] or [72].
3. The interested reader is invited to compare Caflisch's theorem with an earlier result by Nishida [102], who obtained the compressible Euler limit of the Boltzmann equation by an abstract Cauchy–Kowalewski argument (in the style of Nirenberg and Ovsyannikov, see, for instance, [101]). Nishida's result is as follows: consider the scaled Boltzmann equation (9.1) with an initial data F^{in} analytic in x with enough decay in v that is a perturbation of some absolute Maxwellian. Then, there exists a family of solutions of the scaled Boltzmann equation parametrized by $\varepsilon > 0$ that lives on some interval of time independent of $\varepsilon > 0$. This family of solutions converges to the Maxwellian built on an analytic solution of the compressible Euler system in the vanishing ε limit. However, the lifespan of Nishida's family of solutions of the scaled Boltzmann equation is not known to coincide with the blow-up time of the limiting smooth solution of the compressible Euler system. See also [117] for an improved variant of Nishida's result.

10. The relative entropy method: Application to the incompressible Euler limit

As explained in Section 9, all methods based on asymptotic expansions in the Knudsen number apply only to situations where the solutions of both the Boltzmann equation and the limiting hydrodynamic equation are smooth. The present section introduces a new method for cases where only the solution of the target equation (i.e., the hydrodynamic model) is smooth.

We shall explain how this method applies to the incompressible Euler limit of the Boltzmann equation. Consider the Boltzmann equation in the incompressible Euler scaling

$$
\begin{aligned}
&\varepsilon\,\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon^q}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t,x,v)\in\mathbb{R}_+^*\times\mathbb{T}^3\times\mathbb{R}^3,\\
&F_\varepsilon|_{t=0} = \mathcal{M}\big(1, \varepsilon u^{\mathrm{in}}, 1\big),
\end{aligned}
\tag{10.1}
$$

where $q > 1$ and $u^{\mathrm{in}} \equiv u^{\mathrm{in}}(x)$ is a divergence-free vector field on $\mathbb{T}^3$. Here, the collision kernel b is supposed to satisfy assumption (3.55) (i.e., to come from a hard cut-off potential) as well as the additional condition

$$
\inf_{(z,\omega)\in\mathbb{R}^3\times\mathbb{S}^2} \frac{b(z,\omega)}{|(z/|z|)\cdot\omega|} > 0. \tag{10.2}
$$

Throughout this section, we denote by M the centered reduced Gaussian distribution in v, i.e.,

$$
M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}}\mathrm{e}^{-|v|^2/2}.
$$

Next, we introduce a new concept of limit that is especially well adapted to all incompressible hydrodynamic limits of the Boltzmann equation.

DEFINITION 10.1 (Bardos, Golse and Levermore [10]). A family $g_\varepsilon \equiv g_\varepsilon(x,v)$ of $L^1_{\mathrm{loc}}(\mathbb{T}^3\times\mathbb{R}^3; M\,\mathrm{d}x\,\mathrm{d}v)$ is said to converge entropically to g at rate ε if

- $1+\varepsilon g_\varepsilon \geqslant 0$ a.e. on $\mathbb{T}^3\times\mathbb{R}^3$ for each $\varepsilon > 0$,
- $g_\varepsilon \to g$ weakly in $L^1_{\mathrm{loc}}(\mathbb{T}^3\times\mathbb{R}^3; M\,\mathrm{d}x\,\mathrm{d}v)$,
- and

$$
\frac{1}{\varepsilon^2}H\big(M(1+\varepsilon g_\varepsilon)\big|M\big) \to \frac{1}{2}\iint_{\mathbb{T}^3\times\mathbb{R}^3} g(x,v)^2\,\mathrm{d}x\,\mathrm{d}v
$$

as $\varepsilon \to 0$.

After these preliminaries, we can state the incompressible Euler limit theorem.

THEOREM 10.2 (Saint-Raymond [109]). *Assume that $u^{\mathrm{in}} \in H^3(\mathbb{T}^3)$ is a divergence-free vector field, and let u be the maximal solution of the incompressible Euler equations*

$$
\begin{aligned}
&\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0, \qquad \mathrm{div}_x\, u = 0,\\
&u|_{t=0} = u^{\mathrm{in}}
\end{aligned}
$$

on $[0, T) \times \mathbb{T}^3$. *For* $\varepsilon > 0$, *let* F_ε *be a renormalized solution relative to* M *of the scaled Boltzmann equation* (10.1). *Then, for each* $t \in [0, T)$,

$$\frac{F_\varepsilon(t,x,v) - M(v)}{\varepsilon M(v)} \to u(t,x) \cdot v$$

entropically at rate ε *as* $\varepsilon \to 0$.

The relative entropy method was used for the first time to derive incompressible hydrodynamic models from the Boltzmann equation in Chapter 2 of [18], [53] and in [87] – however, the results obtained in these references were incomplete since the proofs used additional controls not known to be satisfied by renormalized solutions of the Boltzmann equation.

Of course, if u is a 2D–3C solution of the incompressible Euler equations (see Section 7.5), then $T = +\infty$ and the incompressible Euler limit is global.

Let us now describe the main ideas in the relative entropy method.

First, assuming that F_ε is a classical solution to the scaled Boltzmann equation (10.1), we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varepsilon^2} H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon u,1)}) &= -\frac{1}{\varepsilon^2}\int_{\mathbb{T}^3} D(u) : \int_{\mathbb{R}^3} (v - \varepsilon u)^{\otimes 2} F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \\ &\quad + \frac{1}{\varepsilon}\int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Now, this identity is not known to be true if F_ε is a renormalized solution. What is known instead is the following variant of it (see Theorem 8.10); for each $t \in [0, T)$, one has

$$\begin{aligned}&\frac{1}{\varepsilon^2} H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon u,1)})(t) + \frac{1}{\varepsilon}\int_{\mathbb{T}^3} \operatorname{trace}\big(m_\varepsilon(t)\big) \\ &\quad \leqslant -\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^3} D(u) : \int_{\mathbb{R}^3} (v - \varepsilon u)^{\otimes 2} F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^3} D(u) : m_\varepsilon(s)\,\mathrm{d}s.\end{aligned} \tag{10.3}$$

The key argument is the following lemma.

LEMMA 10.3. *For each* $T' \in [0, T)$ *there exists* $C_{T'} \geqslant 1$ *such that*

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3} \big|D(u) : (v - \varepsilon u)^{\otimes 2} F_\varepsilon\big| \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant \frac{C_{T'}}{\varepsilon^2}\int_0^t \big\|D(u)\big\|_{L^\infty} H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon u,1)})(s)\,\mathrm{d}s + \mathrm{o}(1)\end{aligned}$$

uniformly in $t \in [0, T']$ *as* $\varepsilon \to 0$.

Define

$$X_\varepsilon(t) = \frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)})(t) + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \mathrm{trace}\big(m_\varepsilon(t)\big);$$

it follows from (10.3) and Lemma 10.3 that, for each $T' \in [0, T)$ and each $t \in [0, T']$, one has

$$\begin{aligned} X_\varepsilon(t) &\leqslant C_{T'} \big\| D(u) \big\|_{L^\infty([0,T']\times\mathbb{T}^3)} \int_0^t X_\varepsilon(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + \mathrm{o}(1)_{L^\infty([0,T'])}. \end{aligned}$$

Hence

$$\begin{aligned} X_\varepsilon(t) &\leqslant \mathrm{e}^{C_{T'} \| D(u) \|_{L^\infty([0,T']\times\mathbb{T}^3)}} \left| \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \right| \\ &\quad + \mathrm{o}(1)_{L^\infty([0,T'])}. \end{aligned} \tag{10.4}$$

With this inequality, one concludes as follows. The sequence of fluctuations

$$\frac{1}{\varepsilon}(F_\varepsilon - M) \text{ is relatively compact in } w^* - L^\infty\big(\mathbb{R}_+; w - L^1\big(\big(1 + |v|^2\big)\,\mathrm{d}x\,\mathrm{d}v\big)\big)$$

so that, modulo extraction of a subsequence

$$\int_{\mathbb{R}^3} F_\varepsilon \,\mathrm{d}v \to 1 \qquad \text{in } L^\infty\big(\mathbb{R}_+; L^1\big(\mathbb{T}^3\big)\big)$$

and

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^3} v F_\varepsilon \,\mathrm{d}v \to U \quad \text{in } w^* - L^\infty\big(\mathbb{R}_+; w - L^1\big(\mathbb{T}^3\big)\big).$$

Because of the local conservation of mass that is satisfied by the renormalized solution F_ε, one has

$$\mathrm{div}_x\, U = 0.$$

Hence

$$\frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot (U - u)\,\mathrm{d}x\,\mathrm{d}s = 0$$

so that

$$X_\varepsilon(t) \to 0 \quad \text{as } \varepsilon \to 0 \text{ for each } t \in \big[0, T'\big].$$

By convexity and weak limit, one has

$$\frac{1}{2}\big\|(U-u)(t)\big\|^2_{L^2(\mathbb{T}^3)} \leqslant \varliminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon u,1)})(t)$$

for each $t \in [0, T']$. Therefore, passing to the limit in (10.4) shows that $U = u$, as announced.

We shall not describe the proof of Lemma 10.3, which is really technical. In this proof, one has to master the difficulties created by the high velocity tails of the number density; this is done by using decay estimates due to Grad and Caflisch [23] on the gain term of the linearized collision operator, we refer to [109] for a complete proof.

But the difficulties in the proof of Lemma 10.3 are special to the Boltzmann equation; the main line of the relative entropy method is as described above. This method is due to Yau [121] who used it for the hydrodynamic limit of Ginzburg–Landau models.

11. Applications of the moment method

The moment method is based on compactness results which are used to pass to the limit in the local conservation laws of mass momentum and energy as the Knudsen number vanishes. This method does not require estimates other than the natural bounds on mass, energy, entropy and entropy production. On principle, it could therefore be used when both the solutions of the scaled Boltzmann equation and of the limiting hydrodynamic equations are not known to be regular.

First, we state the various theorems on hydrodynamic limits that can be proved in this way.

11.1. *The acoustic limit*

We start from the Boltzmann equation in the acoustic scaling posed in the periodic box

$$\begin{aligned} &\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t,x,v)\in\mathbb{R}_+^*\times\mathbb{T}^3\times\mathbb{R}^3,\\ &F_\varepsilon|_{t=0} = F_\varepsilon^{\mathrm{in}}. \end{aligned} \tag{11.1}$$

Assume that b satisfies the weak cut-off assumption (8.1) as well as the bound

$$0 < \int_{\mathbb{S}^2} b(z,\omega)\,\mathrm{d}\omega \leqslant C_b\big(1+|z|^2\big)^\beta \quad \text{a.e., } z\in\mathbb{R}^3, \tag{11.2}$$

for some $\beta \in [0, 1]$.

We assume that

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} F_\varepsilon^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 1,$$

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} vF_\varepsilon^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 0, \tag{11.3}$$
$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 F_\varepsilon^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = \frac{3}{2}.$$

Set M to be the centered, reduced Gaussian distribution

$$M(v) = \frac{1}{(2\pi)^{3/2}}\mathrm{e}^{-|v|^2/2}, \tag{11.4}$$

so that

$$\int_{\mathbb{R}^3} M\,\mathrm{d}x\,\mathrm{d}v = 1,$$
$$\int_{\mathbb{R}^3} vM\,\mathrm{d}x\,\mathrm{d}v = 0,$$
$$\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 M\,\mathrm{d}x\,\mathrm{d}v = \frac{3}{2}.$$

The acoustic limit of the Boltzmann equation (11.1) is given by the following theorem.

THEOREM 11.1 (Golse and Levermore [52]). *Let $\delta_\varepsilon > 0$ be such that*

$$\delta_\varepsilon \to 0 \quad \textit{and} \quad \delta_\varepsilon|\ln\delta_\varepsilon|^{\beta/2} = \mathrm{o}\big(\varepsilon^{1/2}\big)$$

as $\varepsilon \to 0$. Assume that

$$\frac{F_\varepsilon^{\mathrm{in}}(x,v) - M}{\delta_\varepsilon M} \to \rho^{\mathrm{in}}(x) + u^{\mathrm{in}}(x)\cdot v + \theta^{\mathrm{in}}(x)\frac{1}{2}\big(|v|^2 - 3\big)$$

entropically at rate δ_ε. For each $\varepsilon > 0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation (11.1) *with initial data $F_\varepsilon^{\mathrm{in}}$.*

Then, for each $t \geqslant 0$, the family

$$\frac{F_\varepsilon(t,x,v) - M}{\delta_\varepsilon M} \to \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 3\big)$$

entropically at rate δ_ε, where (ρ,u,θ) is the solution of the acoustic system

$$\partial_t\rho + \mathrm{div}_x\, u = 0,$$
$$\partial_t u + \nabla_x(\rho+\theta) = 0, \qquad (t,x)\in\mathbb{R}_+^*\times\mathbb{T}^3,$$
$$\frac{3}{2}\,\partial_t\theta + \mathrm{div}_x\, u = 0,$$

with initial data

$$(\rho, u, \theta)|_{t=0} = \left(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\right).$$

Notice that our assumptions on the family of initial data imply that

$$\left(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\right) \in L^2\left(\mathbb{T}^3\right),$$

and that

$$\int_{\mathbb{T}^3} \rho^{\mathrm{in}}(x)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} u^{\mathrm{in}}(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\mathrm{in}}(x)\,\mathrm{d}x = 0.$$

In particular, the existence and uniqueness theory for the acoustic system described in Section 7.1.2 applies here, and the solution (ρ, u, θ) satisfies

$$(\rho, u, \theta) \in C\left(\mathbb{R}_+; L^2\left(\mathbb{T}^3\right)\right)$$

and

$$\int_{\mathbb{T}^3} \rho^{\mathrm{in}}(x)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} u^{\mathrm{in}}(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\mathrm{in}}(x)\,\mathrm{d}x = 0.$$

11.2. *The Stokes–Fourier limit*

We start from the Boltzmann equation in the Stokes scaling posed in the periodic box

$$\begin{aligned} &\varepsilon\,\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3, \\ &F_\varepsilon|_{t=0} = F_\varepsilon^{\mathrm{in}}. \end{aligned} \tag{11.5}$$

Assume that the collision kernel b comes from a hard cut-off potential, i.e., that it satisfies (5.2) for some $\alpha \in [0, 1]$. Assume further that, for each $\varepsilon > 0$, the initial data $F_\varepsilon^{\mathrm{in}}$ satisfies the relations (11.3).

THEOREM 11.2 (Golse and Levermore [52]). *Let $\delta_\varepsilon > 0$ be such that*

$$\delta_\varepsilon \to 0 \quad \textit{and} \quad \delta_\varepsilon |\ln \delta_\varepsilon|^\alpha = \mathrm{o}(\varepsilon)$$

as $\varepsilon \to 0$. Assume that

$$\frac{F_\varepsilon^{\mathrm{in}}(x, v) - M}{\delta_\varepsilon M} \to u^{\mathrm{in}}(x)\cdot v + \theta^{\mathrm{in}}(x)\frac{1}{2}\left(|v|^2 - 5\right)$$

entropically at rate δ_ε, where u^{in} satisfies

$$\operatorname{div}_x u^{\mathrm{in}} = 0$$

and where M is the centered, reduced Gaussian distribution (11.4). *For each $\varepsilon > 0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation* (11.5) *with initial data $F_\varepsilon^{\mathrm{in}}$.*

Then, for each $t \geqslant 0$, the family

$$\frac{F_\varepsilon(t,x,v) - M}{\delta_\varepsilon M} \to u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2 - 5\big)$$

entropically at rate δ_ε, where (u,θ) is a solution of the Stokes–Fourier system

$$\begin{aligned}
&\partial_t u + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{T}^3,\\
&\partial_t \theta = \kappa \Delta_x \theta,
\end{aligned}$$

with initial data

$$(u,\theta)|_{t=0} = \big(u^{\mathrm{in}}, \theta^{\mathrm{in}}\big).$$

The viscosity and heat conductivity are given by

$$\nu = \frac{1}{10}\int_{\mathbb{R}^3} \widehat{A} : AM \, \mathrm{d}v, \qquad \kappa = \frac{2}{15}\int_{\mathbb{R}^3} \widehat{B}\cdot BM \, \mathrm{d}v,$$

see also formula (6.9) *for expressions of these quantities in terms of the functions* **a** *and* **b** *defined in* (6.12).

We recall that $\widehat{A}$ and $\widehat{B}$ are defined in terms of

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I, \qquad B = \frac{1}{2}\big(|v|^2 - 5\big)v$$

by

$$\begin{aligned}
&\mathcal{L}_M \widehat{A} = A, \qquad \widehat{A} \perp \operatorname{Ker}\mathcal{L}_M,\\
&\mathcal{L}_M \widehat{B} = B, \qquad \widehat{B} \perp \operatorname{Ker}\mathcal{L}_M.
\end{aligned}$$

Our assumptions on the family of initial data imply that

$$\big(u^{\mathrm{in}}, \theta^{\mathrm{in}}\big) \in L^2\big(\mathbb{T}^3\big),$$

while

$$\int_{\mathbb{T}^3} u^{\mathrm{in}}(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\mathrm{in}}(x)\,\mathrm{d}x = 0.$$

Hence, the existence and uniqueness theory for the Stokes–Fourier system described in Section 7.1.1 applies here, and the solution (u,θ) satisfies

$$(u,\theta)\in C\big(\mathbb{R}_+;L^2\big(\mathbb{T}^3\big)\big)\cap C^\infty\big(\mathbb{R}_+^*\times\mathbb{R}^3\big)$$

and

$$\int_{\mathbb{T}^3} u^{\mathrm{in}}(x)\,\mathrm{d}x=0\quad\text{and}\quad\int_{\mathbb{T}^3}\theta^{\mathrm{in}}(x)\,\mathrm{d}x=0.$$

11.3. *The Navier–Stokes–Fourier limit*

We start from the Boltzmann equation in the Navier–Stokes scaling, posed in the Euclidean space, with Maxwellian equilibrium at infinity

$$\begin{aligned}
&\varepsilon\,\partial_t F_\varepsilon+v\cdot\nabla_x F_\varepsilon=\frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon,F_\varepsilon),\quad (t,x,v)\in\mathbb{R}_+^*\times\mathbb{R}^3\times\mathbb{R}^3,\\
&F_\varepsilon(t,x,v)\to M\quad |x|\to+\infty,\\
&F_\varepsilon|_{t=0}=F_\varepsilon^{\mathrm{in}},
\end{aligned}\tag{11.6}$$

where M is the centered, reduced Gaussian distribution (11.4).

Assume that the collision kernel b comes from a hard cut-off potential, i.e., that it satisfies (5.2) for some $\alpha\in[0,1]$.

THEOREM 11.3 (Golse and Saint-Raymond [60,61]). *Assume that*

$$\frac{F_\varepsilon^{\mathrm{in}}(x,v)-M}{\varepsilon M}\to u^{\mathrm{in}}(x)\cdot v+\theta^{\mathrm{in}}(x)\frac{1}{2}\big(|v|^2-5\big)$$

entropically at rate ε, where u^{in} satisfies

$$\operatorname{div}_x u^{\mathrm{in}}=0.$$

For each $\varepsilon>0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation (11.5) *with initial data $F_\varepsilon^{\mathrm{in}}$.*

Then the family

$$\left(\frac{1}{\varepsilon}\int_{\mathbb{R}^3} vF_\varepsilon(t,x,v)\,\mathrm{d}v,\ \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\left(\frac{1}{3}|v|^2-1\right)\big(F_\varepsilon(t,x,v)-M\big)\,\mathrm{d}v\right)$$

is weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}^3)$, and each of its limit points as $\varepsilon\to0$ is

$$(u,\theta)\in C\big(\mathbb{R}_+,w-L^2\big(\mathbb{R}^3\big)\big)\cap L^2\big(\mathbb{R}_+;H^1\big(\mathbb{R}^3\big)\big),$$

a solution of the Navier–Stokes–Fourier system

$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{T}^3,$$
$$\partial_t \theta + \operatorname{div}_x(u\theta) = \kappa \Delta_x \theta,$$

with initial data

$$(u,\theta)|_{t=0} = \left(u^{\mathrm{in}}, \theta^{\mathrm{in}}\right).$$

The viscosity and heat conductivity are given by

$$\nu = \frac{1}{10}\int_{\mathbb{R}^3} \widehat{A} : AM \,\mathrm{d}v, \qquad \kappa = \frac{2}{15}\int_{\mathbb{R}^3} \widehat{B} \cdot BM \,\mathrm{d}v$$

again, see (6.9) *for expressions of these quantities in terms of the functions* **a** *and* **b** *defined in* (6.12). *Moreover, this solution* (u,θ) *satisfies, for each* $t > 0$, *the inequality*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{5}{4}\|\theta(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} \left(\nu|\nabla_x u|^2 + \frac{5}{2}\kappa|\nabla_x \theta|^2\right) \mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \frac{1}{2}\|u^{\mathrm{in}}\|_{L^2}^2 + \frac{5}{4}\|\theta^{\mathrm{in}}\|_{L^2}^2.$$

In particular, if $\theta^{\mathrm{in}} = 0$, this theorem shows that any weak limit point of

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^3} vF_\varepsilon(t,x,v)\,\mathrm{d}v$$

in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3)$ as $\varepsilon \to 0$ is a Leray solution of the Navier–Stokes equations with initial data u^{in}.

This theorem explains the following observation by Lions: "[...] the global existence of [renormalized] solutions [...] can be seen as the analogue for Boltzmann's equation to the pioneering work on the Navier–Stokes equations by J. Leray" (see [83], p. 432).

11.4. *Sketch of the proof of the Navier–Stokes–Fourier limit by the moment method*

The proof of the Navier–Stokes–Fourier limit theorem above involves many ideas developed in a sequence of papers over the past 15 years:

- the BGL program was defined in [10]; this reference provided the general entropy and entropy production estimates used to control the number density fluctuation and its distance to local equilibrium; as a result, the evolution Stokes and the steady Navier–Stokes motion equations were derived under the assumption that the renormalized solutions of the Boltzmann equation considered satisfy the local conservation of momentum as well as a nonlinear compactness estimate (for the Navier–Stokes limit) that will be described below in more details;

- under the same assumptions as in [10], Lions and Masmoudi [86] were able to derive the evolution Navier–Stokes motion equations, by a kind of "compensated compactness" argument bearing on fast oscillating acoustic waves; they also introduced a slightly modified notion of renormalized solution which led them to a complete derivation of the evolution Stokes motion equation;
- in [11], it was observed for the first time that the local conservation law of momentum could be *proved* in the hydrodynamic limit, thereby relieving the need for assuming that local conservation law at the level of the renormalized solutions of the Boltzmann equation; this led to a complete proof of the acoustic limit for bounded collision kernels; a more complete understanding of how the local conservation laws of both momentum and energy could be proved in the hydrodynamic limit was eventually reached in [52]; the latter reference provided an essentially optimal derivation of the Stokes motion and energy equations as well as a derivation of the acoustic system that allowed for the most general hard cut-off potentials; however, the acoustic system was established only under some unphysical restriction on the scaling of the number density fluctuation;
- in [106,108], Saint-Raymond gave a complete derivation of both the Navier–Stokes motion and energy equations for the BGK model with constant relaxation time; her proof was based on obtaining for the first time some weaker analogue of the nonlinear compactness assumption used in [10];
- finally, a complete derivation of the incompressible Navier–Stokes motion and heat equations from the Boltzmann equation was proposed for the first time in [60] for bounded collision kernels (such as occurring in the case of cut-off Maxwell molecules); this reference used all the methods constructed in the previous works mentioned above, together with a new velocity averaging method specific to the L^1 case and that amplified Saint-Raymond's observation in [106]; this result was later extended to all hard cut-off potentials (including hard spheres) in [61].

It is this last reference that we describe below; although its scope is more general than that of [60], it involves a new idea for handling unbounded collision kernels that actually simplifies the discussion in [60].

11.4.1. *A priori estimates.* In this subsection we quickly list the a priori estimates on the family F_ε of solutions of the Boltzmann equation that are uniform in $\varepsilon > 0$. As can be seen from Theorem 8.10, the only such estimate comes from the DiPerna–Lions variant of Boltzmann's H-theorem,

$$\begin{aligned} H\big(F_\varepsilon(t)\big|M\big) + \frac{1}{4\varepsilon^2}\int_0^t \int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} &\big(F'_\varepsilon F'_{\varepsilon *} - F_\varepsilon F_{\varepsilon *}\big) \\ &\times \ln\left(\frac{F'_\varepsilon F'_{\varepsilon *}}{F_\varepsilon F_{\varepsilon *}}\right) b \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant H\big(F_\varepsilon^{\mathrm{in}}\big|M\big). \end{aligned} \tag{11.7}$$

We shall further transform (11.7) with the two following inequalities.

Pointwise inequalities. For each ξ and $\eta \in \mathbb{R}_+^*$,

$$\left(\sqrt{\xi}-1\right)^2 \leqslant \frac{1}{4}(\xi \ln \xi - \xi + 1) \tag{11.8}$$

while

$$\left(\sqrt{\xi}-\sqrt{\eta}\right)^2 \leqslant \frac{1}{4}(\xi-\eta)(\ln \xi - \ln \eta). \tag{11.9}$$

Since $(F_\varepsilon - M)/(\varepsilon M)$ converges entropically at rate ε, one has

$$H\left(F_\varepsilon^{\mathrm{in}}\middle|M\right) \leqslant C^{\mathrm{in}}\varepsilon^2. \tag{11.10}$$

This bound and the relative entropy inequality (11.7) entail the following entropy bound for each $t > 0$

$$H\left(F_\varepsilon(t)\middle|M\right) \leqslant C^{\mathrm{in}}\varepsilon^2 \tag{11.11}$$

and the following entropy production bounds

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}^3}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \left(F_\varepsilon' F_{\varepsilon*}' - F_\varepsilon F_{\varepsilon*}\right) \ln\left(\frac{F_\varepsilon' F_{\varepsilon*}'}{F_\varepsilon F_{\varepsilon*}}\right) b \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant 4C^{\mathrm{in}}\varepsilon^4. \end{aligned} \tag{11.12}$$

Introducing the relative number density and relative number density fluctuations,

$$G_\varepsilon = \frac{F_\varepsilon}{M} \quad \text{and} \quad g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}, \tag{11.13}$$

the two pointwise inequalities (11.8) and (11.9) convert the entropy and entropy production bounds into the two uniform a priori estimates

$$\int_{\mathbb{R}^3} \left\langle \left(\sqrt{G_\varepsilon(t)}-1\right)^2 \right\rangle \mathrm{d}x \leqslant \frac{1}{4} C^{\mathrm{in}}\varepsilon^2 \tag{11.14}$$

and

$$\int_0^{+\infty}\int_{\mathbb{R}^3} \left\langle\!\left\langle \left(\sqrt{G_\varepsilon' G_{\varepsilon*}'} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)^2 \right\rangle\!\right\rangle \mathrm{d}x\,\mathrm{d}t \leqslant C^{\mathrm{in}}\varepsilon^4. \tag{11.15}$$

The importance of the two a priori bounds above (11.14) and (11.15) in the derivation of the Navier–Stokes limit cannot be overestimated. In fact, various analogues of these bounds were used earlier in the context of nonlinear diffusion limits, see [12] and especially [58].

The importance of the entropy and entropy dissipation bounds for hydrodynamic limits of diffusive type was noticed in these works. We recall that

$$\langle \phi \rangle = \int_{\mathbb{R}^3} \phi(v) M(v) \, \mathrm{d}v,$$

and we further introduce the notation

$$\langle\!\langle \Phi \rangle\!\rangle = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \Phi(v, v_*, \omega) \, \mathrm{d}\mu(v, v_*, \omega),$$

where

$$\mathrm{d}\mu(v, v_*, \omega) = b(v - v_*, \omega) M(v) \, \mathrm{d}v M(v_*) \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

In the sequel, we outline the main ideas in the derivation of the Navier–Stokes motion equation in Theorem 11.3, since the derivation of the heat equation is essentially analogous.

Of course, this proof more or less follows the formal argument presented in Section 6.1. However, several key properties of the solutions to the scaled Boltzmann equation used in this formal argument – such as, for instance, the local conservation laws of momentum and energy – are not known to be satisfied by renormalized solutions. Hence the proof sketched below differs noticeably from the formal argument in several places, while the general idea remains essentially the same.

11.4.2. *Normalizing functions.* As explained in Section 8.2, the Boltzmann equation can be equivalently renormalized with any admissible nonlinearity whose derivative saturates the quadratic growth of the collision integral.

Throughout the proof of the Navier–Stokes limit theorem, we shall essentially use two kinds of normalizing nonlinearities

- compactly supported nonlinearities that coincide with the identity near the reference Maxwellian state, and
- variants of the maximal, i.e., square-root renormalization.

Nonlinearities of the first kind are used to define the renormalized form of the Boltzmann equation in which one passes to the vanishing ε limit, while the square-root normalization is used to establish compactness properties of the family of solutions to the scaled Boltzmann equation.

The first kind of normalizing nonlinearities is defined through the class of bump functions $\gamma \in C^\infty(\mathbb{R}_+)$ such that

$$\gamma|_{[0,3/2]} \equiv 1, \qquad \gamma|_{[2,+\infty)} \equiv 0, \qquad \gamma \text{ is nonincreasing on } \mathbb{R}_+. \tag{11.16}$$

The Boltzmann equation is then renormalized with the nonlinearity

$$\Gamma(Z) = (Z - 1)\gamma(Z);$$

later on, we denote

$$\hat{\gamma}(Z) = \frac{\mathrm{d}}{\mathrm{d}Z}\big((Z-1)\gamma(Z)\big) = \Gamma'(Z). \tag{11.17}$$

The scaled Boltzmann equation renormalized with Γ is put in the form

$$\partial_t(g_\varepsilon\gamma_\varepsilon) + \frac{1}{\varepsilon}v\cdot\nabla_x(g_\varepsilon\gamma_\varepsilon) = \frac{1}{\varepsilon^3}\hat{\gamma}_\varepsilon\mathcal{Q}_M(G_\varepsilon, G_\varepsilon), \tag{11.18}$$

where we have denoted

$$\gamma_\varepsilon = \gamma(G_\varepsilon), \qquad \hat{\gamma}_\varepsilon = \hat{\gamma}(G_\varepsilon),$$

and where $\mathcal{Q}$ designates the Boltzmann collision integral intertwined with the multiplication by M (see (6.6)),

$$\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG).$$

Later on, we shall pass to the limit in the momentum equation deduced from (11.18).

The second class of normalizing nonlinearities that we shall use to establish compactness properties of the number density fluctuations G_ε is defined as

$$\Gamma_\zeta(Z) = \sqrt{\zeta + Z}, \quad \zeta > 0,$$

where the parameter ζ will later be adapted to ε.

11.4.3. *Governing equations for moments of* g_ε. As explained in Theorem 8.10, renormalized solutions to the Boltzmann equation satisfy the local conservation of mass (i.e., the continuity equation); in terms of the number density fluctuation g_ε, this local conservation law is expressed as

$$\varepsilon\,\partial_t\langle g_\varepsilon\rangle + \mathrm{div}_x\langle vg_\varepsilon\rangle = 0. \tag{11.19}$$

Now, the entropy bound (11.11) implies that

$$\big(1+|v|^2\big)g_\varepsilon \text{ is relatively compact in } w-L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x; L^1(M\,\mathrm{d}v)\big). \tag{11.20}$$

Before saying a few words on (11.20), let us explain how we use it. Modulo extraction of a subsequence, one has

$$g_\varepsilon \to g \quad \text{in } w-L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x; L^1\big(\big(1+|v|^2\big)M\,\mathrm{d}v\big)\big)$$

and hence

$$\langle g_\varepsilon\rangle \to \langle g\rangle \quad \text{and} \quad \langle vg_\varepsilon\rangle \to \langle vg\rangle \quad \text{in } w-L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x).$$

Passing to the limit as $\varepsilon \to 0$ in (11.19) leads to

$$\operatorname{div}_x \langle vg \rangle = 0,$$

so that, denoting

$$u = \langle vg \rangle$$

the relation above is the incompressibility condition in the Navier–Stokes equations, i.e.,

$$\operatorname{div}_x u = 0.$$

Let us go back to (11.20) and explain how it follows from the entropy bound (11.11), see [10] for more details on this. Define

$$h(z) = (1+z)\ln(1+z) - z, \quad z > -1.$$

In terms of h, the entropy bound (11.11) is expressed as

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3} \bigl\langle h\bigl(\varepsilon g_\varepsilon(t)\bigr)\bigr\rangle \,\mathrm{d}x \leqslant C^{\mathrm{in}}.$$

If the entropy bound (11.11) was equivalent to an $L^p(M\,\mathrm{d}v\,\mathrm{d}x)$ bound for some $p > 1$, Hölder's inequality would imply that $(1+|v|)^2 g_\varepsilon$ is bounded in $L^\infty(\mathrm{d}t; L^r(M\,\mathrm{d}v\,\mathrm{d}x))$ for some $r > 1$, since $(1+|v|^2) \in L^q(M\,\mathrm{d}v)$ for each $q \in (1, +\infty)$. However, the entropy control (11.11) on g_ε is weaker than an $L^p(M\,\mathrm{d}v\,\mathrm{d}x)$ bound. But (11.20) follows from a careful use of Young's inequality

$$p|z| \leqslant \frac{\alpha}{\varepsilon^2} h(\varepsilon z) + \frac{1}{\alpha} h^*(p), \quad p > 0, z > -1, 0 < \varepsilon < \alpha,$$

where

$$h^*(p) = \mathrm{e}^p - p - 1$$

designates the Legendre dual of h. The compactness (11.20) follows from replacing z with g_ε and p with $\frac{1}{4}(1+|v|^2)$ in the inequality above, letting then $\alpha \to 0$ in the inequality so obtained.

Let us now explain how the motion equation in the Navier–Stokes system is derived from the Boltzmann equation. This is of course the main part in the proof, and it involves several technicalities.

In particular, we shall need truncations in the velocity variable at a level that is tied to ε. For each function $\xi \equiv \xi(v)$ and each $K > 6$, we define

$$\xi_{K_\varepsilon}(v) = \xi(v)\mathbb{1}_{|v|^2 \leqslant K|\ln \varepsilon|}. \tag{11.21}$$

Multiplying each side of the scaled, renormalized Boltzmann equation (11.18) by each component of v_{K_ε} and averaging in v leads to

$$\partial_t \langle v g_\varepsilon \gamma_\varepsilon \rangle + \mathrm{div}_x\, \mathbf{F}_\varepsilon(A) + \nabla_x \frac{1}{\varepsilon}\left\langle \frac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle = \mathbf{D}_\varepsilon(v), \tag{11.22}$$

where $\mathbf{F}_\varepsilon(A)$ is the truncated, renormalized traceless part of the momentum flux

$$\mathbf{F}_\varepsilon(A) = \frac{1}{\varepsilon}\langle A_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \tag{11.23}$$

while $\mathbf{D}_\varepsilon(v)$ is the momentum conservation defect

$$\mathbf{D}_\varepsilon(v) = \frac{1}{\varepsilon^3}\big\langle\!\big\langle v_{K_\varepsilon} \hat{\gamma}_\varepsilon \big(G'_\varepsilon G'_{\varepsilon *} - G_\varepsilon G_{\varepsilon *}\big)\big\rangle\!\big\rangle. \tag{11.24}$$

Notice that truncating large velocities in the number density, or large values thereof (which is what the renormalization procedure does) break the symmetries in the collision integral leading to the local conservation of momentum (see Proposition 3.1): this accounts for the defect $\mathbf{D}_\varepsilon(v)$ on the right-hand side of (11.22). As $\varepsilon \to 0$, $v_{K_\varepsilon} \to v$ while $G_\varepsilon \to 1$ so that $\hat{\gamma}_\varepsilon \to 1$; hence, the missing symmetries are restored in the integrand defining $\mathbf{D}_\varepsilon(v)$. Hence, one can hope that $\mathbf{D}_\varepsilon(v) \to 0$ as $\varepsilon \to 0$.

In fact, the strategy for establishing the Navier–Stokes limit theorem consists of the following three steps.

Step 1. Prove that, modulo extraction of a subsequence

$$\langle v g_\varepsilon \gamma_\varepsilon \rangle \to \langle v g \rangle = u \quad \text{in } w - L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x),$$

while

$$P\langle v g_\varepsilon \gamma_\varepsilon \rangle \to u \quad \text{in } C\big(\mathbb{R}_+; \mathcal{D}'\big(\mathbb{R}^3\big)\big),$$

where P denotes the Leray projection, i.e., the orthogonal projection on divergence-free vector fields in $L^2(\mathbb{R}^3)$.

Step 2. Likewise, prove that

$$\mathbf{D}_\varepsilon(v) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x).$$

Step 3. Finally prove that

$$P\big(\mathrm{div}_x\, \mathbf{F}_\varepsilon(A)\big) \to P\,\mathrm{div}_x(u \otimes u) - \nu \Delta_x u \quad \text{in } \mathcal{D}'\big(\mathbb{R}^*_+ \times \mathbb{R}^3\big).$$

Once these three steps are completed, one applies P to both sides of (11.22), which gives

$$\partial_t P\langle v g_\varepsilon \gamma_\varepsilon \rangle + P\big(\mathrm{div}_x\, \mathbf{F}_\varepsilon(A)\big) = P\mathbf{D}_\varepsilon(v). \tag{11.25}$$

Taking limits in each term as $\varepsilon \to 0$ shows that u satisfies the Navier–Stokes motion equation. As for the initial condition, observe that it is guaranteed by the uniform convergence in t, i.e., by the second statement in Step 1.

11.4.4. *Vanishing of the momentum conservation defect.* We start with Step 2, i.e., we explain how to prove the following proposition.

PROPOSITION 11.4. *Under the same assumptions as in Theorem* 11.3,

$$\mathbf{D}_\varepsilon(v) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x).$$

First, we start from the elementary formula

$$\begin{aligned} &G'_\varepsilon G'_{\varepsilon*} - G_\varepsilon G_{\varepsilon*} \\ &\quad = \Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} + \sqrt{G_\varepsilon G_{\varepsilon*}}\Big) \\ &\quad = \Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)^2 + 2\sqrt{G_\varepsilon G_{\varepsilon*}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big) \end{aligned}$$

and split the momentum conservation defect as

$$\mathbf{D}_\varepsilon(v) = \mathbf{D}^1_\varepsilon(v) + \mathbf{D}^2_\varepsilon(v)$$

with

$$\mathbf{D}^1_\varepsilon(v) = \frac{1}{\varepsilon^3}\Big\langle\!\Big\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)^2\Big\rangle\!\Big\rangle$$

and

$$\mathbf{D}^2_\varepsilon(v) = \frac{2}{\varepsilon^3}\Big\langle\!\Big\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\sqrt{G_\varepsilon G_{\varepsilon*}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)\Big\rangle\!\Big\rangle.$$

That $\mathbf{D}^1_\varepsilon(v) \to 0$ in $L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ follows from the entropy production estimate (11.15). Setting

$$\Xi_\varepsilon = \frac{1}{\varepsilon^2}\sqrt{G_\varepsilon G_{\varepsilon*}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)$$

we further split $\mathbf{D}^2_\varepsilon(v)$ as

$$\begin{aligned} \mathbf{D}^2_\varepsilon(v) = &-\frac{2}{\varepsilon}\langle\!\langle v\mathbb{1}_{|v|^2 > K_\varepsilon}\hat{\gamma}_\varepsilon\Xi_\varepsilon\rangle\!\rangle + \frac{2}{\varepsilon}\Big\langle\!\Big\langle v\hat{\gamma}_\varepsilon\big(1 - \hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon*}\big)\Xi_\varepsilon\Big\rangle\!\Big\rangle \\ &+ \frac{1}{\varepsilon}\Big\langle\!\Big\langle (v + v_*)\hat{\gamma}_\varepsilon\hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon*}\Xi_\varepsilon\Big\rangle\!\Big\rangle. \end{aligned}$$

The first term is easily mastered by the entropy production estimate (11.15) and the following classical estimate on the tail of Gaussian integrals

$$\int_{\mathbb{R}^N} \mathrm{e}^{-|v|^2/2}|v|^a \mathbb{1}_{|v|^2>R}\,\mathrm{d}v = \mathrm{O}\big(R^{(a+N)/2-1}\mathrm{e}^{-R/2}\big) \quad \text{as } R \to +\infty.$$

Observe that the integrand in the third term has the same symmetries as the original collision integrand (before truncation in $|v|$ and renormalization). It is also mastered by a combination of the entropy production estimate (11.15) with the Gaussian tail estimate above.

The most difficult part in the analysis of the momentum conservation defect is by far the second term in the decomposition of $\mathbf{D}_\varepsilon^2(v)$ above. That it vanishes in $L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ as $\varepsilon \to 0$ ultimately relies upon the following estimate.

Nonlinear compactness estimate.

$$\big(1+|v|\big)^\alpha \left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2 \text{ is uniformly integrable and tight on } [0,T]\times K\times\mathbb{R}^3 \tag{11.26}$$

for the measure $\mathrm{d}t\,\mathrm{d}x\,M\,\mathrm{d}v$, for each $T>0$ and each compact $K\subset\mathbb{R}^3$, where α is the relative velocity exponent that appears in the hard cut-off assumption (3.55) on the collision kernel b.

We shall not give further details on the proof that $\mathbf{D}_\varepsilon^2(v)\to 0$ in $L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$, which is based on the above nonlinear compactness estimate together with the entropy production bound (11.15).

Let us however say a few words on the nonlinear compactness estimate itself. The relative entropy bound (11.11) is essentially as good as an $L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$ bound on g_ε *on the set of* (t,x,v)*'s such that* $G_\varepsilon(t,x,v)=\mathrm{O}(1)$. Elsewhere, it essentially reduces to an $\mathrm{O}(\varepsilon)$ bound in $L^\infty(\mathrm{d}t; L^1(M\,\mathrm{d}v\,\mathrm{d}x))$, which is quite not enough for the Navier–Stokes limit. This is why the first works on this limit *assumed* some variant of this nonlinear compactness estimate. For instance, in either [10] or [86], it was assumed that

$$\big(1+|v|^2\big)\frac{g_\varepsilon^2}{1+G_\varepsilon} \text{ is uniformly integrable and tight on } [0,T]\times K\times\mathbb{R}^3, \tag{11.27}$$

whereas all that was known on this quantity was the estimate

$$\big(1+|v|^2\big)\frac{g_\varepsilon^2}{1+G_\varepsilon} = \mathrm{O}\big(|\ln\varepsilon|\big) \quad \text{in } L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x; L^1(M\,\mathrm{d}v)\big)$$

(see [10]). This led to a decomposition of the number density fluctuation as

$$g_\varepsilon = g_\varepsilon^\flat + \varepsilon g_\varepsilon^\sharp,$$

where the "good" part of the fluctuation is

$$g_\varepsilon^\flat = \frac{2g_\varepsilon}{1+G_\varepsilon} = \mathrm{O}(1) \quad \text{in } L^\infty\big(\mathrm{d}t;\, L^2(M\,\mathrm{d}v\,\mathrm{d}x)\big),$$

while the "bad" part is

$$g_\varepsilon^\sharp = \frac{g_\varepsilon^2}{1+G_\varepsilon} = \mathrm{O}(1) \quad \text{in } L^\infty\big(\mathrm{d}t;\, L^1(M\,\mathrm{d}v\,\mathrm{d}x)\big).$$

In later works – for instance in [106,108] and [60] – this decomposition was slightly modified as follows. Pick a bump function $\gamma \in C_c^\infty(\mathbb{R}_+^*)$ such that

$$\gamma|_{[\frac{3}{4},\frac{5}{4}]} \equiv 1, \quad \operatorname{supp}(\gamma) \subset \left[\frac{1}{2}, \frac{3}{2}\right] \text{ and } 0 \leqslant \gamma \leqslant 1,$$

and define

$$g_\varepsilon^\flat = g_\varepsilon \gamma(G_\varepsilon), \qquad g_\varepsilon^\sharp = \frac{1-\gamma(G_\varepsilon)}{\varepsilon} g_\varepsilon.$$

It was proved in [60] that

$$\big|g_\varepsilon^\flat\big|^2 \text{ is uniformly integrable and tight on } [0,T] \times K \times \mathbb{R}^3 \tag{11.28}$$

for the measure $\mathrm{d}t\,\mathrm{d}x\,M\,\mathrm{d}v$, while

$$g_\varepsilon^\sharp = \mathrm{O}\left(\frac{1}{\ln|\ln\varepsilon|}\right) \quad \text{in } L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x;\, L^1(M\,\mathrm{d}v)\big). \tag{11.29}$$

Observe the difference between these last two controls and (11.27): with the new definition of $g_\varepsilon^\flat$ and $g_\varepsilon^\sharp$, it is no longer true that $|g_\varepsilon^\flat|^2 \leqslant C g_\varepsilon^\sharp$, while $(\frac{g_\varepsilon}{1+G_\varepsilon})^2 \leqslant \frac{g_\varepsilon^2}{1+G_\varepsilon}$, so that (11.27) actually entailed that the square of the good part in the old flat-sharp decomposition is uniformly integrable, even with a quadratic weight in v. In fact, the techniques in [60] did not allow adding a quadratic weight in v as in (11.27), so that this compactness assumption remained unproved; fortunately, it was possible to complete the proof of the Navier–Stokes limit for cut-off Maxwell molecules with only the bounds (11.28)–(11.29), and the weighted estimate

$$\big(1+|v|\big)^s\big(1-\gamma(G_\varepsilon)\big) = \mathrm{O}\left(\frac{\varepsilon^2}{\sqrt{\ln|\ln\varepsilon|}}\right) \quad \text{in } L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x;\, L^1(M\,\mathrm{d}v)\big). \tag{11.30}$$

This control shows that the set where the bad part of the number density fluctuation dominates is small in weighted v-space. There is a definite lack of symmetry between the controls (11.29), bearing on large values of g_ε, and (11.30), bearing on large $|v|$'s. This lack of symmetry is remedied in the most recent variant (11.26) of these nonlinear compactness estimates (see [61]), we shall return to this when sketching the proof of (11.26).

11.4.5. *The asymptotic momentum flux.* With the vanishing of conservation defects (Step 2 in the proof of the Navier–Stokes limit) settled in the previous section, we turn our attention to Step 3, i.e., passing to the limit in the divergence of the momentum flux modulo gradients. This is by far the most difficult part of our analysis, and does require several preparations. In the present subsection, we reduce the momentum flux to some asymptotic normal form, to which we eventually apply compactness results to be described later.

LEMMA 11.5. *Let Π be the $L^2(M\,\mathrm{d}v)$-orthogonal projection on* $\operatorname{Ker}\mathcal{L}$; *then, under the same assumptions as in Theorem* 11.3,

$$\mathbf{F}_\varepsilon(A) = \left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle - 2\left\langle \widehat{A}\frac{1}{\varepsilon^2}\mathcal{Q}_M\big(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}\big)\right\rangle + o(1)_{L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)},$$

where we recall that the tensor field $\widehat{A}$ is defined by

$$\widehat{A}\perp\operatorname{Ker}\mathcal{L}_M\quad\textit{and}\quad\mathcal{L}_M\big(\widehat{A}\big)=A=v\otimes v-\frac{1}{3}|v|^2I.$$

The proof of this lemma is based upon splitting the momentum flux as

$$\begin{aligned}\mathbf{F}_\varepsilon(A) &= \frac{1}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon\frac{G_\varepsilon-1}{\varepsilon}\right\rangle\\ &= \left\langle A_{K_\varepsilon}\gamma_\varepsilon\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle + \frac{2}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle\\ &= \mathbf{F}^1_\varepsilon(A)+\mathbf{F}^2_\varepsilon(A),\end{aligned}$$

as a consequence of the elementary identity

$$\begin{aligned}\frac{1}{\varepsilon}(G_\varepsilon-1) &= \frac{1}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big)\big(\sqrt{G_\varepsilon}+1\big)\\ &= \frac{1}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big)^2+\frac{2}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big).\end{aligned}$$

Then, one applies the following corollary of the nonlinear compactness estimate (11.26).

COROLLARY 11.6. *Under the same assumptions as in Theorem* 11.3,

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}-\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\to 0\quad\textit{in } L^2\big(\mathrm{d}t\,\mathrm{d}x;L^2\big(\big(1+|v|^\alpha\big)M\,\mathrm{d}v\big)\big)$$

as $\varepsilon\to 0$.

With the corollary above, one can show that the term $\mathbf{F}_\varepsilon^1(A)$ in the decomposition of the momentum flux is asymptotically close to

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle$$

(notice that the high velocity truncation is disposed of since $\Pi(\sqrt{G_\varepsilon}-1)/\varepsilon$ has at most polynomial growth in v as $|v|\to+\infty$). That the second term $\mathbf{F}_\varepsilon^2(A)$ is asymptotically close to

$$\left\langle \widehat{A}\frac{1}{\varepsilon^2}\mathcal{Q}_M\big(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}\big)\right\rangle,$$

as $\varepsilon\to 0^+$, uses Lemma 6.2.

Next, we explain how Lemma 11.5 is used in the proof of the Navier–Stokes limit. To begin with, since

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\simeq\frac{1}{2}g_\varepsilon\gamma_\varepsilon,$$

one has

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle\simeq\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\otimes\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle-\frac{1}{3}\big|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\big|^2 I.$$

On the other hand, the entropy production estimate (11.15) implies that, modulo extraction of a subsequence, one has

$$\frac{1}{\varepsilon^2}\big(\sqrt{G_\varepsilon' G_{\varepsilon*}'}-\sqrt{G_\varepsilon G_{\varepsilon*}}\big)\to q$$

in $w-L^2(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mu)$. Passing to the limit in the scaled, renormalized Boltzmann equation (11.18) entails the relation

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{S}^2} q\,b(v-v_*,\omega)M\,\mathrm{d}v_*\,\mathrm{d}\omega\\&\quad=v\cdot\nabla_x g=\frac{1}{2}A:\nabla_x u+\text{terms that are odd in } v.\end{aligned}$$

Eventually we arrive at the following asymptotic form of the momentum flux.

PROPOSITION 11.7. *Under the same assumptions as in Theorem* 11.3, *one has*

$$\begin{aligned}\mathbf{F}_\varepsilon(A)&=\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\otimes\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle-\frac{1}{3}\big|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\big|^2 I\\&\qquad-\nu\big(\nabla_x u+(\nabla_x u)^T\big)+\mathrm{o}(1)_{L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)},\end{aligned}$$

where

$$u = \langle vg \rangle \quad \textit{and} \quad g = \lim_{\varepsilon \to 0} g_\varepsilon \quad \textit{in } w - L^1_{\mathrm{loc}}\big(\mathrm{d}t\,\mathrm{d}x; L^1(M\,\mathrm{d}v)\big).$$

11.4.6. *Strong compactness arguments.* In order to pass to the limit in the quadratic term $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ and to conclude that

$$P\,\mathrm{div}_x\left(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - \frac{1}{3}\big|\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\big|^2 I\right)$$
$$\to P\,\mathrm{div}_x\left(u \otimes u - \frac{1}{3}|u|^2 I\right)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$ as $\varepsilon \to 0$, the weak convergence properties of g_ε established so far are clearly insufficient. One needs instead some *strong* compactness properties on the family $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$.

(a) *Strong compactness in the x-variable.* Velocity averaging is the natural way to obtain compactness in the space variable x for kinetic equations in the parabolic scaling (11.6).

For the purpose of studying the compactness of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ in the x-variable, we use the following variant of the L^2-based velocity averaging theorem.

LEMMA 11.8. *Let ϕ_ε be a bounded family in $L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x; L^2(M\,\mathrm{d}v))$ such that $|\phi_\varepsilon|^2$ is locally uniformly integrable on $\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3$ for the Lebesgue measure. Assume that*

$$(\varepsilon\,\partial_t + v \cdot \nabla_x)\phi_\varepsilon \textit{ is bounded in } L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v).$$

Then, for each $\psi \in L^2(M\,\mathrm{d}v)$, the family $\langle \phi_\varepsilon \psi \rangle$ is relatively compact in $L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ with respect to the x-variable, meaning that, for each $T > 0$ and each compact $K \subset \mathbb{R}^3$, one has

$$\iint_{[0,T]\times K} \big|\langle \phi_\varepsilon \psi \rangle(t, x+y) - \langle \phi_\varepsilon \psi \rangle(t,x)\big|^2\,\mathrm{d}t\,\mathrm{d}x \to 0$$

as $y \to 0$ uniformly in ε.

See [60] for the proof, which is somewhat similar to the L^1 case of velocity averaging recalled in Section 8.2.2 (especially Proposition 8.6).

Now, we apply the lemma above to

$$\phi_\varepsilon = \frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}$$

since

$$(\varepsilon\,\partial_t + v \cdot \nabla_x)\phi_\varepsilon = \frac{1}{\varepsilon^2}\frac{\mathcal{Q}(G_\varepsilon, G_\varepsilon)}{2\sqrt{\varepsilon^c + G_\varepsilon}} = \mathrm{O}(1)_{L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v)}$$

for $c \in (1, 2)$, by the entropy production estimate (11.15). Since

$$\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon} \simeq \frac{1}{2} g_\varepsilon \gamma_\varepsilon,$$

applying the velocity averaging lemma above leads to the following compactness ("in the x-variable") result.

PROPOSITION 11.9. *Under the same assumptions as in Theorem* 11.3, *for each* $T > 0$ *and* $K \subset \mathbb{R}^3$ *compact, one has*

$$\iint_{[0,T]\times K} \left| \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x + y) - \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x) \right|^2 \mathrm{d}t \, \mathrm{d}x \to 0$$

uniformly in ε *as* $y \to 0$.

(b) *Strong compactness in the t-variable.* It remains to obtain compactness in the time variable. As we shall see, the solenoidal part of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ is strongly compact in the t-variable, but its orthogonal complement, which is a gradient field, is not because of high frequency oscillations in t.

PROPOSITION 11.10. *Under the assumptions of Theorem* 11.3, *modulo extraction of a subsequence, one has*

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u$$

in $C(\mathbb{R}_+; w - L^2_x)$ *and in* $L^2_{\mathrm{loc}}(\mathrm{d}t \, \mathrm{d}x)$ *as* $\varepsilon \to 0$.

PROOF. Indeed, Proposition 11.9 and the translation invariance of the Leray projection P together with the fact that P is a pseudo-differential operator of order 0 imply that

$$\iint_{[0,T]\times K} \left| P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x + y) - P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x) \right|^2 \mathrm{d}t \, \mathrm{d}x \to 0 \tag{11.31}$$

uniformly in ε as $y \to 0$. On the other hand, the conservation law (11.25) implies that

$$\partial_t \int_{\mathbb{R}^3} P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot \xi \, \mathrm{d}x = \mathrm{O}(1) \quad \text{in } L^1_{\mathrm{loc}}(\mathrm{d}t), \tag{11.32}$$

for each compactly supported, solenoidal vector field $\xi \in H^3(\mathbb{R}^3)$, since we know from Lemma 11.5 and the bounds (11.14) and (11.15), that $\mathbf{F}_\varepsilon(A)$ is bounded in $L^1_{\mathrm{loc}}(\mathrm{d}t \, \mathrm{d}x)$. Also,

$$g_\varepsilon \gamma_\varepsilon \leqslant \left(1 + \sqrt{2}\right) \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}$$

so that (11.14) implies that

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle = \mathrm{O}(1) \quad \text{in } B\big(\mathbb{R}_+; L^2\big(\mathbb{R}^3\big)\big), \tag{11.33}$$

where $B(X, Y)$ denotes the class of bounded maps from X to Y.

Since the class of H^3, compactly supported solenoidal vector fields is dense in that of all H^3 solenoidal vector fields (see Appendix A of [85]), (11.33) and (11.32) imply that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \text{ is relatively compact in } C\big(\mathbb{R}_+; w - L^2\big(\mathbb{R}^3\big)\big), \tag{11.34}$$

by a variant of Ascoli's theorem that can be found in Appendix C of [85].

As for the $L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ compactness, observe that (11.34) implies that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \text{ is relatively compact in } L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x),$$

where χ_δ designates any mollifying sequence and $\star$ is the convolution in the x-variable only. Hence

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \to Pu \cdot Pu \star \chi_\delta$$

in $w - L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ as $\varepsilon \to 0$. By (11.31),

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \to P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$$

in $L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ uniformly in ε as $\delta \to 0$. With this, we conclude that

$$\big|P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\big|^2 \to |Pu|^2 \quad \text{in } w - L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$$

which implies that $P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to Pu$ strongly in $L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$. $\square$

Next, consider

$$\nabla_x \pi_\varepsilon = \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle.$$

Since

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u \quad \text{in } w - L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x) \quad \text{and} \quad \mathrm{div}_x\, u = 0$$

one has

$$\nabla_x \pi_\varepsilon \to 0 \quad \text{in } w - L^2_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x) \tag{11.35}$$

as $\varepsilon \to 0$. Decompose then

$$\begin{aligned}
&P\,\mathrm{div}_x\big(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\big)\\
&\quad = P\,\mathrm{div}_x\big(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\big) + P\,\mathrm{div}_x\big(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \nabla_x \pi_\varepsilon\big)\\
&\qquad + P\,\mathrm{div}_x\big(\nabla_x \pi_\varepsilon \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\big) + P\,\mathrm{div}_x(\nabla_x \pi_\varepsilon \otimes \nabla_x \pi_\varepsilon).
\end{aligned}$$

By Proposition 11.10, the first term converges to $P\,\mathrm{div}_x(u\otimes u)$ in the sense of distributions, while the second and third terms converge to 0 in the sense of distributions because of (11.35).

As for the last term, let $\zeta_\delta=\xi_\delta\star\xi_\delta\star\xi_\delta$, where ξ_δ is an approximate identity. One can prove that

$$\varepsilon\,\partial_t\zeta_\delta\star_x\nabla_x\pi_\varepsilon+\nabla_x\zeta_\delta\star_x\left\langle\frac{1}{3}|v|^2_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\right\rangle\to 0\quad\text{in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+;H^1_{\mathrm{loc}}(\mathbb{R}^3)\big),$$
$$\varepsilon\,\partial_t\zeta_\delta\star_x\left\langle\frac{1}{3}|v|^2_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\right\rangle+\frac{5}{3}\Delta_x\zeta_\delta\star_x\pi_\varepsilon\to 0\quad\text{in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+;H^1_{\mathrm{loc}}(\mathbb{R}^3)\big)$$

as a result of (11.22), the vanishing of momentum and energy conservation defects (see Proposition 11.4 for the momentum, and proceed analogously for the energy) and the fact that $\mathbf{F}_\varepsilon(A)$ is bounded in $L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)$ (see Lemma 11.5, and the bounds (11.14) and (11.15)). From the above system, Lions and Masmoudi observed in [86] that

$$\begin{aligned}&\mathrm{div}_x(\nabla_x\zeta_\delta\star_x\pi_\varepsilon\otimes\nabla_x\zeta_\delta\star_x\pi_\varepsilon)\\&\quad=\frac{1}{2}\nabla_x\left(|\nabla_x\zeta_\delta\star_x\pi_\varepsilon|^2-\frac{5}{3}\zeta_\delta\star_x\left\langle\frac{1}{3}|v|^2_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\right\rangle^2\right)+\mathrm{o}(1)_{L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)}.\end{aligned}$$

Together with the uniform compactness "in the x-variable" proved in Proposition 11.9 and (11.31) this implies that

$$P\,\mathrm{div}_x(\nabla_x\pi_\varepsilon\otimes\nabla_x\pi_\varepsilon)\to 0$$

in the sense of distributions. Collecting the observations above, we have just proved the following proposition.

PROPOSITION 11.11. *Under the assumptions of Theorem* 11.3, *modulo extraction of a subsequence*, *one has*

$$P\,\mathrm{div}_x\big(\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\otimes\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\big)\to P\,\mathrm{div}_x(u\otimes u)$$

in the sense of distributions on $\mathbb{R}^*_+\times\mathbb{R}^3$ *as* $\varepsilon\to 0$.

With this proposition, we have completed all three steps in the proof of the Navier–Stokes limit (Theorem 11.3).

11.5. *The nonlinear compactness estimate*

It only remains to prove the nonlinear compactness estimate (11.26), on which the two most important steps in the proof of the Navier–Stokes limit – i.e., the vanishing of conservation defects and the limiting form of the momentum flux – are based.

This nonlinear compactness estimate results from new ideas on velocity averaging in L^1; consistently with our presentation of this subject in Section 8.2.2, we shall describe these ideas on the steady transport equation.

We start with a definition of the notion of partial uniform integrability in a product space.

DEFINITION 11.12. Let μ and ν be positive regular Borel measures on $\mathbb{R}^N$; we say that a family ϕ_ε of elements of $L^1(\mathbb{R}_x^N \times \mathbb{R}_y^N; \mathrm{d}\mu(x)\,\mathrm{d}\nu(y))$ is uniformly integrable in the variable y if

$$\int_{\mathbb{R}^N} \left(\sup_{\nu(A)<\eta} \int_A \left|\phi_\varepsilon(x,y)\right| \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) \to 0$$

as $\eta \to 0$, uniformly in ε.

EXAMPLE. Assume that ν is a finite measure; then, for each $p > 1$, any bounded family in $L^1(\mathrm{d}\mu(x); L^p(\mathrm{d}\nu(y)))$ is uniformly integrable in y. Indeed, if ϕ_ε is any such family, one has

$$\int_{\mathbb{R}^N} \left(\sup_{\nu(A)<\eta} \int_A \left|\phi_\varepsilon(x,y)\right| \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) \leqslant \eta^{1/p'} \int_{\mathbb{R}^N} \left\|\phi_\varepsilon(x,\cdot)\right\|_{L^q(\mathrm{d}\nu)} \mathrm{d}x,$$

where $p' = p/(p-1)$, by applying Hölder's inequality to the inner integral.

As usual, we shall say that a family $\phi_\varepsilon \in L^1(\mathbb{R}_x^N \times \mathbb{R}_y^N; \mathrm{d}\mu(x)\,\mathrm{d}\nu(y))$ is locally uniformly integrable in y if, for each compact $K \subset \mathbb{R}^N \times \mathbb{R}^N$, the family $\mathbb{1}_K \phi_\varepsilon$ is uniformly integrable in y.

With this notion of partial uniform integrability, we can formulate an important improvement of the L^1-variant of velocity averaging stated in Proposition 8.6 and Theorem 8.7.

THEOREM 11.13 (Golse and Saint-Raymond [59]). *Let f_ε be a bounded family in $L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N; \mathrm{d}x\,\mathrm{d}y)$ such that*

- *the family $v \cdot \nabla_x f_\varepsilon$ is bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N; \mathrm{d}x\,\mathrm{d}y)$, and*
- *the family f_ε is locally uniformly integrable in the variable v.*

Then

(1) *the family f_ε is locally uniformly integrable (in both variables (x, v)), and*
(2) *for each compactly supported $\phi \in L^\infty(\mathbb{R}^N)$, the family of averages*

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathrm{d}x)$.

This result amplifies an earlier remark by Saint-Raymond who observed in [106] that, under the extra assumption that f_ε is bounded in $L^1_x(L^\infty_y)$, the family of averages

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v$$

is locally uniformly integrable.

SKETCH OF THE PROOF OF THEOREM 11.13. We shall explain how to prove Saint-Raymond's result under the assumptions of the theorem above.

Step 1. Let $\chi \equiv \chi(t,x,v)$ be the solution of the free transport equation

$$\partial_t\chi + v\cdot\nabla_x\chi = 0, \quad (x,v)\in\mathbb{R}^N\times\mathbb{R}^N, t>0,$$
$$\chi(0,x,v) = \mathbb{1}_A(x), \quad (x,v)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Clearly, $\chi(t,x,v) = \mathbb{1}_A(x-tv)$ for each $t>0$; it can therefore be put in the form

$$\chi(t,x,v) = \mathbb{1}_{A_{t,x}}(v), \quad t>0, (x,v)\in\mathbb{R}^N\times\mathbb{R}^N,$$

where, for each $t>0$,

$$A_{t,x} = \left\{v\in\mathbb{R}^N \,\middle|\, x-tv\in A\right\}.$$

Further, $A_{t,x}$ is measurable and, for each $t>0$ and $x\in\mathbb{R}^N$, one has

$$|A_{t,x}| = \int_{\mathbb{R}^N}\chi(t,x,v)\,\mathrm{d}v = \int_{\mathbb{R}^N}\mathbb{1}_A(x-tv)\,\mathrm{d}v = \frac{1}{t^N}\int_{\mathbb{R}^N}\mathbb{1}_A(z)\,\mathrm{d}z = \frac{|A|}{t^N}.$$

Step 2. Without loss of generality, assume that f_ε and ϕ_ε in the statement of the theorem are nonnegative, and that all the f_ε's are supported in the same compact K of $\mathbb{R}^N\times\mathbb{R}^N$. Then

$$\begin{aligned}
&\int_A\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^N}\int_{A_{t,x}} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad - \int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\chi(s,x,v)v\cdot\nabla_x f_\varepsilon(x,v)\phi(v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s
\end{aligned} \tag{11.36}$$

as can be seen by integrating by parts the second integral on the right-hand side of the equality above.

Pick $\eta > 0$ arbitrarily small; the second integral on the right-hand side of (11.36) satisfies

$$\left| \int_0^t \iint_{\mathbb{R}^N \times \mathbb{R}^N} \chi(s,x,v) v \cdot \nabla_x f_\varepsilon(x,v) \phi(v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \right| \leqslant t \| v \cdot \nabla_x f_\varepsilon \|_{L^1} \|\phi\|_{L^\infty}$$

and therefore can be made less than η by choosing

$$0 < t < \frac{\eta}{1 + \| v \cdot \nabla_x f_\varepsilon \|_{L^1} \|\phi\|_{L^\infty}}.$$

For this $t > 0$, the first integral on the right-hand side of (11.36) satisfies

$$\int_{\mathbb{R}^N} \int_{A_{t,x}} f_\varepsilon(x,v) \phi(v) \, \mathrm{d}v \, \mathrm{d}x \to 0$$

as $|A| \to 0$ uniformly in ε, since f_ε is uniformly integrable in v and $|A_{t,x}| = |A|/t^N$, as established in Step 1.

Therefore, for each $\eta > 0$, there exists $\alpha > 0$ such that $|A| < \alpha$ implies that

$$\int_A \int_{\mathbb{R}^N} f_\varepsilon(x,v) \phi(v) \, \mathrm{d}v \, \mathrm{d}x \leqslant 2\eta$$

uniformly in $\varepsilon > 0$, which entails that the family of averages

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v) \phi(v) \, \mathrm{d}v$$

is locally uniformly integrable. □

The nonlinear compactness estimate (11.26) will of course be obtained from statement (1) in Theorem 11.13. In fact, we first observe that, for each $c \in (1,2)$,

$$\phi_\varepsilon^\delta = \left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon} \right)^2 \gamma \left(\varepsilon\delta \left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon} \right) \right)$$

satisfies

$$\phi_\varepsilon^\delta = \mathrm{O}(1) \quad \text{in } L_t^\infty\big(L^1(M \, \mathrm{d}v \, \mathrm{d}x)\big),$$

while

$$(\varepsilon \, \partial_t + v \cdot \nabla_x) \phi_\varepsilon^\delta = \mathrm{O}(1) \quad \text{in } L^1_{\mathrm{loc}}(\mathrm{d}t \, \mathrm{d}x \, M \, \mathrm{d}v).$$

We next let $\delta \to 0$ and remove the ε^c from under the square root (in that order) so that

$$\phi_\varepsilon^\delta \simeq \left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \right)^2.$$

In order to apply Theorem 11.13 to ϕ_ε^δ, it remains to prove that this family is uniformly integrable in the v-variable. In fact, we prove the following results.

PROPOSITION 11.14. *Under the assumptions of Theorem* 11.3, *for each* $T > 0$ *and each compact* $K \subset \mathbb{R}^3$, *the family*

$$(1+|v|)^\alpha \left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2$$

is uniformly integrable in the v*-variable on* $[0,T] \times K \times \mathbb{R}^3$ *for the measure* $\mathrm{d}t\,\mathrm{d}x\,M\,\mathrm{d}v$. (*We recall that* α *is the relative velocity exponent in the hard cut-off assumption on the collision kernel* b, *see* (3.55).)

This proposition improves upon the result in [60], that applied to cut-off Maxwell molecules only (i.e., to the case where $\alpha = 0$). Its proof is fairly technical, so that we shall just sketch the main idea in it.

Start from the identity

$$\begin{aligned} &\mathcal{L}_M\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right) \\ &\quad = \varepsilon \mathcal{Q}_M\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}, \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right) - \frac{1}{\varepsilon}\mathcal{Q}_M\left(\sqrt{G_\varepsilon}, \sqrt{G_\varepsilon}\right). \end{aligned} \tag{11.37}$$

Next, we recall the Bardos–Caflisch–Nicolaenko spectral gap for $\mathcal{L}_M$ in weighted space (see Theorem 3.11)

$$\langle \phi \mathcal{L}_M \phi \rangle \geqslant C_0 \left\langle (1+|v|)^\alpha \phi^2 \right\rangle, \quad \phi \in (\operatorname{Ker} \mathcal{L}_M)^\perp$$

together with the following continuity estimate for $\mathcal{Q}$ (see [56])

$$\left\| \mathcal{Q}_M(\phi,\phi) \right\|_{L^2((1+|v|)^{-\alpha} M\,\mathrm{d}v)} \leqslant C \|\phi\|_{L^2(M\,\mathrm{d}v)} \|\phi\|_{L^2((1+|v|)^\alpha M\,\mathrm{d}v)}.$$

Using both estimates in the identity above leads to the following control

$$\begin{aligned} &\left(1 - \mathrm{O}(\varepsilon)\left\| \frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \right\|_{L^2(M\,\mathrm{d}v)}\right) \left\| \frac{\sqrt{G_\varepsilon}-1}{\varepsilon} - \Pi \frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \right\|_{L^2((1+|v|)^\alpha M\,\mathrm{d}v)} \\ &\quad \leqslant \mathrm{O}(\varepsilon)_{L^2_{t,x}} + \mathrm{O}(\varepsilon) \left\| \frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \right\|^2_{L^2(M\,\mathrm{d}v)}. \end{aligned}$$

This control suggests that

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \text{ is close to its hydrodynamic projection } \Pi \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$$

precisely in that weighted L^2 space that appears in the statement of Proposition 11.14. Since the hydrodynamic projection $\Pi \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$ is as regular in v as one can hope for (being a quadratic polynomial in v), this eventually entails uniform integrability in v once the difficulties related to the (t,x) dependence in the estimate above have been handled. As we already indicated above, the remaining part of the proof is too technical to be described here, and we refer the interested reader to [61] for a complete argument.

Instead of giving more details on this proof, we comment on the differences between the nonlinear compactness estimate (11.28)–(11.30) obtained in [60] in the case of cut-off Maxwell molecules, and the most recent variant of such controls obtained in [61], namely (11.26). In fact, the main difference between these controls lies in the method for obtaining uniform integrability in v.

In [60], the decomposition (11.37) was replaced with

$$G_\varepsilon = \left(G_\varepsilon - \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon\rangle}\right) + \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon\rangle} \tag{11.38}$$

(in fact, with a more technical variant of (11.38) involving several truncations), where $\mathcal{A}^+$ was the gain part of a fictitious collision operator

$$\mathcal{A}^+(\phi,\phi) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} \phi'\phi'_*\big|\cos(v-v_*,\omega)\big|\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Roughly speaking the term

$$\left(G_\varepsilon - \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon\rangle}\right)$$

was easily controlled by the entropy production bound, while the term

$$\frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon\rangle}$$

improved the regularity and decay in v due to estimates by Grad [62] and Caflisch [23] – that are linear analogues of the result by Lions on the smoothing properties of the gain part in the Boltzmann collision integral (Lemma 3.10).

However, the main drawback of the decomposition (11.38) was that it naturally involved the quantity

$$g_\varepsilon\gamma_\varepsilon + \frac{1-\gamma_\varepsilon}{\varepsilon}$$

(see (2.17) in [60]), thereby leading to a dissymmetry between the roles of large v's and large values of g_ε in the estimates (11.28)–(11.30). This, as we already mentioned, was an obstruction in upgrading (11.28)–(11.29) with a $(1+|v|)^\alpha$ weight as in (11.26).

The method used in [61] essentially differs from that of [60]; to begin with, we do not attempt to establish that the number density F_ε is close in some sense to its associated local equilibrium M_{F_ε} (i.e., the Maxwellian with same local density, energy and momentum as F_ε), an idea that is very natural in the context of the BGK model of the Boltzmann equation, see [106,108].

Instead, we seek to prove that some variant of the number density fluctuation, namely the quantity $\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \simeq \frac{1}{2} g_\varepsilon$, becomes close to its associated infinitesimal Maxwellian, i.e., its $L^2(M\,\mathrm{d}v)$-orthogonal projection on $\operatorname{Ker}\mathcal{L}_M$. This new approach naturally leads to using the Dirichlet form of the linearized collision operator, whose relative coercivity transversally to $\operatorname{Ker}\mathcal{L}_M$ provides exactly the $(1+|v|)^\alpha$ weight needed in the proof.

12. Conclusions and open problems

This survey of the Boltzmann equation and its hydrodynamic limits remains incomplete in several respects. As already mentioned in the Introduction, we have chosen to emphasize global theories (such as the DiPerna–Lions theory for the Boltzmann equation, or the Leray theory for the Navier–Stokes equations) for the evolution problem posed on a spatial domain without boundaries. More realistic formulations of the hydrodynamic limits of the Boltzmann equation would certainly involve boundary value problems. At the formal level, the influence of boundaries on hydrodynamic limits is discussed at length in Sone's authoritative monograph [114].

More specifically, hydrodynamic limits of boundary value problems may, in some particular cases, involve boundary layer equations whose mathematical study is a subject of its own and has inspired a considerable amount of literature. These boundary layers are meant to forget the specific dependence of the boundary data upon the velocity variable so as to match it with the inner form of the number density, which, as explained in this survey, is a local Maxwellian or infinitesimal Maxwellian in the hydrodynamic limit. This is obviously not the place for a description of that theory, to begin with because these boundary layers are essentially steady (instead of evolution) problems. The interested reader could start with Chapter 5, Section 5 of [27] and then get acquainted with the mathematical theory of half-space problems as exposed in [6,26,31,56,57,92] and [118]. These works deal with linearized, or at best, weakly nonlinear problems. Nonlinear boundary layer equations with nonzero net mass flux at the boundary – typically in the case of a gas condensing on, or evaporating from a solid boundary – have been studied in depth numerically by Sone, Aoki and their collaborators, see Chapter 7 of [114] and the references therein. At the formal level, their theory is expected to agree with a general theory of boundary conditions for the Euler system of compressible fluids, yet to be formulated at the time of this writing.

In some cases however, boundary layer are not leading order effects. This is the case of the incompressible Stokes or Navier–Stokes limit of the Boltzmann equation for a gas in a container with specular or diffuse reflection, or Maxwell's accommodation condition at the boundary; see [95], which extends to such boundary value problems the results in [52]. The analogue of the DiPerna–Lions theory for the boundary value problem is due to Mischler [96] and is by no means a straightforward extension of [37]. It requires rather subtle analytical tools because natural boundary conditions for the Boltzmann equation – except in the case of specular reflection of gas molecules at the boundary – do not interact well with the renormalization procedure which is essential in controlling the collision integral.

Let us conclude with a few open problems, in addition to those already mentioned in the body of this article.

It would be of considerable interest to derive the global BV solutions of the compressible Euler system constructed by T.-P. Liu from the Boltzmann equation. As in the case of the incompressible Euler limit of the Boltzmann equation, the entropy production bound entailed by Boltzmann's H-theorem does not balance the action of the streaming operator on the number density: the compactness of hydrodynamic moments of the number density is probably to be sought in some stability property of BV solutions of the compressible Euler system. Most likely, such a theory should use Bressan's remarkable ideas in that direction (see [19,20]).

As an indication of the level of difficulty of this problem, let us simply mention that the hydrodynamic limit of the Boltzmann equation leading to a single Riemann problem has not been established so far, except in the case of a single, weak shock wave. Shock profiles for the Boltzmann equation were formally discussed in [25]; for the hard sphere gas, a complete construction of these profiles in the weakly nonlinear regime can be found in a series of important papers by Nicolaenko [98–100]. All these constructions were based on a deep understanding of the algebra of the linearized Boltzmann equation in connection with the underlying saddle point structure of subsonic states in gas dynamics. They suggested some topological structure that remains to be unraveled and could ultimately lead to a construction of shock profiles of arbitrary strength. The extension to other cut-off molecular interactions is due to Caflisch and Nicolaenko [24]. However, all these constructions led to solutions of the Boltzmann equation whose positivity was not established. This latter problem was solved only very recently by Liu and Yu [89].

Another open problem would be to improve Theorem 11.1, by relaxing the unphysical assumption made on the size of the number density fluctuations δ_ε to reach the physically natural condition that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. This would probably require more information on the local conservation laws of momentum and energy for renormalized solutions of the Boltzmann equation. Such information would most likely be an important prerequisite for making progress on the compressible Euler limit.

Finally, we have only considered evolution problems in this survey. Although steady problems are beyond the scope of the present book, they are perhaps even more important in some applications (such as aerodynamics). For instance, it is well known that, for any divergence free force field $f \equiv f(x) \in L^2(\Omega;\mathbb{R}^3)$, the steady incompressible Navier–Stokes equations in a smooth, bounded open domain $\Omega \subset \mathbb{R}^3$

$$\begin{aligned} &-\nu\Delta_x u = f - \nabla_x p - (u\cdot\nabla_x)u, \qquad \operatorname{div}_x u = 0, \quad x \in \Omega,\\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{12.1}$$

has at least one classical solution $u \equiv u(x) \in H^2(\Omega,\mathbb{R}^3)$, obtained by a Leray–Schauder fixed point argument (see, for instance, [73]). Unfortunately, the parallel theory for the Boltzmann equation is not as advanced: for one thing, at the time of this writing, there is no analogue of the DiPerna–Lions result for the steady Boltzmann equation with a prescribed external force field. The reader interested in those matters is advised to read the survey article by Maslova [93]; see also her book [94]; among classical references on the subject, there are some papers by Guiraud (see [64–66]), and more recent work by Arkeryd and Nouri (see, for instance, [3]). See also [116] for the case of a gas flow modeled by the Boltzmann equation around a convex body. Finally, a rather exhaustive description of such steady problems and their hydrodynamic limits (at the formal level) may be found in Sone's book [114].

In spite of the difficulties inherent to the steady Boltzmann equation, the fact that the solutions of (12.1) are more regular than what is known in the case of the evolution problem could be of considerable help, at least in the context of the hydrodynamic limit.

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CHAPTER 4

Long-Time Behavior of Solutions to Hyperbolic Equations with Hysteresis

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Abstract

For a quasilinear hyperbolic system with Dirichlet boundary conditions and with hysteretic constitutive law describing waves in elastoplastic solids, we give an overview of results on existence, uniqueness and asymptotic stability of solutions if either initial data or the time-periodicity condition are prescribed. Convexity in the hysteresis diagrams implies the exis-

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tence of a second-order dissipation term which in turn prevents the system from formation of shocks.

Introduction

The main problem in quasilinear hyperbolic equations consists in the fact that shock-free solutions exist, in general, only for small times even if the data are smooth, and weak solutions are difficult to handle. A recent overview can be found, e.g., in [4]. The situation is substantially different if the nonlinearity in the constitutive law has a hysteresis character. In many cases, shocks do not occur and the solution remains regular. The monograph [14] was an attempt to give an explanation of this fact. It was shown that in the p-system, trajectories of entropy solutions to the Riemann problem tend to follow the convex hull of the constitutive graph. If now the constitutive graph consists of *convex hysteresis loops*, it is natural to expect that the solution trajectories will follow the hysteresis branches and, in terms of gas dynamics, only rarefaction takes place. There is a substantial difference between viscous (rate-dependent) and plastic (rate-independent) dissipation in hyperbolic equations. While the former transforms the problem into essentially a parabolic one with unbounded speed of propagation, the latter preserves the hyperbolic character of the balance equations and the propagation speed is bounded independently of the hysteresis term, see Proposition 1.0.2.

Existence, uniqueness and qualitative properties of solutions to the wave equation with elastoplastic hysteresis were studied in [14] for a fairly general class of hysteresis operators and under mixed boundary conditions, where displacement is prescribed on one end and stress on the other end of the space interval. The material presented here is mostly new. Besides the interaction with additional lower-order nonlinearities, we consider the case of Dirichlet boundary conditions, where the displacement (or velocity, depending on the setting) is prescribed on both ends. This is more difficult, since the a priori estimate for the space derivative of the stress depends on the sup-norm of the stress for which no boundary condition is available. The estimation technique thus has to use more refined arguments based on special properties of the hysteresis memory. In order to reduce the complexity, we restrict ourselves to the so-called *Prandtl–Ishlinskii hysteresis operator*, although much of the results remain valid for a larger class of convex hysteresis operators, and an example is presented in Section 1.4.

If convexity of the hysteresis loops is lost, like in Maxwell's equations for large amplitude electromagnetic waves in ferromagnetic materials, then the regular behavior cannot be expected and shocks are again likely to occur. The only publication on this subject seems to be [26], where existence of weak solutions on a bounded time interval has been established for a very general class of hysteresis operators.

The text is organized as follows. In Section 1 we state and solve the problem of existence, uniqueness and asymptotic stability of regular solutions to an initial–boundary value problem for a wave equation with a Prandtl–Ishlinskii stress–strain law. We also show that the regularity is preserved under nonlinear perturbations of the constitutive law if convexity is not violated. The time-periodic problem is investigated in Section 2 and we again prove results of existence, uniqueness and asymptotic stability for this case. The last Section 3 is a collection of known results on the Prandtl–Ishlinskii model. This part has an auxiliary character and has only be included in order to keep the exposition as self-contained as possible.

1. An initial–boundary value problem

We consider the model problem

$$\begin{cases} \partial_t v = \partial_x \sigma + f(\sigma, v, x, t), \\ \partial_t \varepsilon = \partial_x v, \\ \varepsilon = \mathcal{F}[\lambda, \sigma] \end{cases} \tag{1.0.1}$$

describing, e.g., longitudinal oscillations of a homogeneous elastoplastic beam, where $(x,t) \in Q_T :=]0,1[\times]0,T[$, and the mass density is normalized to 1. We assume that $\mathcal{F}$ is a hysteresis operator and f is a function, both satisfying Hypothesis 1.0.1. For system (1.0.1) we prescribe initial and boundary conditions

$$v(x,0) = v^0(x), \qquad \varepsilon(x,0) = \varepsilon^0(x) \quad \text{for } x \in]0,1[, \tag{1.0.2}$$

$$v(0,t) = v(1,t) = 0 \quad \text{for } t \in]0,T[. \tag{1.0.3}$$

HYPOTHESIS 1.0.1. (i) *$\mathcal{F}$ is a Prandtl–Ishlinskii operator of the form* (3.5.2) *with nondecreasing generating function $h : [0,\infty[\to]0,\infty[$, and with a given initial configuration $\lambda \in C([0,1]; \Lambda_K)$ for some $K > 0$.*

(ii) *$f : \mathbb{R}^2 \times Q_T \to \mathbb{R}$ is a given function such that $f(\sigma, v, \cdot, \cdot), \partial_t f(\sigma, v, \cdot, \cdot) : Q_T \to \mathbb{R}$ are measurable for all $(\sigma, v) \in \mathbb{R}^2$, $f(\cdot,\cdot,x,t), \partial_t f(\cdot,\cdot,x,t) : \mathbb{R}^2 \to \mathbb{R}$ are continuous for a.e. $(x,t) \in Q_T$, $f^0 := f(0,0,\cdot,\cdot) \in C([0,T]; L^2(0,1))$, and there exist functions $\alpha_f \in L^1(0,T)$ and $\beta_f \in L^1(0,T; L^2(0,1))$ such that the inequalities*

$$\begin{cases} \left| f(\sigma_1, v_1, x, t) - f(\sigma_2, v_2, x, t) \right| \leqslant \alpha_f(t) \big(|\sigma_1 - \sigma_2| + |v_1 - v_2| \big), \\ \left| \partial_t f(\sigma, v, x, t) \right| \leqslant \beta_f(x,t), \end{cases} \tag{1.0.4}$$

hold for a.e. $(x,t) \in Q_T$ and every $\sigma, \sigma_1, \sigma_2, v, v_1, v_2 \in \mathbb{R}$.

(iii) $v^0, \varepsilon^0 \in W^{1,2}(0,1)$, $v^0(0) = v^0(1) = 0$.

In the case without hysteresis, i.e., $h(r) = h(0)$ for all $r \geqslant 0$, system (1.0.1) is semilinear hyperbolic with wave propagation speed

$$c_0 = \frac{1}{\sqrt{h(0)}}. \tag{1.0.5}$$

We now show that even if hysteresis is present, the speed of propagation is bounded by the same constant c_0 *independently of the initial data and of the hysteresis dissipation*, so that the hyperbolic character of the problem is not violated, at variance with the case where viscosity is included into the model.

PROPOSITION 1.0.2. *Let Hypothesis* 1.0.1 *be fulfilled, and let c_0 be the constant in* (1.0.5). *Let there exist an interval $[x_1, x_2] \subset]0,1[$ such that the data $\sigma^0, v^0, \lambda, f$ satisfy $\sigma^0(x) = v^0(x) = \lambda(x,\cdot) \equiv 0$ for $x \in [x_1, x_2]$, $f(0,0,x,t) = 0$ for a.e. $(x,t) \in \Omega := \{(x,t) \in Q_T; x_1 + c_0 t < x < x_2 - c_0 t\}$. Then every solution (v,σ) of* (1.0.1) *vanishes in $\overline{\Omega}$.*

PROOF. We apply the classical energy method proposed by Courant and Hilbert, see [5]. Put $U(x,t) = \mathcal{U}[\lambda,\sigma](x,t)$, where $\mathcal{U}$ is the potential energy operator associated with $\mathcal{F}$ according to (3.4.1). We have the pointwise inequality

$$U(x,t) \geqslant \frac{1}{2c_0^2}\sigma^2(x,t) \quad \text{a.e.} \tag{1.0.6}$$

For $t \in [0,T]$ set

$$A(t) = R\int_0^t \alpha_f(t')\,\mathrm{d}t', \quad \text{where } R = 1 + \sqrt{1+c_0^2}. \tag{1.0.7}$$

For a.e. $(x,t) \in \Omega$, we have by (3.4.2), (3.4.3) and (1.0.4) that

$$\begin{aligned}
&\partial_t\left(\mathrm{e}^{-A(t)}\left(\frac{1}{2}v^2 + U\right)\right) - \partial_x\left(\mathrm{e}^{-A(t)}v\sigma\right)\\
&\quad = \mathrm{e}^{-A(t)}\left(-R\alpha_f(t)\left(\frac{1}{2}v^2 + U\right) + v(\partial_t v - \partial_x\sigma) + \partial_t U - \sigma\,\partial_x v\right)\\
&\quad \leqslant \mathrm{e}^{-A(t)}\left(-R\alpha_f(t)\left(\frac{1}{2}v^2 + U\right) + vf(\sigma,v,x,t)\right)\\
&\quad \leqslant \alpha_f(t)\mathrm{e}^{-A(t)}\left(-R\left(\frac{1}{2}v^2 + \frac{1}{2c_0^2}\sigma^2\right) + |v|\big(|\sigma| + |v|\big)\right) \leqslant 0.
\end{aligned}$$

For an arbitrary $\tau \in [0, (x_2 - x_1)/(2c_0)] \cap [0,T]$, we denote $\Omega_\tau = \{(x,t) \in \Omega; t < \tau\}$. The Green theorem yields

$$\begin{aligned}
0 &\geqslant \iint_{\Omega_\tau}\left(\partial_t\left(\mathrm{e}^{-A(t)}\left(\frac{1}{2}v^2 + U\right)\right) - \partial_x\left(\mathrm{e}^{-A(t)}v\sigma\right)\right)\mathrm{d}x\,\mathrm{d}t\\
&= \int_{x_1+c_0\tau}^{x_2-c_0\tau} \mathrm{e}^{-A(\tau)}\left(\frac{1}{2}v^2 + U\right)(x,\tau)\,\mathrm{d}x\\
&\quad + \int_{x_1}^{x_1+c_0\tau} \mathrm{e}^{-A((x-x_1)/c_0)}\left(\frac{1}{2}v^2 + U + \frac{1}{c_0}v\sigma\right)\left(x, \frac{x-x_1}{c_0}\right)\mathrm{d}x\\
&\quad + \int_{x_2-c_0\tau}^{x_2} \mathrm{e}^{-A((x_2-x)/c_0)}\left(\frac{1}{2}v^2 + U - \frac{1}{c_0}v\sigma\right)\left(x, \frac{x_2-x}{c_0}\right)\mathrm{d}x.
\end{aligned}$$

All three integrals on the right-hand side of the above inequality are nonnegative by (1.0.6). This is only possible if both v and σ vanish in $\overline{\Omega}$, which completes the proof. $\square$

1.1. *Existence and uniqueness of solutions*

THEOREM 1.1.1. *Let Hypothesis* 1.0.1 *be fulfilled. Then there exists a unique solution* $(v, \sigma, \varepsilon) \in C(\overline{Q}_T; \mathbb{R}^3)$ *of system* (1.0.1)–(1.0.3) *such that* $\partial_t v$, $\partial_x v$, $\partial_t \sigma$, $\partial_x \sigma$, $\partial_t \varepsilon$ *belong to the space* $L^\infty(0, T; L^2(0, 1))$, *and* (1.0.1) *holds almost everywhere in* Q_T.

PROOF. The uniqueness argument is straightforward. We consider two solutions $(v_1, \sigma_1, \varepsilon_1)$, $(v_2, \sigma_2, \varepsilon_2)$, and put $\bar{v} = v_1 - v_2$, $\bar{\sigma} = \sigma_1 - \sigma_2$, $\bar{\varepsilon} = \varepsilon_1 - \varepsilon_2$. The hypothesis yields that $\bar{v}(x, 0) = \bar{\varepsilon}(x, 0) = 0$ for all $x \in [0, 1]$. From Proposition 3.2.1 it follows that $\sigma_i = \widehat{\mathcal{F}}[\mu, \varepsilon_i]$ for $i = 1, 2$, hence also $\bar{\sigma}(x, 0) = 0$. For a.e. $(x, t) \in Q_T$, we have

$$\begin{cases} \partial_t \bar{v} = \partial_x \bar{\sigma} + f(\sigma_1, v_1, x, t) - f(\sigma_2, v_2, x, t), \\ \partial_t \bar{\varepsilon} = \partial_x \bar{v}. \end{cases} \tag{1.1.1}$$

Testing the first equation in (1.1.1) by $\bar{v}$, the second by $\bar{\sigma}$, and using (1.0.4), we obtain

$$\int_0^1 (\bar{v}\, \partial_t \bar{v} + \bar{\sigma}\, \partial_t \bar{\varepsilon})\, \mathrm{d}x \leqslant \alpha_f(t) \int_0^1 \big(|\bar{v}| + |\bar{\sigma}|\big)|\bar{v}|\, \mathrm{d}x. \tag{1.1.2}$$

Let c_0 be as in (1.0.5). From (3.3.2) and the elementary inequality $2pq \leqslant \delta p^2 + q^2/\delta$ for $p, q, \delta > 0$, it follows for $t \geqslant 0$ that

$$\begin{aligned} &\int_0^1 \left(|\bar{v}|^2 + \frac{1}{c_0^2}|\bar{\sigma}|^2\right)(t)\, \mathrm{d}x \\ &\quad \leqslant \big(1 + \sqrt{1 + c_0^2}\big) \int_0^t \alpha_f(\tau) \int_0^1 \left(|\bar{v}|^2 + \frac{1}{c_0^2}|\bar{\sigma}|^2\right)(\tau)\, \mathrm{d}x\, \mathrm{d}\tau, \end{aligned} \tag{1.1.3}$$

and the Gronwall argument yields that $\bar{v} = \bar{\sigma} = 0$ a.e.

The existence statement will be proved by space semidiscretization. For $n \in \mathbb{N}$ we consider the system of equations

$$\dot{v}_j = n(\sigma_{j+1} - \sigma_j) + f_j(\sigma_j, v_j, t), \qquad j = 1, \ldots, n-1, \tag{1.1.4}$$

$$\dot{\varepsilon}_j = n(v_j - v_{j-1}), \qquad j = 1, \ldots, n, \tag{1.1.5}$$

$$\varepsilon_j = \mathcal{F}[\lambda_j, \sigma_j], \qquad j = 1, \ldots, n, \tag{1.1.6}$$

coupled with "boundary conditions"

$$v_0(t) = v_n(t) = 0, \tag{1.1.7}$$

and initial conditions

$$v_j(0) = v_j^0, \qquad \varepsilon_j(0) = \varepsilon_j^0, \tag{1.1.8}$$

where we set

$$f_j(\sigma, v, t) = n \int_{(j-1)/n}^{j/n} f(\sigma, v, x, t)\,\mathrm{d}x, \qquad \lambda_j(r) = n \int_{(j-1)/n}^{j/n} \lambda(x, r)\,\mathrm{d}x, \tag{1.1.9}$$

$$v_j^0 = n \int_{(j-1)/n}^{j/n} v^0(x)\,\mathrm{d}x, \qquad \varepsilon_j^0 = n \int_{(j-1)/n}^{j/n} \varepsilon^0(x)\,\mathrm{d}x \tag{1.1.10}$$

for all admissible values of arguments and indices, except for the compatibility condition $v_0^0 = v_n^0 = 0$. It follows from Propositions 3.2.1 and 3.2.2 that (1.1.6) can be written in the form

$$\sigma_j = \widehat{\mathcal{F}}[\mu_j, \varepsilon_j] \tag{1.1.11}$$

with Lipschitz continuous operators $\widehat{\mathcal{F}}[\mu_j, \cdot] : C[0, T] \to C[0, T]$. System (1.1.4)–(1.1.5) is of the type

$$\dot{z}(t) = \Phi[z, \cdot](t), \quad z(0) = z^0, \tag{1.1.12}$$

for an unknown function $z : [0, T] \to \mathbb{R}^{2n-1}$, $z = (v_1, \ldots, v_{n-1}, \varepsilon_1, \ldots, \varepsilon_n)$, with a mapping $\Phi : C([0, T]; \mathbb{R}^{2n-1}) \times [0, T] \to C([0, T]; \mathbb{R}^{2n-1})$ given by the right-hand side of (1.1.4)–(1.1.5). We solve (1.1.12) as a fixed point problem in $C([0, T]; \mathbb{R}^{2n-1})$ for the operator

$$S[z](t) = z^0 + \int_0^t \Phi[z, \cdot](\tau)\,\mathrm{d}\tau. \tag{1.1.13}$$

By Hypothesis 1.0.1 and Proposition 3.2.2, there exists $a \in L^1(0, T)$ such that, for all $z_1, z_2 \in C([0, T]; \mathbb{R}^{2n-1})$ and $t \in [0, T]$, we have

$$\big|\Phi[z_1, \cdot](t) - \Phi[z_2, \cdot](t)\big| \leqslant a(t) \|z_1 - z_2\|_{[0,t]}, \tag{1.1.14}$$

hence

$$\big|S[z_1] - S[z_2]\big|(t) \leqslant \int_0^t a(\tau) \|z_1 - z_2\|_{[0,\tau]}\,\mathrm{d}\tau. \tag{1.1.15}$$

For $t \in [0, T]$ set $A(t) = \exp(\int_0^t a(\tau)\,\mathrm{d}\tau)$ and

$$\|z\|_{A,[0,t]} = \max_{\tau \in [0,t]} \left(\frac{1}{A(\tau)} \|z\|_{[0,\tau]} \right) \quad \text{for } z \in C\big([0, T]; \mathbb{R}^{2n-1}\big). \tag{1.1.16}$$

In particular, $\|\cdot\|_{A,[0,T]}$ is a norm in $C([0,T];\mathbb{R}^{2n-1})$ which is equivalent to $\|\cdot\|_{[0,T]}$. By (1.1.15), we have for all $t \in [0,T]$ that

$$\begin{aligned}
&\frac{1}{A(t)}\big\|S[z_1]-S[z_2]\big\|_{[0,t]} \\
&\quad\leqslant \frac{1}{A(t)}\int_0^t a(\tau)A(\tau)\frac{1}{A(\tau)}\|z_1-z_2\|_{[0,\tau]}\,\mathrm{d}\tau \\
&\quad\leqslant \frac{A(t)-1}{A(t)}\|z_1-z_2\|_{A,[0,t]} \leqslant \frac{A(T)-1}{A(T)}\|z_1-z_2\|_{A,[0,T]}.
\end{aligned}\tag{1.1.17}$$

We see that S is a contraction on $C([0,T];\mathbb{R}^{2n-1})$ endowed with norm $\|\cdot\|_{A,[0,T]}$, and its unique fixed point is a solution to (1.1.4)–(1.1.8).

We now derive estimates which enable us to pass to the limit as $n \to \infty$. To this end, we differentiate (1.1.4)–(1.1.5) and test by $\dot{v}_j$ and $\dot{\sigma}_j$, respectively. From Hypothesis 1.0.1 we obtain

$$\ddot{v}_j\dot{v}_j \leqslant n(\dot{\sigma}_{j+1}-\dot{\sigma}_j)\dot{v}_j + \big(\beta_j(t)+\alpha_f(t)\big(|\dot{\sigma}_j|+|\dot{v}_j|\big)\big)|\dot{v}_j|,\tag{1.1.18}$$

$$\ddot{\varepsilon}_j\dot{\sigma}_j = n(\dot{v}_j-\dot{v}_{j-1})\dot{\sigma}_j,\tag{1.1.19}$$

where $\beta_j(t) = n\int_{(j-1)/n}^{j/n}\beta_f(x,t)\,\mathrm{d}x$. The boundary conditions (1.1.7) yield

$$\sum_{j=1}^{n-1}(\dot{\sigma}_{j+1}-\dot{\sigma}_j)\dot{v}_j + \sum_{j=1}^{n}(\dot{v}_j-\dot{v}_{j-1})\dot{\sigma}_j = 0,\tag{1.1.20}$$

and by virtue of Theorem 3.4.1 we have for all $t \in\,]0,T]$ that

$$\int_0^t \ddot{\varepsilon}_j\dot{\sigma}_j\,\mathrm{d}\tau \geqslant \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(t-) - \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(0+).\tag{1.1.21}$$

From (1.1.5) and (1.1.8), it follows that $\dot{\varepsilon}_j(0+) = \dot{\varepsilon}_j(0) = n(v_j^0 - v_{j-1}^0)$. Using (1.1.11) and (3.2.11) we obtain

$$\begin{aligned}
\big|\dot{\sigma}_j(0+)\big| &= \lim_{t\to 0+}\frac{1}{t}\big|\sigma_j(t)-\sigma_j(0)\big| \\
&\leqslant \lim_{t\to 0+}\frac{2}{h(0)t}\big\|\varepsilon_j-\varepsilon_j(0)\big\|_{[0,t]} \\
&= \frac{2}{h(0)}\big|\dot{\varepsilon}_j(0)\big|,
\end{aligned}\tag{1.1.22}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n}(\dot{\sigma}_j\dot{\varepsilon}_j)(0+) \leqslant \frac{2n}{h(0)}\sum_{j=1}^{n}\big|v_j^0-v_{j-1}^0\big|^2.\tag{1.1.23}$$

The right-hand side of (1.1.23) can be estimated independently of n. To do so, we decompose the sum as

$$n\sum_{j=1}^{n}\left|v_j^0 - v_{j-1}^0\right|^2 = n\left(\left|v_1^0\right|^2 + \left|v_{n-1}^0\right|^2 + \sum_{j=2}^{n-1}\left|v_j^0 - v_{j-1}^0\right|^2\right). \tag{1.1.24}$$

Formula (1.1.10) yields

$$\begin{aligned} n\left|v_1^0\right|^2 &= n^3\left|\int_0^{1/n}\int_0^{(1/n)-x} \partial_x v^0\big(x'\big)\,\mathrm{d}x'\,\mathrm{d}x\right|^2 \\ &\leqslant n\int_0^{1/n}\int_0^{(1/n)-x}\left|\partial_x v^0\big(x'\big)\right|^2\mathrm{d}x'\,\mathrm{d}x, \end{aligned}$$

and similarly,

$$n\left|v_{n-1}^0\right|^2 \leqslant n\int_0^{1/n}\int_{((n-1)/n)-x}^{1}\left|\partial_x v^0\big(x'\big)\right|^2\mathrm{d}x'\,\mathrm{d}x.$$

For $j = 2, \dots, n-1$, we have

$$\begin{aligned} n\left|v_j^0 - v_{j-1}^0\right|^2 &= n^3\left|\int_0^{1/n}\int_{((j-1)/n)-x}^{(j/n)-x} \partial_x v^0\big(x'\big)\,\mathrm{d}x'\,\mathrm{d}x\right|^2 \\ &\leqslant n\int_0^{1/n}\int_{((j-1)/n)-x}^{(j/n)-x}\left|\partial_x v^0\big(x'\big)\right|^2\mathrm{d}x'\,\mathrm{d}x. \end{aligned}$$

From the above computations it follows that

$$n\sum_{j=1}^{n}\left|v_j^0 - v_{j-1}^0\right|^2 \leqslant \int_0^1\left|\partial_x v^0\big(x'\big)\right|^2\mathrm{d}x'. \tag{1.1.25}$$

We similarly have

$$\frac{1}{n}\sum_{j=1}^{n-1}\left|\dot v_j(0)\right|^2 \leqslant \frac{1}{n}\sum_{j=1}^{n-1}\Big(n\left|\sigma_{j+1}(0) - \sigma_j(0)\right| + \left|f_j\big(\sigma_j(0), v_j(0), 0\big)\right|\Big)^2. \tag{1.1.26}$$

From (1.1.11) and Propositions 3.2.1 and 3.2.2 it follows that

$$\left|\sigma_{j+1}(0) - \sigma_j(0)\right| \leqslant \frac{2}{h(0)}\left|\varepsilon_{j+1}^0 - \varepsilon_j^0\right|$$

and arguing as in (1.1.25) we obtain

$$n\sum_{j=1}^{n-1}\left|\sigma_{j+1}(0)-\sigma_j(0)\right|^2 \leqslant \left(\frac{2}{h(0)}\right)^2 \int_0^1 \left|\partial_x \varepsilon^0(x)\right|^2 \mathrm{d}x. \tag{1.1.27}$$

We further have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\left|f_j\big(\sigma_j(0), v_j(0), 0\big)\right|^2 \leqslant \int_0^1 \left|f\big(\sigma(x,0), v^0(x), x, 0\big)\right|^2 \mathrm{d}x \tag{1.1.28}$$

and

$$\frac{1}{n}\sum_{j=1}^{n-1}\beta_j^2(t) \leqslant \int_0^1 \beta_f^2(x,t)\,\mathrm{d}x. \tag{1.1.29}$$

Let us introduce auxiliary functions

$$W_n(t) = \frac{1}{n}\sum_{j=1}^{n-1}\dot{v}_j^2(t), \qquad S_n(t) = \frac{1}{n}\sum_{j=1}^{n}\dot{\sigma}_j^2(t). \tag{1.1.30}$$

In the following series of estimates, C denotes any positive constant independent of n and T, and C_T is a constant independent of n and possibly dependent on T.

We integrate (1.1.18) and (1.1.19) from 0 to t. Combining the inequality $\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) \geqslant h(0)\dot{\sigma}_j^2(t)$ (which follows from (3.2.17)) with (1.1.20)–(1.1.29), we obtain

$$\begin{aligned} &S_n(t) + W_n(t) \\ &\quad\leqslant C\left(1 + \int_0^t \sqrt{W_n(\tau)\int_0^1 \beta_f^2(x,\tau)\,\mathrm{d}x} + \alpha(\tau)\big(S_n(\tau)+W_n(\tau)\big)\,\mathrm{d}\tau\right). \end{aligned} \tag{1.1.31}$$

Using the inequality $W_n^{1/2} \leqslant (1+W_n)/2$ and the hypothesis that the function $\tilde{\alpha}(t) = (\int_0^1 \beta_f^2(x,t)\,\mathrm{d}x)^{1/2} + \alpha_f(t)$ belongs to $L^1(0,T)$, we obtain from (1.1.31) that

$$S_n(t) + W_n(t) \leqslant C_T\left(1 + \int_0^t \Big(\tilde{\alpha}(\tau)\big(S_n(\tau)+W_n(\tau)\big)\Big)\,\mathrm{d}\tau\right) \tag{1.1.32}$$

for almost all $t \in\,]0,T[$, and Gronwall's lemma yields that

$$S_n(t) + W_n(t) \leqslant C_T \quad \text{a.e.} \tag{1.1.33}$$

We now define piecewise linear and piecewise constant approximations of (v, σ) by the formulæ

$$\hat{v}^{(n)}(x,t) = v_{j-1}(t) + n\left(x - \frac{j-1}{n}\right)\bigl(v_j(t) - v_{j-1}(t)\bigr), \tag{1.1.34}$$

$$\hat{\sigma}^{(n)}(x,t) = \sigma_{j-1}(t) + n\left(x - \frac{j-1}{n}\right)\bigl(\sigma_j(t) - \sigma_{j-1}(t)\bigr), \tag{1.1.35}$$

$$\bar{v}^{(n)}(x,t) = v_{j-1}(t), \tag{1.1.36}$$

$$\bar{\sigma}^{(n)}(x,t) = \sigma_j(t), \tag{1.1.37}$$

$$\underline{\sigma}^{(n)}(x,t) = \sigma_{j-1}(t), \tag{1.1.38}$$

$$\bar{\lambda}^{(n)}(x,t) = \lambda_j(t), \tag{1.1.39}$$

$$\bar{f}(\sigma, v, x, t) = f_{j-1}(\sigma, v, t), \tag{1.1.40}$$

$$\bar{\varepsilon}^{(n)} = \mathcal{F}\bigl[\bar{\lambda}^{(n)}, \bar{\sigma}^{(n)}\bigr], \tag{1.1.41}$$

for $x \in [(j-1)/n, j/n[$, $j = 1, \dots, n$, continuously extended to $x = 1$, where we set $f_0(\sigma, v, t) = 0$. Equations (1.1.4) and (1.1.5) then have the form

$$\partial_t \bar{v}^{(n)} = \partial_x \hat{\sigma}^{(n)} + \bar{f}\bigl(\underline{\sigma}^{(n)}, \bar{v}^{(n)}, x, t\bigr), \tag{1.1.42}$$

$$\partial_t \bar{\varepsilon}^{(n)} = \partial_x \hat{v}^{(n)}. \tag{1.1.43}$$

From (1.1.34) it follows that the functions $\partial_t \hat{v}^{(n)}$, $\partial_t \hat{\sigma}^{(n)}$, $\partial_t \bar{v}^{(n)}$, $\partial_t \bar{\sigma}^{(n)}$ are bounded in $L^\infty(0, T; L^2(0,1))$ independently of n. We further have

$$\begin{aligned}
&\int_0^1 \bigl|\bar{f}\bigl(\underline{\sigma}^{(n)}, \bar{v}^{(n)}, x, t\bigr)\bigr|^2 \,\mathrm{d}x \\
&\quad = \frac{1}{n}\sum_{j=1}^{n-1} f_j^2(\sigma_j, v_j, t) \\
&\quad \leqslant \frac{1}{n}\sum_{j=1}^{n-1}\left(f_j\bigl(\sigma_j(0), v_j(0), 0\bigr) + \int_0^t \frac{\partial}{\partial\tau}\bigl(f_j(\sigma_j, v_j, \tau)\bigr)\,\mathrm{d}\tau\right)^2 \\
&\quad \leqslant C_T,
\end{aligned} \tag{1.1.44}$$

hence also $\partial_x \hat{v}^{(n)}, \partial_x \hat{\sigma}^{(n)}$ are bounded in $L^\infty(0, T; L^2(0,1))$ independently of n. Denote $H^{\infty,2}(Q_T) = \{u \in L^2(Q_T); \partial_t u, \partial_x u \in L^\infty(0, T; L^2(0,1))\}$. By compact embedding $H^{\infty,2}(Q_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T)$ we find functions $v, \sigma \in H^{\infty,2}(Q_T)$ and a subsequence of $\{(\hat{v}^{(n)}, \hat{\sigma}^{(n)})\}$ (still indexed by n) such that

$$\left.\begin{array}{l} \partial_t \hat{\sigma}^{(n)} \to \partial_t \sigma \\ \partial_x \hat{\sigma}^{(n)} \to \partial_x \sigma \\ \partial_t \hat{v}^{(n)} \to \partial_t v \\ \partial_x \hat{v}^{(n)} \to \partial_x v \end{array}\right\} \quad \text{weakly-star in } L^\infty\big(0, T; L^2(0,1)\big), \tag{1.1.45}$$

$$\left.\begin{array}{l} \hat{\sigma}^{(n)} \to \sigma \\ \hat{v}^{(n)} \to v \end{array}\right\} \quad \text{uniformly in } C(Q_T). \tag{1.1.46}$$

We furthermore have for $x \in [(j-1)/n, j/n[$ that

$$\begin{aligned} \big|\hat{\sigma}^{(n)}(x,t) - \underline{\sigma}(x,t)\big|^2 &\leqslant \big|\sigma_j(t) - \sigma_{j-1}(t)\big|^2 \leqslant \sum_{j=1}^{n} \big|\sigma_j(t) - \sigma_{j-1}(t)\big|^2 \\ &\leqslant \frac{1}{n} \big\|\partial_x \hat{\sigma}^{(n)}\big\|^2_{L^\infty(0,T;L^2(0,1))}, \end{aligned}$$

and similarly,

$$\begin{aligned} \big|\hat{\sigma}^{(n)}(x,t) - \bar{\sigma}(x,t)\big|^2 &\leqslant \frac{1}{n} \big\|\partial_x \hat{\sigma}^{(n)}\big\|^2_{L^\infty(0,T;L^2(0,1))}, \\ \big|\hat{v}^{(n)}(x,t) - \bar{v}(x,t)\big|^2 &\leqslant \frac{1}{n} \big\|\partial_x \hat{v}^{(n)}\big\|^2_{L^\infty(0,T;L^2(0,1))}, \end{aligned}$$

hence

$$\left.\begin{array}{l} \bar{\sigma}^{(n)} \to \sigma \\ \underline{\sigma}^{(n)} \to \sigma \\ \bar{v}^{(n)} \to v \end{array}\right\} \quad \text{uniformly in } L^\infty\big(0, 1; C[0,T]\big). \tag{1.1.47}$$

The results of Section 3.5 enable us to pass to the limit in (1.1.41)–(1.1.43) and check that (v, σ) is a solution of (1.0.1)–(1.0.3). The fact that ε is continuous with respect to x follows directly from Proposition 3.2.2 with $\lambda_i(r) = \lambda(x_i, r)$ and $w_i(t) = \sigma(x_i, t)$, $i = 1, 2$, for any $x_1, x_2 \in [0,1]$. □

1.2. *Global boundedness*

In order to investigate the asymptotic behavior of solutions as $t \to \infty$, we first establish conditions under which the solutions remain globally bounded. In particular, we assume that the right-hand side of (1.0.1) is independent of σ. In other words, we consider the system

$$\begin{cases} \partial_t v = \partial_x \sigma + f(v, x, t), \\ \partial_t \varepsilon = \partial_x v, \\ \varepsilon = \mathcal{F}[\lambda, \sigma] \end{cases} \tag{1.2.1}$$

under the following hypotheses.

HYPOTHESIS 1.2.1. *The right-hand side* $f : \mathbb{R} \times \,]0,1[\, \times \,]0,\infty[\, \to \mathbb{R}$ *of* (1.2.1) *is such that the functions* $f(v,\cdot,\cdot), \partial_t f(v,\cdot,\cdot) :]0,1[\, \times \,]0,\infty[\, \to \mathbb{R}$ *are measurable for all* $v \in \mathbb{R}$, $f(\cdot,x,t), \partial_t f(\cdot,x,t) : \mathbb{R} \to \mathbb{R}$ *are continuous for a.e.* $(x,t) \in \,]0,1[\, \times \,]0,\infty[$, $f^0 := f(0,\cdot,\cdot) \in L^\infty(0,\infty; L^2(0,1))$, *and there exist a constant* $\gamma_f > 0$ *and a function* $\beta_f \in L^\infty(0,\infty; L^2(0,1))$ *such that for almost all arguments we have*

$$-\gamma_f \leqslant \partial_v f(v,x,t) \leqslant 0, \tag{1.2.2}$$

$$\left|\partial_t f(v,x,t)\right| \leqslant \beta_f(x,t). \tag{1.2.3}$$

HYPOTHESIS 1.2.2. *Let h and κ be the functions associated with the operator $\mathcal{F}$ according to* (3.2.1) *and* (3.4.8). *For $p > 0$ set*

$$\mu(p) = \max\left\{h(p), \frac{h^{3/4}(p)}{\kappa^{1/2}(p)}\right\}, \tag{1.2.4}$$

and assume that

$$\lim_{p\to\infty} \frac{\mu^2(p)h(p)}{p^2} = 0. \tag{1.2.5}$$

HYPOTHESIS 1.2.3. *The function H defined by* (3.2.3) *satisfies the implication*

$$\begin{aligned} &\exists L, m, R_0 > 0, \forall r > 0, \\ &\quad R \geqslant \max\{R_0, Lr\} \quad \Longrightarrow \quad 2H(R-r) - H(R) \geqslant mR. \end{aligned} \tag{1.2.6}$$

REMARK 1.2.4. Condition (1.2.2) says that “negative friction” is excluded. The hysteresis dissipation is not strong enough to keep the solution away from resonance if the energy supply becomes dominant. On the other hand, the results remain valid even if no friction ($\partial_v f = 0$) is present.

The functions $h(p)$ and $\kappa(p)$ characterize the *slope* and the *curvature* of the hysteresis branches, respectively. Condition (1.2.5) is fulfilled if, for example, there exist $0 < h_* \leqslant h^*$, $r_0 > 0$ and $-1 < \alpha < 2/3$, such that

$$h_* \max\{r_0, r\}^{\alpha-1} \leqslant h'(r) \leqslant h^* \max\{r_0, r\}^{\alpha-1} \quad \text{a.e.} \tag{1.2.7}$$

A sufficient condition for (1.2.6) reads, for instance,

$$\exists \delta_0 > 0, \quad \limsup_{r\to\infty} \frac{H((1+\delta_0)r)}{H(r)} < 2. \tag{1.2.8}$$

Indeed, (1.2.8) can be rewritten in the form

$$\begin{aligned} &\exists \delta_0, m, q_0 > 0, \forall \delta \in]0, \delta_0], \\ &\quad q \geqslant q_0 \quad \Longrightarrow \quad \frac{H((1+\delta)q)}{H(q)} \leqslant 2 - \frac{(1+\delta_0)m}{h(0)}. \end{aligned} \tag{1.2.9}$$

Then (1.2.6) follows from (1.2.9) with $L = 1 + 1/\delta_0$, $r = \delta q$, $R_0 = (1 + \delta_0)q_0$, $R = (1+\delta)q$.

A variant of l'Hôpital's rule implies that condition (1.2.8) is in turn satisfied if, for example, $h(r)$ is concave for large r.

The existence and uniqueness of a global solution to problem (1.2.1) coupled with (1.0.2)–(1.0.3) under Hypotheses 1.0.1 and 1.2.1 follow from Theorem 1.1.1. The aim of this subsection is to prove the following global boundedness result.

THEOREM 1.2.5. *Let Hypotheses* 1.0.1 *and* 1.2.1–1.2.3 *hold. Then there exists a constant* $C > 0$ *independent of* t *such that the solution* (v, σ) *to* (1.2.1), (1.0.2)–(1.0.3) *satisfies a.e. the conditions*

$$\int_0^1 \big((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + (\partial_x \sigma)^2\big)(x,t)\,\mathrm{d}x \leqslant C, \tag{1.2.10}$$

$$\big|v(x,t)\big| + \big|\sigma(x,t)\big| \leqslant C. \tag{1.2.11}$$

The proof of Theorem 1.2.5 is split into two parts. We first prove Lemma 1.2.6. Since no boundary conditions for σ are prescribed, the transition from Lemma 1.2.6 to (1.2.10)–(1.2.11) is not straightforward and a deeper result on hysteresis memory structure will have to be established in the subsequent Lemma 1.2.7.

In accordance with the previous notation, we define for $t > 0$ the sets $Q_t = [0,1] \times [0,t]$ and for $u \in L^\infty(Q_t)$ put

$$\|u\|_{Q_t} = \sup \operatorname{ess}\big\{\big|u(x,\tau)\big|;\,(x,\tau) \in Q_t\big\}. \tag{1.2.12}$$

LEMMA 1.2.6. *Let Hypotheses* 1.0.1, 1.2.1 and 1.2.2 *hold, and let* (v, σ) *be the solution to* (1.2.1), (1.0.2)–(1.0.3). *Then for every* $\delta > 0$ *there exists* $p_0 > 0$ *such that for all* $p \geqslant p_0$ *the following implication holds true for a.e.* $t \geqslant 0$,

$$\|\sigma\|_{Q_t} \leqslant p \quad \Longrightarrow \quad \frac{1}{p^2}\int_0^1 \big((\partial_t \sigma)^2 + (\partial_x \sigma)^2\big)(x,t)\,\mathrm{d}x \leqslant \delta. \tag{1.2.13}$$

PROOF. Taking into account the convergences (1.1.45)–(1.1.47), it suffices to consider the discrete system

$$\dot{v}_j = n(\sigma_{j+1} - \sigma_j) + f_j(v_j, t), \qquad j = 1, \dots, n-1, \tag{1.2.14}$$

$$\dot{\varepsilon}_j = n(v_j - v_{j-1}), \qquad j = 1, \dots, n, \tag{1.2.15}$$

$$\varepsilon_j = \mathcal{F}[\lambda_j, \sigma_j], \qquad j = 1, \dots, n, \tag{1.2.16}$$

analogous to (1.1.4)–(1.1.6) with boundary and initial conditions (1.1.7)–(1.1.8), and prove that for every $\delta > 0$ there exists $p_0 > 0$ independent of n and t such that for all $p \geqslant p_0$ the following implication holds:

$$\begin{aligned} &\max_{j=1,\dots,n} \|\sigma_j\|_{[0,t]} \leqslant p \\ &\quad\Longrightarrow \quad \frac{1}{p^2}\left(\frac{1}{n}\sum_{j=1}^{n} \dot{\sigma}_j^2(t) + n\sum_{j=1}^{n-1}(\sigma_{j+1} - \sigma_j)^2(t)\right) \leqslant \delta. \end{aligned} \tag{1.2.17}$$

To do so, we define for $j = 1, \dots, n-1$ auxiliary functions

$$G_j(v, t) = v f_j(v, t) - \int_0^v f_j\big(v', t\big)\,\mathrm{d}v' \quad \text{for } (v, t) \in \mathbb{R} \times [0, \infty[, \tag{1.2.18}$$

and set

$$E(t) = \frac{1}{2n}\left(\sum_{j=1}^{n-1} \dot{v}_j^2(t) + \sum_{j=1}^{n} \dot{\varepsilon}_j(t)\dot{\sigma}_j(t)\right), \tag{1.2.19}$$

$$S(t) = \frac{1}{n}\sum_{j=1}^{n} \big|\dot{\sigma}_j(t)\big|^3, \tag{1.2.20}$$

$$W(t) = \frac{1}{n}\sum_{j=1}^{n-1} \dot{v}_j^2(t), \tag{1.2.21}$$

$$Z(t) = \frac{1}{n}\sum_{j=1}^{n-1}\Big(G_j\big(v_j(t), t\big) - v_j(t)\dot{v}_j(t)\Big). \tag{1.2.22}$$

We have, by definition,

$$-\gamma_f v_j^2(t) \leqslant G_j\big(v_j(t), t\big) \leqslant 0. \tag{1.2.23}$$

The boundary condition (1.1.7) and equation (1.1.5) yield

$$\frac{1}{n}\sum_{j=1}^{n-1} v_j^2(t) \leqslant n\sum_{j=1}^{n}(v_j - v_{j-1})^2(t) = \frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j^2(t). \tag{1.2.24}$$

Assume now that $\max_{j=1,\dots,n}\|\sigma_j\|_{[0,T]} \leqslant p$ for some $T>0$ and $p \geqslant K$. From (3.2.16) we obtain for $t\in[0,T]$ that

$$\left|\dot{\varepsilon}_j(t)\right| \leqslant h(p)\left|\dot{\sigma}_j(t)\right| \quad \text{for } j=1,\dots,n. \tag{1.2.25}$$

This inequality, together with (3.2.17), implies that

$$\frac{1}{n}\sum_{j=1}^{n-1} v_j^2(t) \leqslant h(p)\frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j(t)\dot{\sigma}_j(t). \tag{1.2.26}$$

We now fix a constant $c^* > 0$ such that

$$\left|Z(t)\right| \leqslant c^* h(p) E(t) \tag{1.2.27}$$

for all $t\in[0,T]$. Using (1.2.14)–(1.2.15) yields for a.e. $t\in\,]0,T[$ that

$$\begin{aligned}\dot{Z}(t) + W(t) &= \frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) + \frac{1}{n}\sum_{j=1}^{n-1}(\partial_t G_j - v_j\,\partial_t f_j)(t)\\ &\leqslant \frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) + \frac{1}{n}\sum_{j=1}^{n-1}\left|v_j(t)\right|\left|\beta_j(t)\right|,\end{aligned} \tag{1.2.28}$$

where β_j is as in (1.1.18), hence, by (1.2.25) and (1.2.26),

$$\begin{aligned}\dot{Z}(t) + \frac{1}{2}W(t) + E(t) &\leqslant \frac{3}{2n}\sum_{j=1}^{n}\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) + C\left(\frac{1}{n}\sum_{j=1}^{n-1}v_j^2(t)\right)^{1/2}\\ &\leqslant Ch(p)\bigl(1+S^{2/3}(t)\bigr),\end{aligned} \tag{1.2.29}$$

where C denotes as before any positive constant independent of n, T and p. The counterpart of (1.1.18)–(1.1.19) reads

$$\ddot{v}_j\dot{v}_j \leqslant n(\dot{\sigma}_{j+1} - \dot{\sigma}_j)\dot{v}_j + \left|\beta_j(t)\right|\left|\dot{v}_j(t)\right|, \tag{1.2.30}$$

$$\ddot{\varepsilon}_j\dot{\sigma}_j = n(\dot{v}_j - \dot{v}_{j-1})\dot{\sigma}_j, \tag{1.2.31}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n-1}\ddot{v}_j(t)\dot{v}_j(t)+\frac{1}{n}\sum_{j=1}^{n}\ddot{\varepsilon}_j(t)\dot{\sigma}_j(t)\leqslant CW^{1/2}(t)\quad\text{a.e.}\tag{1.2.32}$$

From Theorem 3.4.1 it follows that for all $0\leqslant s<t\leqslant T$ we have

$$E(t-)-E(s+)+\frac{1}{4}\kappa(p)\int_s^t S(\tau)\,\mathrm{d}\tau\leqslant C\int_s^t W^{1/2}(\tau)\,\mathrm{d}\tau.\tag{1.2.33}$$

Let c^* be as in (1.2.27). Inequalities (1.2.29) and (1.2.33) yield for all $0\leqslant s<t$ that

$$\begin{aligned}&\big(Z+2c^*\mu(p)E\big)(t-)-\big(Z+2c^*\mu(p)E\big)(s+)\\&\qquad+\frac{1}{2}c^*\mu(p)\kappa(p)\int_s^t S(\tau)\,\mathrm{d}\tau+\int_s^t\left(\frac{1}{2}W(\tau)+E(\tau)\right)\mathrm{d}\tau\\&\quad\leqslant C\int_s^t\big(\mu(p)W^{1/2}(\tau)+h(p)\big(1+S^{2/3}(\tau)\big)\big)\,\mathrm{d}\tau,\end{aligned}\tag{1.2.34}$$

hence

$$\begin{aligned}&\big(Z+2c^*\mu(p)E\big)(t-)-\big(Z+2c^*\mu(p)E\big)(s+)+\int_s^t E(\tau)\,\mathrm{d}\tau\\&\quad\leqslant C\left(\mu^2(p)+h(p)+\frac{h^3(p)}{\kappa^2(p)\mu^2(p)}\right)(t-s)\\&\quad\leqslant C^*\big(1+\mu^2(p)\big)(t-s),\end{aligned}\tag{1.2.35}$$

where C^* is some frozen value of C, and set

$$\begin{aligned}E_1(t)&=Z(t)+2c^*\mu(p)E(t),\\E_2(t)&=\int_0^t E(\tau)\,\mathrm{d}\tau-C^*\big(1+\mu^2(p)\big)t.\end{aligned}\tag{1.2.36}$$

By (1.2.35), the function E_1+E_2 is nonincreasing, hence for every nonnegative absolutely continuous test function $\eta(t)$ and every $t\in[0,T]$, we have

$$\int_0^t(E_1+E_2)(\tau)\dot{\eta}(\tau)\,\mathrm{d}\tau\geqslant\big(E_1(t-)+E_2(t)\big)\eta(t)-E_1(0+)\eta(0),\tag{1.2.37}$$

or equivalently,

$$E_1(t-)\eta(t)\leqslant E_1(0+)\eta(0)+\int_0^t E_1(\tau)\dot{\eta}(\tau)\,\mathrm{d}\tau-\int_0^t\dot{E}_2(\tau)\eta(\tau)\,\mathrm{d}\tau.\tag{1.2.38}$$

We choose

$$\eta(t) = \mathrm{e}^{qt}, \quad q = \frac{1}{3c^* \mu(p)}. \tag{1.2.39}$$

Using (1.1.23)–(1.1.28), we estimate $E_1(0+)$ by $C(1+\mu(p))$, so that (1.2.38) yields

$$\begin{aligned} E_1(t-)\mathrm{e}^{qt} &\leqslant C\big(1+\mu(p)\big) + \int_0^t (qE_1 - E)(\tau)\mathrm{e}^{q\tau}\,\mathrm{d}\tau \\ &\quad + \frac{1}{q}C^*\big(1+\mu^2(p)\big)\big(\mathrm{e}^{qt}-1\big). \end{aligned} \tag{1.2.40}$$

We have $qE_1(\tau) - E(\tau) = qZ(\tau) - \frac{1}{3}E(\tau) \leqslant 0$, hence

$$E_1(t-) \leqslant C\big(1+\mu(p)\big)\mathrm{e}^{-qt} + \frac{1}{q}C^*\big(1+\mu^2(p)\big), \tag{1.2.41}$$

that is,

$$E(t) \leqslant \frac{1}{c^*\mu(p)}E_1(t) \leqslant C\big(1+\mu^2(p)\big) \quad \text{for a.e. } t \in [0,T]. \tag{1.2.42}$$

Using (1.2.26) we have

$$\frac{1}{n}\sum_{j=1}^{n-1} v_j^2(t) \leqslant Ch(p)\big(1+\mu^2(p)\big), \tag{1.2.43}$$

and (1.2.14) yields

$$n\sum_{j=1}^{n}(\sigma_{j+1}-\sigma_j)^2(t) \leqslant Ch(p)\big(1+\mu^2(p)\big) \quad \text{for } t \in [0,T], \tag{1.2.44}$$

and (1.2.17) follows from (1.2.5) and (3.2.17). This completes the proof of Lemma 1.2.6. □

LEMMA 1.2.7. *Let Hypothesis* 1.2.3 *hold, and let* $\lambda : [0,1] \to \Lambda_K$, $\varepsilon, \sigma : [0,1] \times [0,T] \to \mathbb{R}$ *be continuous mappings,* $\varepsilon = \mathcal{F}[\lambda, \sigma]$. *Assume that there exist constants* c_1, c_2, c_3 *independent of* t *such that*

$$\left|\int_0^1 \varepsilon(x,t)\,\mathrm{d}x\right| \leqslant c_1 \qquad \forall t \in [0,T], \tag{1.2.45}$$

$$\big|\sigma(x,0)\big| \leqslant c_2 \qquad \forall x \in [0,1], \tag{1.2.46}$$

$$\int_0^1 \big|\partial_x \sigma(x,t)\big|\,\mathrm{d}x \leqslant 2c_3 \quad \forall t \in [0,T]. \tag{1.2.47}$$

Put

$$R = \max\left\{Lc_3, \frac{c_1+1}{m}, c_2+1, K, R_0\right\}. \tag{1.2.48}$$

Then $|\sigma(x,t)| < R$ *for all* $(x,t) \in [0,1] \times [0,T]$.

PROOF. Assume that the statement is false. Then there exist x_0, t_0 such that one of the following two alternatives occurs,

(i) $\sigma(x_0,t_0) = R$, $\sigma(x,t) > -R$ for $(x,t) \in [0,1] \times [0,t_0[$;

(ii) $\sigma(x_0,t_0) = -R$, $\sigma(x,t) < R$ for $(x,t) \in [0,1] \times [0,t_0[$.

The two cases are similar, we therefore consider only (i).

For all $x \in [0,1]$ we have

$$\sigma(x,t_0) \geqslant \sigma(x_0,t_0) - \int_0^1 \left|\partial_x \sigma(x,\tau)\right| \mathrm{d}\tau \geqslant R - 2c_3, \tag{1.2.49}$$

hence by definition (3.1.13) of the play,

$$\mathfrak{p}_r[\lambda,\sigma](x,t_0) \geqslant \sigma(x,t_0) - r \geqslant R - 2c_3 - r \tag{1.2.50}$$

for all $r > 0$. By Lemma 3.1.2, we have

$$\mathfrak{p}_r[\lambda,\sigma](x,t) \geqslant \min\left\{\lambda(x,r), -R + r\right\} \tag{1.2.51}$$

for all $r > 0$ and $(x,t) \in [0,1] \times [0,t_0]$. We have

$$\lambda(x,r) = \begin{cases} 0 & \text{for } r \geqslant K, \\ \lambda(x,r) - \lambda(x,K) \geqslant -K + r & \text{for } 0 < r < K, \end{cases} \tag{1.2.52}$$

hence

$$\mathfrak{p}_r[\lambda,\sigma](x,t) \geqslant \min\{0, -R + r\}. \tag{1.2.53}$$

Consequently, combining (1.2.50) with (1.2.53), we obtain

$$\mathfrak{p}_r[\lambda,\sigma](x,t_0) \geqslant \max\left\{R - 2c_3 - r, \min\{0, -R + r\}\right\}, \tag{1.2.54}$$

and (3.2.1) with w replaced by σ yields for every $x \in [0,1]$ that

$$\begin{aligned} \varepsilon(x,t_0) \geqslant{}& h(0)(R - 2c_3) + \int_0^{R-c_3} (R - 2c_3 - r)\,\mathrm{d}h(r) \\ &+ \int_{R-c_3}^{R} (-R + r)\,\mathrm{d}h(r) \\ ={}& 2H(R - c_3) - H(R) \\ \geqslant{}& 1 + c_1, \end{aligned} \tag{1.2.55}$$

which is a contradiction with the assumption (1.2.45). □

We are now ready to pass to the proof of Theorem 1.2.5 and thus conclude this subsection.

PROOF OF THEOREM 1.2.5. By virtue of the boundary conditions for v, the solution to (1.2.1) has the property

$$\int_0^1 \varepsilon(x,t)\,\mathrm{d}x = \int_0^1 \varepsilon^0(x)\,\mathrm{d}x$$

for every $t \geqslant 0$. We therefore can apply Lemma 1.2.7 and find a constant $\overline{C} > 0$ independent of t such that for every $t \geqslant 0$ we have

$$\|\sigma\|_{Q_t} \leqslant \overline{C}\left(1 + \int_0^1 (\partial_x \sigma)^2(x,t)\,\mathrm{d}x\right). \tag{1.2.56}$$

By Lemma 1.2.6 and (1.2.56), we have the implication

$$\begin{aligned} &\forall \delta > 0, \exists p_0 > 0, \forall p \geqslant p_0, \forall t \geqslant 0, \\ &\quad \|\sigma\|_{Q_t} \leqslant p \quad \Longrightarrow \quad \|\sigma\|_{Q_t} \leqslant \overline{C}(1 + \delta p). \end{aligned} \tag{1.2.57}$$

Choosing, for instance, $\delta = 1/(2\overline{C})$, we see that $|\sigma(x,t)|$ cannot exceed the value $2(p_0 + \overline{C})$. Using again Lemma 1.2.6 we obtain uniform $L^2(0,1)$-bounds for $\partial_t \sigma(\cdot,t)$ and $\partial_x \sigma(\cdot,t)$, which in turn (as a consequence of (3.2.16)) imply a uniform $L^2(0,1)$-bound for $\partial_t \varepsilon(\cdot,t)$. The bounds for $\partial_t v$ and $\partial_x v$ follow directly from (1.2.1). □

1.3. *Asymptotic stabilization*

Solutions to hyperbolic equations with linear viscous terms and without forcing asymptotically vanish with exponential rate. This is not the case if hysteresis is the only source of dissipation. In our situation, the decay rate is of the order $1/t$, and Example 1.3.3 confirms that this estimate is optimal. Keeping initial and boundary conditions (1.0.2)–(1.0.3), we consider the system with time-independent external forcing

$$\begin{cases} \partial_t v = \partial_x \sigma + f(v,x), \\ \partial_t \varepsilon = \partial_x v, \\ \varepsilon = \mathcal{F}[\lambda, \sigma] \end{cases} \tag{1.3.1}$$

under the following hypothesis.

HYPOTHESIS 1.3.1. *The function $f : \mathbb{R} \times]0,1[\to \mathbb{R}$ is such that the functions $f(v,\cdot) :]0,1[\to \mathbb{R}$ is measurable for all $v \in \mathbb{R}$, $f(\cdot,x) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $x \in]0,1[$,*

$f^0 := f(0,\cdot) \in L^2(0,1)$, *and there exists a constant* $\gamma_f > 0$ *such that for almost all arguments we have*

$$-\gamma_f \leqslant \partial_v f(v,x) \leqslant 0. \tag{1.3.2}$$

The main result of this subsection reads as follows.

THEOREM 1.3.2. *Let Hypotheses* 1.0.1 *and* 1.3.1 *hold, and let*

$$\lim_{p\to\infty} \frac{h(p)}{p^2} = 0. \tag{1.3.3}$$

Then there exists a constant $C > 0$ *independent of* t *such that the solution* (v,σ) *to* (1.3.1), (1.0.2)–(1.0.3) *satisfies a.e. the conditions*

$$\int_0^1 \big((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + (\partial_x \sigma)^2\big)(x,t)\,\mathrm{d}x \leqslant C, \tag{1.3.4}$$

$$\big|v(x,t)\big| + \big|\sigma(x,t)\big| \leqslant C. \tag{1.3.5}$$

If, moreover, the function κ *from* (3.4.8) *does not vanish on* $[0,\infty[$, *then there exist constants* $\sigma_\infty \in \mathbb{R}$ *and* $C > 0$ *independent of* t *such that*

$$\int_0^1 \big((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + \big(\partial_x \sigma + f^0\big)^2\big)(x,t)\,\mathrm{d}x \leqslant \frac{C}{(1+t)^2}, \tag{1.3.6}$$

$$\big|v(x,t)\big| \leqslant \frac{C}{1+t}, \tag{1.3.7}$$

$$\lim_{t\to\infty} \left|\sigma(x,t) + \int_0^x f^0\big(x'\big)\,\mathrm{d}x' - \sigma_\infty\right| = 0, \tag{1.3.8}$$

and the limit in (1.3.8) *is uniform with respect to* x.

PROOF. We proceed as in the proof of Theorem 1.2.5. relations (1.2.30)–(1.2.33) remain valid with $\beta_j \equiv 0$ and $C = 0$, hence $E(t)$ is nonincreasing in $[0,\infty[$. The value of $E(0+)$ is bounded by a constant according to (1.1.23)–(1.1.28), hence so is $E(t)$. In order to simplify the notation, we argue formally using (1.3.1), having however in mind the discrete system of the form (1.2.14)–(1.2.16). With this convention, we have

$$\int_0^1 \big((\partial_t v)^2 + \partial_t\sigma\,\partial_t\varepsilon\big)(x,t)\,\mathrm{d}x \leqslant C, \tag{1.3.9}$$

where C is some constant independent of $t > 0$. We now fix $T > 0$ and $p > 0$ such that

$$\|\sigma\|_{Q_T} \leqslant p. \tag{1.3.10}$$

By (1.3.1) and (3.2.16)–(3.2.17), we have

$$\int_0^1 (\partial_x v)^2(x,t)\,\mathrm{d}x = \int_0^1 (\partial_t \varepsilon)^2(x,t)\,\mathrm{d}x \leqslant h(p) \int_0^1 (\partial_t \sigma\, \partial_t \varepsilon)(x,t)\,\mathrm{d}x \leqslant Ch(p) \tag{1.3.11}$$

with a constant C (here and in the sequel) independent of $t \in [0,T]$ and $p > 0$, hence

$$\int_0^1 v^2(x,t)\,\mathrm{d}x \leqslant Ch(p) \quad \forall t \in [0,T] \tag{1.3.12}$$

and

$$\begin{aligned}\int_0^1 (\partial_x \sigma)^2(x,t)\,\mathrm{d}x &\leqslant C + \int_0^1 \big((\partial_t v)^2 + \gamma_f v^2\big)(x,t)\,\mathrm{d}x \\ &\leqslant C\big(1 + h(p)\big).\end{aligned} \tag{1.3.13}$$

From Lemma 1.2.7 it follows that $\|\sigma\|_{Q_T} \leqslant C\sqrt{1+h(p)}$. Choosing p sufficiently large we thus obtain from (1.3.3) the global bounds (1.3.4)–(1.3.5).

To prove (1.3.6)–(1.3.7), we pass again to the space-discrete approximations. Note that a uniform upper bound for $|\sigma(x,t)|$ is already available by virtue of (1.3.5). We therefore do not have to consider the dependence on p in (1.2.25)–(1.2.34). Using again the fact that $\beta_j \equiv 0$ in (1.2.28), (1.2.30), and that κ is positive, we obtain the counterpart of (1.2.29) and (1.2.33) in the form

$$\dot Z(t) + E(t) \leqslant CS^{2/3}(t) \quad \text{a.e.}, \tag{1.3.14}$$

$$E(t-) - E(s+) + c\int_s^t S(\tau)\,\mathrm{d}\tau \leqslant 0 \quad \forall 0 \leqslant s < t, \tag{1.3.15}$$

with some constants $c, C > 0$. In agreement with (1.2.27), we now fix some $m > 0$ such that

$$\big|Z(t)\big| \leqslant \frac{1}{8m} E(t) \quad \text{a.e.} \tag{1.3.16}$$

and set

$$E_m(t) := E(t) + \frac{4m}{1+mt} Z(t) \geqslant \frac{1}{2} E(t). \tag{1.3.17}$$

We have for all $0 \leqslant s < t$ that

$$\begin{aligned}&E_m(t-) - E_m(s+) + \int_s^t \left(cS(\tau) + \frac{4m}{1+m\tau} E(\tau) + \frac{4m^2}{(1+m\tau)^2} Z(\tau) \right) \mathrm{d}\tau \\ &\quad \leqslant \int_s^t \frac{4mC}{1+m\tau} S^{2/3}(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{1.3.18}$$

hence, by Hölder's inequality, there exists another constant $C > 0$ such that

$$E_m(t-) - E_m(s+) + \int_s^t \left(\frac{4m}{1+m\tau} E(\tau) + \frac{4m^2}{(1+m\tau)^2} Z(\tau) \right) \mathrm{d}\tau \leqslant \int_s^t \frac{C}{(1+m\tau)^3}\,\mathrm{d}\tau. \tag{1.3.19}$$

In view of (1.3.16), we have

$$4E(\tau) + \frac{4m}{(1+m\tau)} Z(\tau) \geqslant 3E_m(\tau) \quad \text{a.e.}, \tag{1.3.20}$$

hence

$$E_m(t-) - E_m(s+) + \int_s^t \frac{3m}{1+m\tau} E_m(\tau)\,\mathrm{d}\tau \leqslant \int_s^t \frac{C}{(1+m\tau)^3}\,\mathrm{d}\tau \tag{1.3.21}$$

for all $0 \leqslant s < t$. We argue similarly as in (1.2.37). The function

$$t \mapsto E_m(t) + \int_0^t \left(\frac{3m}{1+m\tau} E_m(\tau) - \frac{C}{(1+m\tau)^3} \right) \mathrm{d}\tau$$

is nonincreasing, hence for every nonnegative absolutely continuous test function $\eta(t)$ we have

$$\int_0^t \left(E_m(\tau)\dot{\eta}(\tau) - \left(\frac{3m}{1+m\tau} E_m(\tau) - \frac{C}{(1+m\tau)^3} \right) \eta(\tau) \right) \mathrm{d}\tau \geqslant E_m(t-)\eta(t) - E_m(0+)\eta(0). \tag{1.3.22}$$

For $\eta(t) = (1+mt)^3$ this yields

$$E_m(t-)(1+mt)^3 \leqslant E_m(0+) + Ct. \tag{1.3.23}$$

From (1.3.23) and (1.3.17), we obtain $E(t) \leqslant C(1+t)^{-2}$ a.e., and inequalities (1.3.6) and (1.3.7) easily follow. It remains to prove the convergence (1.3.8) of σ.

We fix constants $R > K$, $F_0 > 0$ such that

$$\int_0^1 \left| f^0(x') \right| \mathrm{d}x' \leqslant F_0, \qquad \left| \sigma(x,t) \right| \leqslant R \quad \forall (x,t) \in [0,1] \times [0,\infty[, \tag{1.3.24}$$

and define auxiliary functions

$$\tilde{\sigma}(x,t) = \sigma(x,t) + \int_0^x f^0(x')\,\mathrm{d}x', \tag{1.3.25}$$

$$\tilde{\lambda}(x,r) = \begin{cases} \lambda(x,r) & \text{for } 0 \leqslant r \leqslant R, \\ P\big[0, \tilde{\lambda}(x,R)\big](r-R) & \text{for } r \geqslant R, \end{cases} \tag{1.3.26}$$

$$\tilde{\varepsilon}(x,t) = \mathcal{F}\big[\tilde{\lambda}, \tilde{\sigma}\big](x,t), \tag{1.3.27}$$

where the mapping P is defined by (3.1.16). Note that $|\tilde{\lambda}(x,R)| \leqslant F_0$, hence $\tilde{\lambda}(x,r) = 0$ for $r \geqslant R + F_0$. We now claim that

$$\partial_t \tilde{\varepsilon}(x,t) = \partial_t \varepsilon(x,t) \quad \text{a.e.} \tag{1.3.28}$$

To check this conjecture, we denote

$$\xi_r(x,t) = \mathfrak{p}_r[\lambda, \sigma](x,t), \qquad \tilde{\xi}_r(x,t) = \mathfrak{p}_r\big[\tilde{\lambda}, \tilde{\sigma}\big](x,t) \tag{1.3.29}$$

for $r > 0$ and $(x,t) \in [0,1] \times [0,\infty[$. By (3.3.1) we have

$$\frac{\partial}{\partial t}\left(\tilde{\xi}_r(x,t) - \xi_r(x,t) - \int_0^x f^0\big(x'\big)\,\mathrm{d}x'\right)^2 \leqslant 0, \tag{1.3.30}$$

hence

$$\begin{aligned} &\left|\tilde{\xi}_r(x,t) - \xi_r(x,t) - \int_0^x f^0\big(x'\big)\,\mathrm{d}x'\right| \\ &\quad \leqslant \left|\tilde{\xi}_r(x,0) - \xi_r(x,0) - \int_0^x f^0\big(x'\big)\,\mathrm{d}x'\right| \end{aligned} \tag{1.3.31}$$

for all admissible values of r, x and t. We have by (3.1.10) and (3.1.16) that

$$\xi_r(x,0) = P\big[\lambda(x,\cdot), \sigma(x,0)\big](r), \qquad \tilde{\xi}_r(x,0) = P\big[\tilde{\lambda}(x,\cdot), \tilde{\sigma}(x,0)\big](r),$$

hence

$$\tilde{\xi}_r(x,0) = \xi_r(x,0) + \int_0^x f^0\big(x'\big)\,\mathrm{d}x' \quad \text{for } 0 < r \leqslant R, \tag{1.3.32}$$

and (1.3.31) implies that

$$\frac{\partial}{\partial t}\tilde{\xi}_r(x,t) = \frac{\partial}{\partial t}\xi_r(x,t) \quad \text{a.e. for } 0 < r \leqslant R. \tag{1.3.33}$$

On the other hand, we have $\lambda(x,r) = 0$ for $r \geqslant R$ and $|\tilde{\lambda}(x,R) - \tilde{\sigma}(x,t)| \leqslant R$, hence $\|m_{\tilde{\lambda}(x,\cdot)}(\tilde{\sigma}(x,\cdot))\|_{[0,t]} \leqslant R$ for all x and t. From Lemma 3.1.2 we conclude that

$$\frac{\partial}{\partial t}\tilde{\xi}_r(x,t) = \frac{\partial}{\partial t}\xi_r(x,t) \quad \text{a.e. for } r > R. \tag{1.3.34}$$

Combining (1.3.33) with (1.3.34), we obtain (1.3.28) from the definition (3.2.1) of the operator $\mathcal{F}$. This enables us to rewrite the system (1.3.1) in the form

$$\begin{cases} \partial_t v = \partial_x \tilde{\sigma} + f(v, x) - f(0, x), \\ \partial_t \tilde{\varepsilon} = \partial_x v \end{cases} \tag{1.3.35}$$

together with identity (1.3.27). In particular, we have for all $t \geqslant 0$ that

$$\int_0^1 \tilde{\varepsilon}(x, t)\, \mathrm{d}x = \int_0^1 \tilde{\varepsilon}(x, 0)\, \mathrm{d}x = \text{const.} \tag{1.3.36}$$

Put $s(t) = \int_0^1 \tilde{\sigma}(x, t)\, \mathrm{d}x$. The estimate

$$\left|\dot{s}(t)\right| \leqslant \frac{C}{1+t},$$

which follows from (1.3.6), is not sufficient for concluding that $s(t)$ converges as $t \to \infty$. To prove that this convergence indeed takes place, we have to use again special properties of the operator $\mathcal{F}$, more precisely Lemma 3.1.2. Set

$$\bar{s} = \limsup_{t\to\infty} s(t), \qquad \underline{s} = \liminf_{t\to\infty} s(t), \tag{1.3.37}$$

and assume that $\bar{s} > \underline{s}$. We fix some $\alpha > 0$ sufficiently small (it will be specified in (1.3.47)), and using (1.3.6) we find $0 < t_0 < t_1 < t_2$ such that

$$\int_0^1 \left|\partial_x \tilde{\sigma}(x, t)\right| \mathrm{d}x \leqslant \alpha, \qquad \underline{s} - \alpha \leqslant s(t) \leqslant \bar{s} + \alpha \quad \text{for } t \geqslant t_0, \tag{1.3.38}$$

$$s(t_1) \leqslant \underline{s} + \alpha, \qquad s(t_2) \geqslant \bar{s} - \alpha. \tag{1.3.39}$$

For all $x \in [0, 1]$ and $t \geqslant t_0$, we have

$$\left|\tilde{\sigma}(x, t) - s(t)\right| \leqslant \int_0^1 \left|\partial_x \tilde{\sigma}(x, t)\right| \mathrm{d}x \leqslant \alpha,$$

hence

$$\tilde{\sigma}(x, t_1) \leqslant \underline{s} + 2\alpha, \qquad \tilde{\sigma}(x, t_2) \geqslant \bar{s} - 2\alpha. \tag{1.3.40}$$

For $r > 0$ set $\lambda_i(x, r) = \mathfrak{p}_r[\tilde{\lambda}, \tilde{\sigma}](x, t_i)$, $i = 1, 2$. On the one hand, we have by definition of the play that

$$\begin{aligned} \lambda_1(x, r) &\leqslant \tilde{\sigma}(x, t_1) + r \leqslant \underline{s} + 2\alpha + r, \\ \lambda_2(x, r) &\geqslant \tilde{\sigma}(x, t_2) - r \geqslant \bar{s} - 2\alpha - r, \end{aligned} \tag{1.3.41}$$

on the other hand, Lemma 3.1.2 and the semigroup property (3.1.21) yield that

$$\lambda_2(x,r) \geqslant \min\bigl\{\lambda_1(x,r), \bar{s} - 2\alpha + r\bigr\}, \tag{1.3.42}$$

hence

$$\lambda_2(x,r) \geqslant \min\bigl\{\lambda_1(x,r), \lambda_1(x,r) - 4\alpha\bigr\} = \lambda_1(x,r) - 4\alpha. \tag{1.3.43}$$

Combining (1.3.43) with (1.3.41), we obtain

$$\lambda_2(x,r) \geqslant \max\bigl\{\lambda_1(x,r), \bar{s} + 2\alpha - r\bigr\} - 4\alpha, \tag{1.3.44}$$

consequently,

$$\begin{aligned} \lambda_2(x,r) - \lambda_1(x,r) &\geqslant \max\bigl\{0, \bar{s} + 2\alpha - r - \lambda_1(x,r)\bigr\} - 4\alpha \\ &\geqslant \max\{0, \bar{s} - \underline{s} - 2r\} - 4\alpha. \end{aligned} \tag{1.3.45}$$

Inserting inequality (1.3.45) into the integral in the definition (3.2.1) of $\mathcal{F}$ (note that h is nondecreasing), we obtain

$$\begin{aligned} &\tilde{\varepsilon}(x,t_2) - \tilde{\varepsilon}(x,t_1) \\ &\quad = h(0)\bigl(\tilde{\sigma}(x,t_2) - \tilde{\sigma}(x,t_1)\bigr) + \int_0^{R+F_0} \bigl(\lambda_2(x,r) - \lambda_1(x,r)\bigr)\,\mathrm{d}h(r) \\ &\quad \geqslant 2H\left(\frac{\bar{s} - \underline{s}}{2}\right) - 4\alpha h(R + F_0). \end{aligned} \tag{1.3.46}$$

Choosing $\alpha > 0$ such that

$$2H\left(\frac{\bar{s} - \underline{s}}{2}\right) - 4\alpha h(R + F_0) \geqslant \alpha, \tag{1.3.47}$$

we obtain

$$\int_0^1 \bigl(\tilde{\varepsilon}(x,t_2) - \tilde{\varepsilon}(x,t_1)\bigr)\,\mathrm{d}x \geqslant \alpha, \tag{1.3.48}$$

in contradiction with (1.3.36). We conclude that

$$\bar{s} = \underline{s} =: \sigma_\infty, \tag{1.3.49}$$

and the proof of Theorem 1.3.2 is complete. □

Note that $\varepsilon(x,t)$ also converges uniformly to some function $\varepsilon_\infty \in C[0,1]$. Indeed, by (3.2.11) we have for $0 \leqslant s < t$ that

$$\bigl|\varepsilon(x,t) - \varepsilon(x,s)\bigr| \leqslant C\bigl\|\sigma(x,\cdot) - \sigma(x,s)\bigr\|_{[s,t]}$$

with a constant $C > 0$ independent of x, t and s. The convergence $\tilde{\sigma}(\cdot, t) \to \sigma_\infty$ is uniform with respect to x, hence also $\varepsilon(\cdot, t)$ converge uniformly. Since all $\varepsilon(\cdot, t)$ are continuous by Theorem 1.1.1, we conclude that ε_∞ is continuous.

EXAMPLE 1.3.3. In order to illustrate the optimality of the $1/t$ decay rate as $t \to \infty$, we consider the following ODE system describing an elastoplastic spring-mass oscillator

$$\dot{v} = -\sigma, \tag{1.3.50}$$

$$\dot{\varepsilon} = v, \tag{1.3.51}$$

$$\varepsilon = \mathcal{F}[0, \sigma], \tag{1.3.52}$$

with initial conditions

$$v(0) = 0, \qquad \varepsilon(0) = \varepsilon_0 > 0, \tag{1.3.53}$$

where $\mathcal{F}[0, \cdot]$ is the Prandtl–Ishlinskii operator (3.2.1) with $\lambda \equiv 0$, and ε_0 is given. In fact, (1.3.50)–(1.3.52) is related to the space-discrete system (1.1.4)–(1.1.6) for $n = 2$ which is of the form

$$\dot{v}_1 = 2(\sigma_2 - \sigma_1), \tag{1.3.54}$$

$$\dot{\varepsilon}_1 = 2v_1, \tag{1.3.55}$$

$$\dot{\varepsilon}_2 = -2v_1, \tag{1.3.56}$$

$$\varepsilon_j = \mathcal{F}[0, \sigma_j], \quad j = 1, 2. \tag{1.3.57}$$

Indeed, assuming $\varepsilon_1(0) = -\varepsilon_2(0)$ and using the fact that both $\mathcal{F}$ and its inverse $\widehat{\mathcal{F}}$ (see Proposition 3.2.1) are odd, we obtain $\varepsilon_1 = -\varepsilon_2, \sigma_1 = -\sigma_2$, so that, after suitable rescaling, system (1.3.54)–(1.3.57) is equivalent to (1.3.50)–(1.3.52). We thus may use the argument of the proof of Theorem 1.3.2 and conclude that there exists a constant $C > 0$ such that

$$\left|v(t)\right| + \left|\sigma(t)\right| \leqslant \frac{C}{1+t} \quad \forall t \geqslant 0. \tag{1.3.58}$$

It will immediately follow from Proposition 1.3.4 that this decay rate is optimal. On the other hand, we show in Remark 1.3.6 that $\varepsilon(t)$ does not asymptotically vanish, but converges as $t \to \infty$ to some positive limit $\varepsilon_\infty > 0$. In mechanical interpretation, this means in agreement with practical experience that the initial deformation is not completely recovered during free elastoplastic oscillations.

PROPOSITION 1.3.4. *Let the generating function h of the Prandtl–Ishlinskii operator $\mathcal{F}$ be locally Lipschitz continuous in $[0, \infty[$, $h(0) > 0$, $h'(r) > 0$ a.e. Then there exist sequences $0 = t_0 < t_1 < t_2 < \cdots$ and $\sigma_0 > \sigma_1 > \sigma_2 > \cdots > 0$ such that the solution (v, σ) to*

(1.3.50)–(1.3.53) *has the properties*

$$\lim_{k\to\infty}(t_k - t_{k-1}) = \inf_{k=1,2,\ldots}(t_k - t_{k-1}) = \pi\sqrt{h(0)}, \tag{1.3.59}$$

$$(-1)^k\sigma \textit{ is increasing in } [t_{k-1}, t_k], \quad (-1)^k\sigma(t_k) = \sigma_k, \tag{1.3.60}$$

$$\exists c > 0, \quad \sigma_k \geqslant \frac{c}{1+k} \quad \forall k \in \mathbb{N}. \tag{1.3.61}$$

The proof will be carried out by induction. The following lemma constitutes a basis for the induction step.

LEMMA 1.3.5. *Under the hypotheses of Proposition* 1.3.4, *let* $t_* \geqslant 0$ *be such that*

$$\sigma_* := \sigma(t_*) > 0, \qquad \dot\sigma(t_*) = 0, \tag{1.3.62}$$

$$\exists r_* \geqslant \sigma_*, \quad \mathfrak{p}_r[0,\sigma](t_*) = \sigma_* - r \quad \textit{for } r \in [0, r_*]. \tag{1.3.63}$$

Then there exists $t^* > t_*$ *such that* $\dot\sigma(t) < 0$ *in* $]t_*, t^*[$, $\dot\sigma(t^*) = 0$ *and* $\sigma^* := \sigma(t^*) \in]-\sigma_*, 0[$.

PROOF. By virtue of (3.2.16)–(3.2.17) we have $v(t_*) = 0$, $\dot v(t_*) < 0$, hence there exists $\bar t > t_*$ such that $\dot\sigma(t) < 0$ in $]t_*, \bar t\,[$. We set

$$t^* = \sup\bigl\{\bar t > t_*; \dot\sigma(t) < 0 \text{ in } \bigr]0, \bar t\,\bigl[, \sigma\bigl(\bar t\bigr) > -\sigma_*\bigr\}. \tag{1.3.64}$$

As in the proof of Proposition 1.0.2, we compute the balance of the total energy $\frac12 v^2 + \mathcal{U}[0,\sigma]$, but in a slightly refined form. Let us first evaluate explicitly $\mathfrak{p}_r[0,\sigma](t)$ for $t \in [t_*, t^*]$. Set $\lambda_*(r) = \mathfrak{p}_r[0,\sigma](t_*)$. Identity (3.1.20) and assumption (1.3.63) yield

$$\mathfrak{p}_r[0,\sigma](t) = \begin{cases} \sigma(t) + r & \text{for } r < \frac12\bigl(\sigma_* - \sigma(t)\bigr), \\ \lambda_*(r) & \text{for } r \geqslant \frac12\bigl(\sigma_* - \sigma(t)\bigr), \end{cases} \tag{1.3.65}$$

and formulæ (3.4.1)–(3.4.3) yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac12 v^2 + \mathcal{U}[0,\sigma]\right)(t) &= -\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{p}_r[0,\sigma](t)\right| r\,\mathrm{d}h(r) \\ &= \dot\sigma(t)\int_0^{(1/2)(\sigma_* - \sigma(t))} r\,\mathrm{d}h(r) \\ &= \dot\sigma(t)\Gamma\left(\frac12\bigl(\sigma_* - \sigma(t)\bigr)\right), \end{aligned} \tag{1.3.66}$$

where we set $\Gamma(s) = sh(s) - H(s)$ for $s \geqslant 0$, H being given by (3.2.3). We denote by $\mathcal{H}$ the function

$$\mathcal{H}(r) = \int_0^r H(s)\,\mathrm{d}s \quad \text{for } r \geqslant 0. \tag{1.3.67}$$

With this notation, we can integrate (1.3.66) from t_* to t and obtain for $t \in [t_*, t^*[$ that

$$\frac{1}{2}v^2(t) - \big(\sigma_* + \sigma(t)\big) H\left(\frac{1}{2}\big(\sigma_* - \sigma(t)\big)\right)$$
$$= -\big(\sigma_* - \sigma(t)\big) H\left(\frac{1}{2}\big(\sigma_* - \sigma(t)\big)\right) + 4\mathcal{H}\left(\frac{1}{2}\big(\sigma_* - \sigma(t)\big)\right). \tag{1.3.68}$$

The function H is strictly convex, hence the right-hand side of (1.3.68) is negative for $t \in]t_*, t^*[$ and $\sigma_* + \sigma(t)$ thus remains negative even for $t \to t^*$. By definition (1.3.64) of t^* only one of the following two cases can occur,

(a) $t^* = \infty$, $\sigma^* := \lim_{t\to\infty} \sigma(t) > -\sigma_*$;

(b) $t^* < \infty$, $\dot\sigma(t^*) = 0$, $\sigma^* := \sigma(t^*) > -\sigma_*$.

Case (a) can easily be excluded. Indeed, we then would have

$$\lim_{t\to\infty} \dot\sigma(t) = \lim_{t\to\infty} \dot\varepsilon(t) = \lim_{t\to\infty} v(t) = 0. \tag{1.3.69}$$

We rewrite (1.3.68) in the form

$$\frac{1}{2}v^2(t) - 2\sigma(t) H\left(\frac{1}{2}\big(\sigma_* - \sigma(t)\big)\right) = 4\mathcal{H}\left(\frac{1}{2}\big(\sigma_* - \sigma(t)\big)\right), \tag{1.3.70}$$

and passing to the limit as $t \to \infty$ we obtain

$$-2\sigma^* H\left(\frac{1}{2}\big(\sigma_* - \sigma^*\big)\right) = 4\mathcal{H}\left(\frac{1}{2}\big(\sigma_* - \sigma^*\big)\right). \tag{1.3.71}$$

This implies that $\sigma^* < 0$, and from (1.3.50) we obtain $\lim_{t\to\infty} \dot v(t) = -\sigma^* > 0$, which contradicts (1.3.69). Consequently, the case (b) takes place together with (1.3.71), and the assertion of Lemma 1.3.5 follows. □

PROOF OF PROPOSITION 1.3.4. We first apply Lemma 1.3.5 at $t_* = 0$. We have by (3.2.4) that $\sigma_0 := \sigma(0) = H(\varepsilon_0) > 0$, hence the conditions are fulfilled with $r_* = \sigma_0$. We conclude that there exists $t_1 > 0$ such that $\dot\sigma(t_1) = 0$, $\sigma_1 := -\sigma(t_1) \in]0, \sigma_0[$, and setting $r_0 := \sigma_0$, $r_1 := \frac{1}{2}(\sigma_0 + \sigma_1) \in]\sigma_1, r_0[$, we obtain from (1.3.65) that

$$\mathfrak{p}_r[0,\sigma](t_1) = \begin{cases} -\sigma_1 + r & \text{for } r < r_1, \\ \sigma_0 - r & \text{for } r \in [r_1, r_0[, \\ 0 & \text{for } r \geqslant r_0. \end{cases} \tag{1.3.72}$$

Recall that the play operator $\mathfrak{p}_r[0,\cdot]$ is odd. We therefore may use again Lemma 1.3.5 for $-\sigma$ instead of σ with $t_* = t_1$, $r_* = r_1$, $\sigma_* = \sigma_1$, and find $t_2 > t_1$ such that $\dot\sigma(t_2) = 0$, $\sigma_2 := \sigma(t_2) \in]0, \sigma_1[$, $\sigma > 0$ in $]t_1, t_2[$, and

$$\mathfrak{p}_r[0,\sigma](t_2) = \begin{cases} \sigma_2 - r & \text{for } r < r_2, \\ \mathfrak{p}_r[0,\sigma](t_1) & \text{for } r \geqslant r_2. \end{cases} \tag{1.3.73}$$

By induction, we now construct a decreasing sequence $\{\sigma_k\}_{k=1}^{\infty}$ of positive numbers and an increasing sequence $0 = t_0 < t_1 < t_2 < \cdots$ such that (1.3.60) holds, and

$$\mathfrak{p}_r[0,\sigma](t_k) = \begin{cases} (-1)^k(\sigma_k - r) & \text{for } r < r_k := \frac{1}{2}(\sigma_{k-1} + \sigma_k), \\ \mathfrak{p}_r[0,\sigma](t_{k-1}) & \text{for } r \geqslant r_k, \end{cases} \tag{1.3.74}$$

for all $r > 0$ and $k \in \mathbb{N}$. By (1.3.71), we furthermore have

$$\sigma_k = \frac{2\mathcal{H}(r_k)}{H(r_k)} =: A(r_k) \quad \forall k \in \mathbb{N}, \tag{1.3.75}$$

hence

$$\sigma_{k-1} - \sigma_k = 2\big(r_k - A(r_k)\big) =: B(r_k). \tag{1.3.76}$$

We can differentiate the function B defined in (1.3.76) and obtain for all $x > 0$ the identity

$$B'(x) = \frac{4}{H^2(x)} \int_0^x H(s)\big(h(x) - h(s)\big)\,\mathrm{d}s. \tag{1.3.77}$$

By hypothesis, h is positive, increasing and Lipschitz continuous on $[0, \sigma_0]$, hence B is increasing, $B(0+) = 0$ and

$$0 < B(x) \leqslant Cx^2 \quad \forall x > 0, \tag{1.3.78}$$

with the convention that, similarly as in previous subsections, C denotes any positive constant independent of k. Using (1.3.75)–(1.3.78) we see, on the one hand, that the limit $\sigma_\infty = \lim_{k\to\infty} \sigma_k$ fulfils $B(\sigma_\infty) = 0$, hence $\sigma_\infty = 0$. On the other hand, we have

$$\frac{1}{\sigma_k} - \frac{1}{\sigma_{k-1}} = \frac{\sigma_{k-1} - \sigma_k}{\sigma_k \sigma_{k-1}} \leqslant C\frac{r_k}{\sigma_k} \leqslant C\frac{r_k H(r_k)}{\mathcal{H}(r_k)}. \tag{1.3.79}$$

The function $x \mapsto (xH(x)/\mathcal{H}(x))$ is bounded in $]0, \sigma_0]$, since its limit at $x \to 0+$ is $1/2$. Hence,

$$\frac{1}{\sigma_k} - \frac{1}{\sigma_{k-1}} \leqslant C \quad \text{for } k \in \mathbb{N}, \tag{1.3.80}$$

and (1.3.61) follows.

It remains to check that (1.3.59) holds. Assume for definiteness that k is even; the other case is similar. For $t \in [t_{k-1}, t_k]$ we have as in (1.3.65) and (1.3.74) that

$$\mathfrak{p}_r[0,\sigma](t) = \begin{cases} \sigma(t) - r & \text{for } r < r(t) := \frac{1}{2}\big(\sigma(t) - \sigma(t_{k-1})\big), \\ \mathfrak{p}_r[0,\sigma](t_{k-1}) & \text{for } r \geqslant r(t), \end{cases} \tag{1.3.81}$$

hence by (3.2.1),

$$\varepsilon(t) - \varepsilon(t_{k-1}) = 2H\big(r(t)\big), \tag{1.3.82}$$

or equivalently,

$$\sigma(t) - \sigma(t_{k-1}) = 2H^{-1}\left(\frac{1}{2}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)\right). \tag{1.3.83}$$

As a consequence of (1.3.50)–(1.3.51) and (1.3.83), the function ε solves the differential equation (according to our notation we have $\sigma(t_{k-1}) = -\sigma_{k-1}$)

$$\ddot{\varepsilon}(t) - \sigma_{k-1} + 2H^{-1}\left(\frac{1}{2}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)\right) = 0. \tag{1.3.84}$$

Testing (1.3.84) by $\dot{\varepsilon}(t)$, integrating from t_{k-1} to t and using the fact that $\dot{\varepsilon}(t_{k-1}) = 0$, $\dot{\varepsilon} > 0$ otherwise, we obtain

$$\dot{\varepsilon}(t) = \sqrt{2\sigma_{k-1}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big) - 8\widehat{\mathcal{H}}\left(\frac{1}{2}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)\right)}, \tag{1.3.85}$$

where

$$\widehat{\mathcal{H}}(r) = \int_0^r H^{-1}(s)\,\mathrm{d}s. \tag{1.3.86}$$

For $t = t_k$ we obtain from (1.3.85), in particular, that

$$\sigma_{k-1}\big(\varepsilon(t_k) - \varepsilon(t_{k-1})\big) = 4\widehat{\mathcal{H}}\left(\frac{1}{2}\big(\varepsilon(t_k) - \varepsilon(t_{k-1})\big)\right). \tag{1.3.87}$$

Set $p_k = \frac{1}{2}(\varepsilon(t_k) - \varepsilon(t_{k-1}))$. Then (1.3.85) can be rewritten as

$$\dot{\varepsilon}(t) = \frac{2\sqrt{2}}{\sqrt{p_k}}\sqrt{\frac{1}{2}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)\widehat{\mathcal{H}}(p_k) - p_k\widehat{\mathcal{H}}\left(\frac{1}{2}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)\right)}. \tag{1.3.88}$$

The substitution

$$s(t) = \frac{1}{2p_k}\big(\varepsilon(t) - \varepsilon(t_{k-1})\big)$$

yields

$$\dot{s}(t) = \frac{\sqrt{2}}{p_k}\sqrt{s(t)\widehat{\mathcal{H}}(p_k) - p_k\widehat{\mathcal{H}}\big(s(t)\big)}, \tag{1.3.89}$$

hence

$$\frac{\sqrt{2}}{p_k}(t_k - t_{k-1}) = \int_0^1 \frac{\mathrm{d}s}{\sqrt{s\widehat{\mathcal{H}}(p_k) - p_k\widehat{\mathcal{H}}(s)}}. \tag{1.3.90}$$

The function $\hat{h} = (H^{-1})'$ is decreasing and

$$s\widehat{\mathcal{H}}(p_k) - p_k\widehat{\mathcal{H}}(s) = s\int_0^{p_k}\int_{sr}^{r} \hat{h}(z)\,\mathrm{d}z\,\mathrm{d}r$$

for $s \in [0, 1]$, hence

$$\frac{\sqrt{2}}{p_k\sqrt{\hat{h}(p_k)}}\int_0^1 \frac{\mathrm{d}s}{\sqrt{s(1-s)}} \geqslant \frac{\sqrt{2}}{p_k}(t_k - t_{k-1}) \geqslant \frac{\sqrt{2}}{p_k\sqrt{\hat{h}(0)}}\int_0^1 \frac{\mathrm{d}s}{\sqrt{s(1-s)}}. \tag{1.3.91}$$

We have by (1.3.82) that $p_k = H(r_k)$, hence $\hat{h}(p_k) = 1/h(r_k)$, and we conclude that

$$\pi\sqrt{h(r_k)} \geqslant t_k - t_{k-1} \geqslant \pi\sqrt{h(0)}. \tag{1.3.92}$$

We already know that $\lim_{k\to\infty} r_k = 0$, and the proof of Proposition 1.3.4 is complete. □

REMARK 1.3.6. We have seen that $\{\varepsilon(t_k)\}$ is an alternating sequence of decreasing local maxima and increasing local minima of ε whose differences p_k tend to 0, hence $\varepsilon_\infty = \lim_{t\to\infty}\varepsilon(t)$ exists. It cannot be expected, however, that $\varepsilon_\infty = 0$. This follows from the identity

$$\varepsilon_\infty = \varepsilon_0 + 2\sum_{k=1}^{\infty}(-1)^k p_k = \sum_{k=1}^{\infty}(p_{2k-2} - 2p_{2k-1} + p_{2k}), \tag{1.3.93}$$

provided we put $p_0 := \varepsilon_0$. We still have $p_k = H(r_k)$ for all $k \geqslant 0$, and the relation

$$r_{2k-2} - 2r_{2k-1} + r_{2k} = \frac{1}{2}\big(B(r_{2k-2}) - B(r_{2k-1})\big) > 0 \tag{1.3.94}$$

holds for every $k \in \mathbb{N}$ by virtue of (1.3.76) and (1.3.77). The function H is increasing and convex, hence

$$p_{2k-1} = H(r_{2k-1}) < H\left(\frac{1}{2}(r_{2k-2} + r_{2k})\right) \leqslant \frac{1}{2}(p_{2k-2} + p_{2k}), \tag{1.3.95}$$

and we see that ε_∞ in (1.3.93) is positive. Similarly, the total energy $E(t) = \frac{1}{2}v^2(t) + \mathcal{U}[0,\sigma](t)$ does not asymptotically vanish. Putting $E_k = E(t_k) = \mathcal{U}[0,\sigma](t_k)$ for $k \geqslant 0$, we obtain by a computation similar as in the proof of Lemma 1.3.5 that

$$E_0 = \mathcal{H}(\sigma_0), \qquad E_{k-1} - E_k = (\sigma_{k-1} - \sigma_k)H(r_k) \quad \text{for } k \in \mathbb{N}. \tag{1.3.96}$$

Since H is strictly convex, we have

$$(\sigma_{k-1} - \sigma_k) H(r_k) < \int_{\sigma_k}^{\sigma_{k-1}} H(s)\,\mathrm{d}s = \mathcal{H}(\sigma_{k-1}) - \mathcal{H}(\sigma_k),$$

hence, using the fact that $E(t)$ is nonincreasing, we have

$$\lim_{t\to\infty} E(t) = E_0 - \sum_{k=1}^{\infty} (E_{k-1} - E_k) > 0. \tag{1.3.97}$$

This fact is not surprising either. A nonzero part of the initial energy is stored in the remanent deformation of the spring, the rest is dissipated into heat.

To conclude this subsection, let us note that if we allow $h'(0+)$ to be infinite, then the decay rate may be faster than $1/t$. The computation in [14], Example III.2.6, shows, however, that it is never exponential. The case where $\mathcal{F}$ in (1.3.52) is replaced by the Preisach operator (3.2.18) is investigated in [15].

1.4. *Quasilinear perturbations*

The results listed in the previous subsections are related to the variational and monotone character of the Prandtl–Ishlinskii operator as linear combination of solution operators to simple variational inequalities. For the existence of solutions, the *convexity of the hysteresis loops* is, however, more substantial than monotonicity. We now show that the existence and regularity result is stable also with respect to *quasilinear perturbations*. In other words, no shocks occur provided the convex–concave hysteresis behavior is preserved. Since monotonicity is lost, the question of uniqueness of solutions is open.

To be more specific, we consider the stress–strain relation in the form

$$\varepsilon = \mathcal{F}\big[\lambda, G(\sigma)\big], \tag{1.4.1}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a smooth "almost linear" increasing function, and $\mathcal{F}$ is the Prandtl–Ishlinskii operator given by (3.2.1). Note that the superposition $\mathcal{F} \circ G$ is the so-called *generalized Prandtl–Ishlinskii operator* as a special case of the Preisach operator (3.2.18), see [9] for the relationship between the functions h, g and ψ. Similar operators play an important role in modelling of piezoelectricity, see [18]. Changing accordingly the notation, we rewrite system (1.0.1) with constitutive law of the type (1.4.1) in a more convenient form with a parameter $\delta > 0$ as

$$\begin{cases} \partial_t v = \partial_x\big(\sigma + \delta g(\sigma)\big) + f(\sigma, v, x, t), \\ \partial_t \varepsilon = \partial_x v, \\ \varepsilon = \mathcal{F}[\lambda, \sigma]. \end{cases} \tag{1.4.2}$$

The existence part of Theorem 1.1.1 holds for δ sufficiently small in the following form.

THEOREM 1.4.1. *Let Hypothesis* 1.0.1 *be fulfilled, let the function* κ *from* (3.4.8) *be positive on* $[0, \infty[$, *and let* g *be a nondecreasing function in* $\mathbb{R}$ *with locally Lipschitz continuous derivative. Then there exists* $\delta_0 > 0$ *such that system* (1.4.2), (1.0.2)–(1.0.3) *for* $0 \leqslant \delta \leqslant \delta_0$ *admits a solution* $(v, \sigma, \varepsilon) \in C(\overline{Q}_T; \mathbb{R}^3)$ *such that* $\partial_t v, \partial_x v, \partial_t \sigma, \partial_x \sigma, \partial_t \varepsilon$ *belong to* $L^\infty(0, T; L^2(0, 1))$, *and* (1.4.1) *holds almost everywhere in* Q_T.

PROOF. We discretize the system (1.4.1) in space in the form similar to (1.1.4)–(1.1.6), more precisely,

$$\dot{v}_j = n(\sigma_{j+1} - \sigma_j) + \delta n\big(g(\sigma_{j+1}) - g(\sigma_j)\big) + f_j(\sigma_j, v_j, t), \quad j = 1, \dots, n-1, \tag{1.4.3}$$

$$\dot{\varepsilon}_j = n(v_j - v_{j-1}), \qquad j = 1, \dots, n, \tag{1.4.4}$$

$$\varepsilon_j = \mathcal{F}[\lambda_j, \sigma_j], \qquad j = 1, \dots, n. \tag{1.4.5}$$

For each fixed n and δ, the existence of solutions to (1.4.3)–(1.4.5) together with conditions (1.1.7)–(1.1.10) is obtained in the same way as in the proof of Theorem 1.1.1. The estimates, however, have to be carried out in a more careful way.

We fix a bound $p_1 \geqslant \max_{j=1,\dots,n} |\sigma_j(0)|$ independent of n, and for each $p > p_1$ we solve (1.4.3)–(1.4.5) with $\delta = \delta(p) > 0$ such that

$$\delta(p) \max_{|s| \leqslant p} g'(s) \leqslant 1, \qquad \delta(p) \operatorname*{sup\,ess}_{|s| \leqslant p} \big|g''(s)\big| \leqslant \frac{\kappa(p)}{4h(p)}. \tag{1.4.6}$$

We define the maximal time $T(n, p) > 0$ for which all $|\sigma_j(t)|$ remain bounded by p, that is,

$$T(n, p) = \max\Big\{t \in [0, T];\ \max_{j=1,\dots,n} \big|\sigma_j(t)\big| \leqslant p\Big\}. \tag{1.4.7}$$

The counterpart of (1.1.18)–(1.1.19) reads

$$\ddot{v}_j \dot{v}_j \leqslant n(\dot{\sigma}_{j+1} - \dot{\sigma}_j)\dot{v}_j + \delta(p) n \frac{\mathrm{d}}{\mathrm{d}t}\big(g(\sigma_{j+1}) - g(\sigma_j)\big)\dot{v}_j + \big(\beta_j(t) + \alpha_f(t)\big(|\dot{\sigma}_j| + |\dot{v}_j|\big)\big)|\dot{v}_j|, \tag{1.4.8}$$

$$\ddot{\varepsilon}_j \dot{\sigma}_j = n(\dot{v}_j - \dot{v}_{j-1})\dot{\sigma}_j, \tag{1.4.9}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n-1} \ddot{v}_j \dot{v}_j + \frac{1}{n}\sum_{j=1}^{n} \big(1 + \delta(p) g'(\sigma_j)\big)\ddot{\varepsilon}_j \dot{\sigma}_j \leqslant \frac{1}{n}\sum_{j=1}^{n-1} \big(\beta_j(t) + \alpha_f(t)\big(|\dot{\sigma}_j| + |\dot{v}_j|\big)\big)|\dot{v}_j|. \tag{1.4.10}$$

We use Theorem 3.4.1 to obtain, for all $0 \leqslant s < t \leqslant T(n,p)$ and all $j = 1, \dots, n$, that

$$\int_s^t \ddot{\varepsilon}_j \dot{\sigma}_j \,\mathrm{d}\tau \geqslant \frac{\kappa(p)}{4} \int_s^t |\dot{\sigma}_j|^3 \,\mathrm{d}\tau + \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(t-) - \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(s+). \tag{1.4.11}$$

We see that the function

$$t \mapsto \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(t) + \frac{\kappa(p)}{4} \int_0^t |\dot{\sigma}_j|^3 \,\mathrm{d}\tau - \int_0^t \ddot{\varepsilon}_j \dot{\sigma}_j \,\mathrm{d}\tau$$

is nonincreasing in $[0, T(n,p)]$. For each absolutely continuous nonnegative test function η and for all $t \in [0, T(n,p)]$, we thus have

$$\begin{aligned} &\int_0^t \left(\frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(\tau)\dot{\eta}(\tau) + \left((\ddot{\varepsilon}_j \dot{\sigma}_j)(\tau) - \frac{\kappa(p)}{4} \big|\dot{\sigma}_j(\tau)\big|^3 \right) \eta(\tau) \right) \mathrm{d}\tau \\ &\quad \geqslant \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(t)\dot{\eta}(t) - \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(0+)\dot{\eta}(0). \end{aligned} \tag{1.4.12}$$

We now set $\eta(t) = 1 + \delta(p) g'(\sigma_j(t))$ in (1.4.12). By hypothesis (1.4.6), we have $1 \leqslant \eta(t) \leqslant 2$ for $t \in [0, T(n,p)]$, hence

$$\begin{aligned} \frac{1}{2}(\dot{\varepsilon}_j \dot{\sigma}_j)(t) &\leqslant (\dot{\varepsilon}_j \dot{\sigma}_j)(0+) + \int_0^t \big(1 + \delta(p) g'\big(\sigma_j(\tau)\big)\big)(\ddot{\varepsilon}_j \dot{\sigma}_j)(\tau) \,\mathrm{d}\tau \\ &\quad + \int_0^t \left(\delta(p) g''\big(\sigma_j(\tau)\big)\big(\dot{\varepsilon}_j \dot{\sigma}_j^2\big)(\tau) - \frac{\kappa(p)}{4} \big|\dot{\sigma}_j(\tau)\big|^3 \right) \mathrm{d}\tau. \end{aligned} \tag{1.4.13}$$

From (3.2.16) and hypothesis (1.4.6), it follows that for all $t \in [0, T(n,p)]$ we have

$$\delta(p) \big| g''\big(\sigma_j(t)\big) \dot{\varepsilon}_j(t) \big| \leqslant \frac{\kappa(p)}{4} \big|\dot{\sigma}_j(t)\big|,$$

hence the last integral on the right-hand side of (1.4.13) is nonpositive. We therefore have

$$\begin{aligned} &\frac{1}{2n} \left(\sum_{j=1}^{n-1} \dot{v}_j^2(t) + \sum_{j=1}^{n} (\dot{\varepsilon}_j \dot{\sigma}_j)(t) \right) \\ &\quad \leqslant \frac{1}{2n} \left(\sum_{j=1}^{n-1} \dot{v}_j^2(0) + 2 \sum_{j=1}^{n} (\dot{\varepsilon}_j \dot{\sigma}_j)(0+) \right) \\ &\qquad + \frac{1}{n} \int_0^t \left(\sum_{j=1}^{n-1} (\ddot{v}_j \dot{v}_j)(\tau) + \frac{1}{n} \sum_{j=1}^{n} \big(1 + \delta(p) g'\big(\sigma_j(\tau)\big)\big)(\ddot{\varepsilon}_j \dot{\sigma}_j)(\tau) \right) \mathrm{d}\tau \end{aligned} \tag{1.4.14}$$

for a.e. $t \in]0, T(n,p)[$. The initial conditions on the right-hand side of (1.4.14) can be estimated independently of n and p similarly as in the proof of Theorem 1.1.1, and as a consequence of (3.2.17) we have $\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) \geqslant h(0)\dot{\sigma}_j^2(t)$ a.e. Combining (1.4.14) with (1.4.10) and using Gronwall's inequality we find a constant C_T independent of n and p (and possibly dependent on T) such that

$$\frac{1}{n}\left(\sum_{j=1}^{n-1} \dot{v}_j^2(t) + \sum_{j=1}^{n} \dot{\sigma}_j^2(t)\right) \leqslant C_T \quad \forall t \in \big[0, T(n,p)\big]. \tag{1.4.15}$$

Arguing as in the proof of Theorem 1.1.1, we find a constant $\overline{C}_T$ independent of n and p such that

$$n\left(\sum_{j=1}^{n} (v_j - v_{j-1})^2(t) + \sum_{j=1}^{n-1} (\sigma_{j+1} - \sigma_j)^2(t)\right) + \max_{j=1,\dots,n} \big|\sigma_j(t)\big| \\ \leqslant \overline{C}_T \quad \forall t \in \big[0, T(n,p)\big]. \tag{1.4.16}$$

We now fix any $p_0 \geqslant \max\{p_1, \overline{C}_T\}$ and set $\delta_0 = \delta(p_0)$. Then we have $T(n,p) = T$ for all $n \in \mathbb{N}$, and the estimates (1.4.15) and (1.4.16) hold a.e. in $[0, T]$. This enables us to complete the proof passing to the limit as $n \to \infty$ similarly as in the proof of Theorem 1.1.1. □

2. Periodic solutions

In this section, we consider the system (1.0.1) with boundary conditions (1.0.3) and with the time-periodicity condition

$$v(x, t+T) = v(x,t), \qquad \sigma(x, t+T) = \sigma(x,t) \tag{2.0.17}$$

for all $(x,t) \in [0,1] \times [0, \infty[$ instead of (1.0.2), where $T > 0$ is a fixed period. Our analysis will be carried out in the spaces $L_T^p = L_{\mathrm{loc}}^p(]0,1[\times]0,\infty[)$ for $1 \leqslant p \leqslant \infty$ and in the space $\mathcal{C}_T^0$ of continuous functions, all satisfying the T-periodicity condition. Having in mind Corollary 3.1.3 which states that outputs of hysteresis operators with periodic inputs may possibly become periodic only after one period, we define the norms

$$\|w\|_p = \left(\int_T^{2T}\int_0^1 \big|w(x,t)\big|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \quad \text{for } w \in L_T^p, 1 \leqslant p < \infty, \tag{2.0.18}$$

$$\|w\|_\infty = \sup\mathrm{ess}\big\{\big|w(x,t)\big|;\, (x,t) \in]0,1[\times]T, 2T[\big\} \quad \text{for } w \in L_T^\infty. \tag{2.0.19}$$

We endow the space $\mathcal{C}_T^0$ with the norm $\|\cdot\|_\infty$ as well. Recall that the compact embeddings

$$\begin{aligned} H_T^{2,3} &:= \big\{w \in L_T^2;\, \partial_x w \in L_T^2, \partial_t w \in L_T^3\big\} \hookrightarrow\hookrightarrow \mathcal{C}_T^0, \\ H_T^{3,2} &:= \big\{w \in L_T^2;\, \partial_x w \in L_T^3, \partial_t w \in L_T^2\big\} \hookrightarrow\hookrightarrow \mathcal{C}_T^0 \end{aligned} \tag{2.0.20}$$

take place, and we fix constants $M, M' > 0$ such that

$$
\begin{aligned}
&\|w\|_\infty \leqslant \left| \int_T^{2T} \int_0^1 w(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| + M\big(\|\partial_x w\|_2 + \|\partial_t w\|_3\big) \\
&\qquad \forall w \in H_T^{2,3}, \\
&\|w\|_\infty \leqslant \left| \int_T^{2T} \int_0^1 w(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| + M'\big(\|\partial_x w\|_3 + \|\partial_t w\|_2\big) \\
&\qquad \forall w \in H_T^{3,2}.
\end{aligned}
\tag{2.0.21}
$$

An estimate for M, M' can be found in [14], Appendix 2.

We find sufficient conditions for the existence and uniqueness of solutions to the Dirichlet-periodic problem and prove its global asymptotic stability.

2.1. *Statement of main results*

In addition to Hypothesis 1.0.1, we impose the following more restrictive assumptions on f.

HYPOTHESIS 2.1.1. *The following conditions hold for all admissible arguments*:

(i) $f(\sigma, v, x, t+T) = f(\sigma, v, x, t)$;

(ii) $f^0, \beta_f \in L_T^2$;

(iii) $|\partial_\sigma f(\sigma, v, x, t)| \leqslant \gamma_f$;

(iv) $-\gamma_f \leqslant \partial_v f(\sigma, v, x, t) \leqslant 0$, *where* $\gamma_f > 0$ *is a fixed constant*.

In this subsection we list the main results on existence (Theorem 2.1.2), uniqueness (Theorem 2.1.3) and asymptotic stability (Theorem 2.1.6) of periodic solutions to system (1.0.1), (1.0.3). Proofs are postponed to the next subsections.

THEOREM 2.1.2. *Let Hypotheses* 1.0.1 *and* 2.1.1 *hold*. *Assume in addition to* (3.2.15) *that the functions h and κ in* (3.2.1) *and* (3.4.8) *satisfy the condition*

$$
\limsup_{p \to \infty} \frac{h(p)}{p\kappa(p)} =: q < \infty \tag{2.1.1}
$$

with

$$
4\sqrt{T}\gamma_f M q \left(1 + \frac{T}{2\pi}\gamma_f\big(1 + \gamma_f \mathrm{e}^{\gamma_f}\big)\right) < 1. \tag{2.1.2}
$$

Then (1.0.1) *with boundary conditions* (1.0.3) *and periodicity conditions* (2.0.17) *admit a solution* $(v, \sigma) \in H_T^{3,2} \times H_T^{2,3}$, *and* (1.0.1) *are satisfied for a.e.* $(x,t) \in]0,1[\times]T, \infty[$.

The situation is similar as in Sections 1.2 and 1.3, cf. Remark 1.2.4. We are able to prove existence only if no negative friction is present. Moreover, uniqueness is obtained only if f is independent of σ, that is, if problem (1.0.1) has the form (1.2.1). On the other hand, we can replace (2.1.1) by a weaker condition (2.1.3).

THEOREM 2.1.3. *Let Hypotheses* 1.0.1 *and* 2.1.1 *hold with f independent of σ. Assume, instead of* (2.1.1), *that*

$$\limsup_{p\to\infty} \frac{h^{3/4}(p)}{p\kappa^{1/2}(p)} =: \tilde{q} < \infty \tag{2.1.3}$$

with

$$2T^{1/4}\sqrt{\|\beta_f\|_2}\,M\tilde{q}\left(1+\frac{T}{2\pi}\gamma_f\right) < 1. \tag{2.1.4}$$

Let further $\rho \in \mathbb{R}$ be given. Then problem (1.2.1) *with boundary conditions* (1.0.3) *and periodicity conditions* (2.0.17) *admits a unique solution $(v,\sigma) \in H_T^{3,2} \times H_T^{2,3}$*, (1.2.1) *are satisfied for a.e. $(x,t) \in]0,1[\times]T,\infty[$ and*

$$\frac{1}{T}\int_T^{2T} \sigma(0,t)\,\mathrm{d}t = \rho. \tag{2.1.5}$$

REMARK 2.1.4. From (2.1.1) it follows that

$$\limsup_{p\to\infty} \frac{h(p)}{p(h(p)-h(p-1))} \leqslant q, \tag{2.1.6}$$

hence

$$\frac{1}{(q+1)p} \leqslant \frac{h(p)-h(p-1)}{h(p)} \leqslant \log h(p) - \log h(p-1) \tag{2.1.7}$$

for p larger than some $p_1 > 0$. We thus obtain that

$$\lim_{p\to\infty} h(p) = \infty, \qquad \lim_{p\to\infty} \frac{\sqrt{h(p)}}{p} = \lim_{p\to\infty} \sqrt{\frac{h(p)}{p\kappa(p)}}\sqrt{\frac{\kappa(p)}{p}} = 0. \tag{2.1.8}$$

In particular, if condition (2.1.1) is satisfied, then (2.1.3) holds with $\tilde{q} = 0$. The class of functions satisfying (2.1.1) is nonempty. For example, for every locally Lipschitz continuous function h such that

$$h_* \max\{r_0, r\}^{\alpha-1} \leqslant h'(r) \leqslant h^* \max\{r_0, r\}^{\alpha-1} \quad \text{a.e.}, \tag{2.1.9}$$

where $0 < h_* \leqslant h^*$, $r_0 > 0$, $\alpha \in]0,1]$ are given numbers, we have for $p > r_0$ that $\kappa(p) \geqslant h_* p^{\alpha-1}$, $h(p) \leqslant h(0) + h^* p^\alpha/\alpha$, hence (2.1.1) holds. For the validity of (2.1.3), it suffices to have for instance

$$h_* \max\{r_0, r\}^{\alpha_1 - 1} \leqslant h'(r) \leqslant h^* \max\{r_0, r\}^{\alpha_2 - 1} \quad \text{a.e.} \tag{2.1.10}$$

with some $\alpha_1 \in]-1, 1]$ and $\alpha_1 \leqslant \alpha_2 \leqslant (2/3)(\alpha_1 + 1)$.

REMARK 2.1.5. In Remark 2.3.1 at the end of Section 2.3, we comment on the nonuniqueness related to the fact that ρ is arbitrary. Also the value of ρ can be determined uniquely if we consider the Dirichlet boundary conditions in displacements instead of velocities.

To conclude this subsection, we state, as a complement to Theorem 2.1.3, a result on asymptotic stability of periodic solutions.

THEOREM 2.1.6. *Let Hypotheses* 1.0.1, 2.1.1 *hold with f independent of σ, and let $\kappa(p) > 0$ for all $p > 0$. Let us define the set*

$$\begin{aligned} B = \big\{&(v,\sigma) \in L^\infty\big(]0,1[\times]0,\infty[\big)^2; \\ &\partial_t v, \partial_t \sigma, \partial_x v, \partial_x \sigma \in L^\infty\big(0,\infty; L^2(0,1)\big), v(0,t) = v(1,t) = 0\big\}, \end{aligned} \tag{2.1.11}$$

and assume that $(\underline{v}, \underline{\sigma}) \in B$ is a solution of the problem

$$\begin{cases} \partial_t \underline{v} = \partial_x \underline{\sigma} + f(\underline{v}, x, t), \\ \partial_t \underline{\varepsilon} = \partial_x \underline{v}, \\ \underline{\varepsilon} = \mathcal{F}[\underline{\lambda}, \underline{\sigma}], \end{cases} \tag{2.1.12}$$

where $\underline{\lambda} \in C([0,1]; \Lambda_K)$ and $K > 0$ are fixed. Then there exists $\lambda \in C([0,1]; \Lambda_{\underline{K}})$ with $\underline{K} = \max\{K, \|\underline{\sigma}\|_\infty\}$ and a periodic solution $(v,\sigma) \in B$ of (1.2.1) *such that*

$$\lim_{t\to\infty} \big(\big|\underline{v}(\cdot,t) - v(\cdot,t)\big|_{L^\infty(0,1)} + \big|\underline{\sigma}(\cdot,t) - \sigma(\cdot,t)\big|_{L^\infty(0,1)}\big) = 0. \tag{2.1.13}$$

REMARK 2.1.7. The assumptions of Theorem 2.1.6 are satisfied for instance under the hypotheses of Theorem 1.2.5. In such a case, condition (2.1.3) automatically holds.

2.2. *Existence*

This subsection is devoted to the proof of Theorem 2.1.2. We fix some $\rho \in \mathbb{R}$ and consider the problem

$$\partial_t v = \partial_x \sigma^* + f\big(\sigma^* + \hat\sigma, v, x, t\big) - \frac{1}{T}\int_T^{2T} f\big(\sigma^* + \hat\sigma, v, x, \tau\big)\,\mathrm{d}\tau, \tag{2.2.1}$$

$$\partial_t \varepsilon^* = \partial_x v, \tag{2.2.2}$$

$$\varepsilon^* = \mathcal{F}\big[\lambda, \sigma^*\big], \tag{2.2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{\sigma}(x) = -\frac{1}{T}\int_T^{2T} f\big(\sigma^* + \hat{\sigma}, v, x, \tau\big)\,\mathrm{d}\tau, \quad \hat{\sigma}(0) = \rho, \tag{2.2.4}$$

together with the T-periodicity condition and

$$v(0,t) = v(1,t) = 0, \quad \int_T^{2T} v(x,t)\,\mathrm{d}t = \int_T^{2T} \sigma^*(x,t)\,\mathrm{d}t = 0 \tag{2.2.5}$$

for all admissible arguments. We observe that if $(v, \sigma^*, \hat{\sigma}, \varepsilon^*)$ is a solution to (2.2.1)–(2.2.5), then (v, σ) with $\sigma = \sigma^* + \hat{\sigma}$ and $\varepsilon = \mathcal{F}[\lambda, \sigma]$ satisfy the conditions of Theorem 2.1.2 since, by Proposition 3.3.1, we have $\partial_t \varepsilon = \partial_t \varepsilon^*$ for a.e. $t > T$.

The solution will be constructed by the Galerkin method. For $j \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$, $t \geqslant 0$, $x \in [0,1]$ we define basis functions

$$e_j(t) = \begin{cases} \sin \frac{2\pi j}{T} t & \text{if } j > 0, \\ \cos \frac{2\pi j}{T} t & \text{if } j \leqslant 0, \end{cases} \tag{2.2.6}$$

$$\varphi_k(x) = \sin k\pi x, \qquad \psi_k(x) = \cos k\pi x. \tag{2.2.7}$$

For all relevant values of j, k, x, t we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} e_j(t) = \frac{2\pi j}{T} e_{-j}(t), \\ &\frac{\mathrm{d}}{\mathrm{d}x}\varphi_k(x) = k\pi\psi_k(x), \\ &\frac{\mathrm{d}}{\mathrm{d}x}\psi_k(x) = -k\pi\varphi_k(x). \end{aligned} \tag{2.2.8}$$

For each fixed $n \in \mathbb{N}$ we set $J_n = \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$, and define functions $v^{(n)}, \sigma^{(n)}, \varepsilon^{(n)}, \hat{\sigma}^{(n)}$ by the formulæ

$$v^{(n)}(x,t) = \sum_{j \in J_n} \sum_{k=1}^{n} v_{jk} e_j(t) \varphi_k(x), \tag{2.2.9}$$

$$\sigma^{(n)}(x,t) = \sum_{j \in J_n} \sum_{k=1}^{n} \sigma_{jk} e_j(t) \psi_k(x), \tag{2.2.10}$$

$$\varepsilon^{(n)}(x,t) = \mathcal{F}\big[\lambda, \sigma^{(n)}\big](x,t), \tag{2.2.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{\sigma}^{(n)}(x) = -\frac{1}{T}\int_T^{2T} f\big(\sigma^{(n)} + \hat{\sigma}^{(n)}, v^{(n)}, x, \tau\big)\,\mathrm{d}\tau, \quad \hat{\sigma}^{(n)}(0) = \rho, \tag{2.2.12}$$

where v_{jk}, σ_{jk} are solutions of the system

$$\int_T^{2T}\int_0^1 \big(\partial_t v^{(n)} - \partial_x \sigma^{(n)} - f\big(\sigma^{(n)} + \hat{\sigma}^{(n)}, v^{(n)}, x, t\big)\big) \\ \times e_j(t)\varphi_k(x)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.2.13}$$

$$\int_T^{2T}\int_0^1 \big(\partial_t \varepsilon^{(n)} - \partial_x v^{(n)}\big) e_j(t)\psi_k(x)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.2.14}$$

for all $j \in J_n$, $k = 0, 1, \ldots, n$. We first have to prove that (2.2.13)–(2.2.14) has a solution. The unknown in the problem is the vector $\mathbf{v} = (v_{jk}, \sigma_{jk})$, $j \in J_n$, $k = 0, 1, \ldots, n$, with $v_{j0} = 0$ for all $j \in J_n$, hence $\mathbf{v}$ can be considered as an element of $\mathbf{V} := \mathbb{R}^{2n\times n} \times \mathbb{R}^{2n\times(n+1)}$. The mappings which with $\mathbf{v} \in \mathbf{V}$ associate the functions $\varepsilon^{(n)} \in \mathcal{C}_T^0$ and $\hat{\sigma}^{(n)} \in C[0,1]$ are continuous, hence the system (2.2.13)–(2.2.14) is of the form

$$\Phi(\mathbf{v}) = 0, \tag{2.2.15}$$

where Φ is a continuous mapping from $\mathbf{V}$ to $\mathbf{V}$. We define a homotopy $\Phi_s : \mathbf{V} \to \mathbf{V}$ with parameter $s \in [0,1]$ by the left-hand side of the system

$$\int_T^{2T}\int_0^1 \big(\partial_t v^{(n)} - \partial_x \sigma^{(n)} - sf\big(\sigma^{(n)} + \hat{\sigma}_s^{(n)}, v^{(n)}, x, t\big)\big) \\ \times e_j(t)\varphi_k(x)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.2.16}$$

$$\int_T^{2T}\int_0^1 \big(\partial_t \varepsilon^{(n)} - \partial_x v^{(n)}\big) e_j(t)\psi_k(x)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.2.17}$$

where

$$\varepsilon^{(n)}(x,t) = \mathcal{F}\big[s\lambda, \sigma^{(n)}\big](x,t), \tag{2.2.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{\sigma}_s^{(n)}(x) = -\frac{s}{T}\int_T^{2T} f\big(\sigma^{(n)} + \hat{\sigma}_s^{(n)}, v^{(n)}, x, \tau\big)\,\mathrm{d}\tau, \quad \hat{\sigma}_s^{(n)}(0) = \rho. \tag{2.2.19}$$

We now check that (2.2.16)–(2.2.19) has no solution on the boundary of a sufficiently large ball independently of $s \in [0,1]$. The operator $\mathcal{F}[0,\cdot]$ corresponding to the initial configuration $\lambda \equiv 0$ is odd, hence also Φ_0 is odd in $\mathbf{V}$, so that its Brouwer degree with respect to this ball and to the point $0 \in \mathbf{V}$ is nonzero. By homotopy, also the degree of $\Phi_1 = \Phi$ is nonzero, hence a solution exists inside the ball. We thus establish the existence of a solution to (2.2.16)–(2.2.19) provided we prove the following statement.

> There exist p_∞, $\tilde{p}_\infty > 0$ independent of $n \in \mathbb{N}$ and $s \in [0,1]$ such that if $\mathbf{v} \in \mathbf{V}$ is a solution of (2.2.16)–(2.2.19) with $\|\sigma^{(n)}\|_\infty = p$, then $p \leqslant p_\infty$ and $\|v^{(n)}\|_\infty \leqslant \tilde{p}_\infty$. (2.2.20)

To prove conjecture (2.2.20), we consider some $p \geqslant K$, where K is as in Hypothesis 1.0.1, some $n \in \mathbb{N}$, $s \in [0,1]$, and a solution of (2.2.16)–(2.2.19) with $\|\sigma^{(n)}\|_\infty \leqslant p$. We test (2.2.16) by $(2\pi j/T)^2 v_{jk}$, (2.2.17) by $(2\pi j/T)^2 \sigma_{jk}$ and sum them up. Integrating by parts and using the T-periodicity we obtain

$$\begin{aligned}
&\int_T^{2T}\int_0^1 \partial_{tt}\varepsilon^{(n)}\,\partial_t\sigma^{(n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_T^{2T}\int_0^1 \partial_t v^{(n)}\,\partial_t\bigl(sf\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.2.21}$$

Similarly, testing (2.2.16) by $-(2\pi j/T)v_{-jk}$ and (2.2.17) by $-(2\pi j/T)\sigma_{-jk}$ yields

$$\begin{aligned}
&\int_T^{2T}\int_0^1 \bigl|\partial_t v^{(n)}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_T^{2T}\int_0^1 \bigl(\partial_t\varepsilon^{(n)}\,\partial_t\sigma^{(n)} + s\,\partial_t v^{(n)} f\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.2.22}$$

By Hypothesis 2.1.1, we have the pointwise relations

$$\partial_t v^{(n)}\,\partial_t\bigl(sf\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr) \leqslant \bigl|\partial_t v^{(n)}\bigr|\bigl(\gamma_f\bigl|\partial_t\sigma^{(n)}\bigr| + |\beta_f|\bigr), \tag{2.2.23}$$

$$\begin{aligned}
&sf\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\,\partial_t v^{(n)} \\
&\quad = \partial_t\bigl(sF\bigl(\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr) - s(\partial_t F)\bigl(\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr) \\
&\qquad + s\bigl(f\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr) - f\bigl(\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr)\,\partial_t v^{(n)},
\end{aligned} \tag{2.2.24}$$

where we set $F(\sigma, v, x, t) = \int_0^v f(\sigma, v', x, t)\,\mathrm{d}v'$. We have $(\partial_t F)(\sigma, v, x, t) \leqslant |v||\beta_f|$ a.e. Combining (2.2.21)–(2.2.24) and using (3.4.10) and (3.2.16) we obtain that

$$\frac{1}{4}\kappa(p)\bigl\|\partial_t\sigma^{(n)}\bigr\|_3^3 \leqslant \bigl\|\partial_t v^{(n)}\bigr\|_2\bigl(\gamma_f\bigl\|\partial_t\sigma^{(n)}\bigr\|_2 + \|\beta_f\|_2\bigr), \tag{2.2.25}$$

$$\bigl\|\partial_t v^{(n)}\bigr\|_2^2 \leqslant h(p)\bigl\|\partial_t\sigma^{(n)}\bigr\|_2^2 + \gamma_f\bigl\|\sigma^{(n)}\bigr\|_2\bigl\|\partial_t v^{(n)}\bigr\|_2 + \|\beta_f\|_2\bigl\|v^{(n)}\bigr\|_2. \tag{2.2.26}$$

We now use the embedding inequalities

$$\begin{aligned}
&\bigl\|\partial_t\sigma^{(n)}\bigr\|_2 \leqslant T^{1/6}\bigl\|\partial_t\sigma^{(n)}\bigr\|_3, \\
&\bigl\|\sigma^{(n)}\bigr\|_2 \leqslant \frac{T}{2\pi}\bigl\|\partial_t\sigma^{(n)}\bigr\|_2, \\
&\bigl\|v^{(n)}\bigr\|_2 \leqslant \frac{T}{2\pi}\bigl\|\partial_t v^{(n)}\bigr\|_2
\end{aligned} \tag{2.2.27}$$

(note that both $\sigma^{(n)}(x,\cdot)$ and $v^{(n)}(x,\cdot)$ have zero average on $[0,T]$) and set

$$\begin{aligned}
x(p) &= \frac{1}{p}\sup\bigl\{\bigl\|\partial_t\sigma^{(n)}\bigr\|_3;\ \bigl\|\sigma^{(n)}\bigr\|_\infty \leqslant p\bigr\},\\
y(p) &= \frac{1}{p}\sup\bigl\{\bigl\|\partial_t v^{(n)}\bigr\|_2;\ \bigl\|\sigma^{(n)}\bigr\|_\infty \leqslant p\bigr\},\\
z(p) &= \frac{1}{p}\sup\bigl\{\bigl\|\partial_x\sigma^{(n)}\bigr\|_2;\ \bigl\|\sigma^{(n)}\bigr\|_\infty \leqslant p\bigr\},
\end{aligned} \tag{2.2.28}$$

where the supremum is taken over all possible solutions of (2.2.16)–(2.2.19) and over all $n\in\mathbb{N}$. From (2.2.25)–(2.2.28) we obtain

$$\frac{1}{4}p\kappa(p)x^3(p) \leqslant y(p)\left(T^{1/6}\gamma_f x(p) + \frac{\|\beta_f\|_2}{p}\right), \tag{2.2.29}$$

$$y^2(p) \leqslant T^{1/3}h(p)x^2(p) + \frac{T}{2\pi}y(p)\left(T^{1/6}\gamma_f x(p) + \frac{\|\beta_f\|_2}{p}\right), \tag{2.2.30}$$

and an elementary computation based on hypothesis (2.1.1) and (2.1.8) yields that

$$\limsup_{p\to\infty}\sqrt{h(p)}x(p) \leqslant 4T^{1/3}\gamma_f q, \qquad \limsup_{p\to\infty} y(p) \leqslant 4T^{1/2}\gamma_f q. \tag{2.2.31}$$

We estimate $z(p)$ using (2.2.16), which yields

$$\begin{aligned}
\bigl\|\partial_x\sigma^{(n)}\bigr\|_2 &\leqslant \bigl\|\partial_t v^{(n)}\bigr\|_2 + \bigl\|f^0\bigr\|_2 + \gamma_f\bigl(\bigl\|\sigma^{(n)}\bigr\|_2 + \bigl\|v^{(n)}\bigr\|_2 + \bigl\|\hat\sigma_s^{(n)}\bigr\|_2\bigr)\\
&\leqslant \left(1+\frac{T}{2\pi}\gamma_f\right)\bigl\|\partial_t v^{(n)}\bigr\|_2 + \frac{T}{2\pi}\gamma_f\bigl\|\partial_t\sigma^{(n)}\bigr\|_2 + \bigl\|f^0\bigr\|_2\\
&\quad + T^{1/2}\gamma_f\left(\int_0^1\bigl|\hat\sigma_s^{(n)}(x)\bigr|^2\,\mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{2.2.32}$$

To estimate the last term on the right-hand side of (2.2.32), we use (2.2.19) and obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\bigl|\hat\sigma_s^{(n)}(x)\bigr| &\leqslant \frac{s}{T}\int_T^{2T}\bigl|f\bigl(\sigma^{(n)}+\hat\sigma_s^{(n)}, v^{(n)}, x, t\bigr)\bigr|\,\mathrm{d}t\\
&\leqslant \gamma_f\bigl|\hat\sigma_s^{(n)}(x)\bigr| + \frac{1}{T}\int_T^{2T}\bigl|f^0(x,t)\bigr|\,\mathrm{d}t\\
&\quad + \frac{\gamma_f}{T}\int_T^{2T}\bigl(\bigl|\sigma^{(n)}\bigr| + \bigl|v^{(n)}\bigr|\bigr)(x,t)\,\mathrm{d}t,
\end{aligned} \tag{2.2.33}$$

hence

$$\max_{x\in[0,1]}\bigl|\hat\sigma_s^{(n)}(x)\bigr| \leqslant \mathrm{e}^{\gamma_f}\bigl(|\rho| + T^{-1/2}\bigl(\bigl\|f^0\bigr\|_2 + \gamma_f\bigl(\bigl\|\sigma^{(n)}\bigr\|_2 + \bigl\|v^{(n)}\bigr\|_2\bigr)\bigr)\bigr). \tag{2.2.34}$$

Combining (2.2.32) with (2.2.34) yields

$$\begin{aligned}\left\|\partial_x\sigma^{(n)}\right\|_2 \leqslant{}& T^{1/2}\gamma_f \mathrm{e}^{\gamma_f}|\rho| + \left(1+\gamma_f\mathrm{e}^{\gamma_f}\right)\left\|f^0\right\|_2\\ &+\left(1+\frac{T}{2\pi}\gamma_f\left(1+\gamma_f\mathrm{e}^{\gamma_f}\right)\right)\left\|\partial_t v^{(n)}\right\|_2\\ &+\gamma_f\frac{T^{7/6}}{2\pi}\left(\mathrm{e}^{\gamma_f}+1\right)\left\|\partial_t\sigma^{(n)}\right\|_3,\end{aligned}\tag{2.2.35}$$

hence in view of (2.2.31),

$$\limsup_{p\to\infty} z(p) \leqslant 4T^{1/2}\gamma_f q\left(1+\frac{T}{2\pi}\gamma_f\left(1+\gamma_f\mathrm{e}^{\gamma_f}\right)\right).\tag{2.2.36}$$

By virtue of (2.1.2), (2.1.8), (2.2.31) and (2.2.36), we may choose $p_\infty > 0$ such that

$$M\left(x(p)+z(p)\right) < 1 \quad \text{for } p \geqslant p_\infty.\tag{2.2.37}$$

In other words, from (2.0.21), (2.2.28) and (2.2.37), it follows that whenever we have a solution of (2.2.16)–(2.2.19), then the implication

$$\left(p\geqslant p_\infty, \left\|\sigma^{(n)}\right\|_\infty \leqslant p\right) \quad\Longrightarrow\quad \left\|\sigma^{(n)}\right\|_\infty < p\tag{2.2.38}$$

holds, hence $\|\sigma^{(n)}\|_\infty < p_\infty$ independently of $n\in\mathbb{N}$. From (2.2.17), (3.2.16) and (2.2.31), we further obtain that $\|\partial_x v^{(n)}\|_3 \leqslant h(p_\infty)\|\partial_t\sigma^{(n)}\|_3 \leqslant p_\infty h(p_\infty)x(p_\infty)$, $\|\partial_t v^{(n)}\|_2 \leqslant p_\infty y(p_\infty)$, hence also $\|v^{(n)}\|_\infty < M' p_\infty(h(p_\infty)x(p_\infty)+y(p_\infty))$ as a consequence of (2.0.21). We thus proved conjecture (2.2.20) which implies that (2.2.16)–(2.2.19) has a solution for every $n\in\mathbb{N}$. Moreover, we have found a bound independent of n for $\sigma^{(n)}$ in $H_T^{2,3}$ and for $v^{(n)}$ in $H_T^{3,2}$. Using the compact embedding (2.0.20), we may find a subsequence (still indexed by n) and some elements $\sigma^*\in H_T^{2,3}$ and $v\in H_T^{3,2}$ such that $\int_T^{2T} v(x,t)\,\mathrm{d}t = \int_T^{2T}\sigma^*(x,t)\,\mathrm{d}t = 0$ a.e., and

$$v^{(n)}\to v, \qquad \sigma^{(n)}\to\sigma^* \qquad \text{uniformly,}\tag{2.2.39}$$

$$\partial_t v^{(n)}\to\partial_t v, \qquad \partial_x\sigma^{(n)}\to\partial_x\sigma^* \quad \text{weakly in } L_T^2,\tag{2.2.40}$$

$$\partial_x v^{(n)}\to\partial_x v, \qquad \partial_t\sigma^{(n)}\to\partial_t\sigma^* \quad \text{weakly in } L_T^3.\tag{2.2.41}$$

We can pass to the limit as $n\to\infty$ in (2.2.11)–(2.2.14) and find $\varepsilon^*\in\mathcal{C}_T^0$, $\hat\sigma\in W^{1,2}(0,1)$ such that

$$\varepsilon^*(x,t) = \mathcal{F}\left[\lambda,\sigma^*\right](x,t),\tag{2.2.42}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat\sigma(x) = -\frac{1}{T}\int_T^{2T} f\left(\sigma^*+\hat\sigma, v, x,\tau\right)\mathrm{d}\tau, \quad \hat\sigma(0)=\rho,\tag{2.2.43}$$

$$\int_T^{2T}\int_0^1 \big(\partial_t v - \partial_x \sigma^* - f\big(\sigma^* + \hat\sigma, v, x, t\big)\big)\vartheta(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.2.44}$$

$$\int_T^{2T}\int_0^1 \big(\partial_t \varepsilon^* - \partial_x v\big)\vartheta(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.2.45}$$

for every test function $\vartheta \in L^2_T$ such that $\int_T^{2T} \vartheta(x,t)\,\mathrm{d}t = 0$ a.e. We now obtain (2.2.1)–(2.2.2) (and thus complete the proof of Theorem 2.1.2) by putting $\vartheta = \partial_t v - \partial_x \sigma^* - f(\sigma^* + \hat\sigma, v, x, t) + (1/T)\int_T^{2T} f(\sigma^* + \hat\sigma, v, x, \tau)\,\mathrm{d}\tau$ in (2.2.44), and $\vartheta = \partial_t \varepsilon^* - \partial_x v$ in (2.2.45).

2.3. *Uniqueness*

In this subsection, we prove Theorem 2.1.3. Since the nonlinearity f is now independent of σ, the counterpart of (2.2.1)–(2.2.5) reads

$$\partial_t v = \partial_x \sigma^* + f(v, x, t) - \frac{1}{T}\int_T^{2T} f(v, x, \tau)\,\mathrm{d}\tau, \tag{2.3.1}$$

$$\partial_t \varepsilon^* = \partial_x v, \tag{2.3.2}$$

$$\varepsilon^* = \mathcal{F}\big[\lambda, \sigma^*\big], \tag{2.3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat\sigma(x) = -\frac{1}{T}\int_T^{2T} f(v, x, \tau)\,\mathrm{d}\tau, \quad \hat\sigma(0) = \rho, \tag{2.3.4}$$

together with the T-periodicity condition and

$$v(0,t) = v(1,t) = 0, \qquad \int_T^{2T} v(x,t)\,\mathrm{d}t = \int_T^{2T} \sigma^*(x,t)\,\mathrm{d}t = 0. \tag{2.3.5}$$

We will not repeat all details of the existence proof which exactly follows the lines of the proof of Theorem 2.1.2. Estimates analogous to (2.2.25)–(2.2.26) for the system (2.3.1)–(2.3.5) have the form

$$\frac{1}{4}\kappa(p)\big\|\partial_t \sigma^{(n)}\big\|_3^3 \leqslant \|\beta_f\|_2\big\|\partial_t v^{(n)}\big\|_2, \tag{2.3.6}$$

$$\big\|\partial_t v^{(n)}\big\|_2^2 \leqslant h(p)\big\|\partial_t \sigma^{(n)}\big\|_2^2 + \|\beta_f\|_2\big\|v^{(n)}\big\|_2 \tag{2.3.7}$$

which, with the notation of (2.2.28), yields similarly as in (2.2.31) that

$$\begin{aligned} &\limsup_{p\to\infty} \sqrt{h(p)}\,x(p) \leqslant 2T^{1/12}\tilde q\sqrt{\|\beta_f\|_2}, \\ &\limsup_{p\to\infty} y(p) \leqslant 2T^{1/4}\tilde q\sqrt{\|\beta_f\|_2}. \end{aligned} \tag{2.3.8}$$

Instead of (2.2.32) we directly have

$$\|\partial_x \sigma^{(n)}\|_2 \leqslant \left(1 + \frac{T}{2\pi}\gamma_f\right)\|\partial_t v^{(n)}\|_2 + \|f^0\|_2, \tag{2.3.9}$$

and the rest of the existence argument is identical to the one in the previous subsection.

To prove the uniqueness, we consider two solutions (v_1, σ_1^*), (v_2, σ_2^*) of (2.3.1)–(2.3.5) (with $\varepsilon_i^*, \hat{\sigma}_i$, $i = 1, 2$, having the corresponding meaning) associated with two different values ρ_1, ρ_2 of ρ in (2.3.4). Set $\bar{v} = v_1 - v_2$, $\bar{\sigma} = \sigma_1^* - \sigma_2^*$, $\bar{\varepsilon} = \varepsilon_1^* - \varepsilon_2^*$. As f is nonincreasing in v, we obtain from (2.3.1)–(2.3.2) that

$$\int_T^{2T}\int_0^1 \partial_t \bar{\varepsilon}\,\bar{\sigma}\,\mathrm{d}x\,\mathrm{d}t = \int_T^{2T}\int_0^1 \bar{v}\big(f(v_1, x, t) - f(v_2, x, t)\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0. \tag{2.3.10}$$

By Proposition 3.3.2, there exists $\sigma^0 \in W^{1,2}(0, 1)$ such that $\bar{\sigma}(x, t) = \sigma^0(x)$ for $t \geqslant T$. In view of (2.3.5), we have $\sigma^0 \equiv 0$, hence $\sigma_1^* = \sigma_2^*$, consequently also $\varepsilon_1^* = \varepsilon_2^*$ and $v_1 = v_2$. We thus have

$$\hat{\sigma}_1(x) = \hat{\sigma}_2(x) = \rho_1 - \rho_2 \tag{2.3.11}$$

for all $x \in [0, 1]$, and the uniqueness follows.

REMARK 2.3.1. The ambiguity due to the arbitrary choice of ρ in (2.3.4) can be removed by considering the Dirichlet boundary conditions in displacements instead of velocities. More specifically, we denote by $(v, \sigma^*, \hat{\sigma}_0)$ the solution of (2.3.1)–(2.3.5) corresponding to $\rho = 0$. We know from (2.3.11) that $(v, \sigma^*, \hat{\sigma}_0 + \rho)$ is then the solution to (2.3.1)–(2.3.5) for any ρ. For $(x, t) \in [0, 1] \times [T, \infty[$ and $\rho \in \mathbb{R}$ set

$$\begin{aligned} \varepsilon^{(\rho)}(x, t) &= \mathcal{F}\big[\lambda, \sigma^* + \hat{\sigma}_0 + \rho\big](x, t), \\ u^{(\rho)}(x, t) &= \int_T^t v\big(x, t'\big)\,\mathrm{d}t' + \int_0^x \varepsilon^{(\rho)}\big(x', T\big)\,\mathrm{d}x'. \end{aligned} \tag{2.3.12}$$

We then have $\partial_t u^{(\rho)} = v$, $\partial_x u^{(\rho)} = \varepsilon^{(\rho)}$, $u^{(\rho)}(x, t + T) = u^{(\rho)}(x, t)$ for all $(x, t) \in [0, 1] \times [T, \infty[$, and $u^{(\rho)}(1, t) = \int_0^1 \varepsilon^{(\rho)}(x, T)\,\mathrm{d}x$. We claim that

$$\exists! \rho \in \mathbb{R}, \quad u^{(\rho)}(1, t) = 0 \quad \forall t \geqslant T. \tag{2.3.13}$$

This conjecture follows from the fact that for $\rho_1 > \rho_2$ we have by (3.3.6) and (2.3.11) that $\varepsilon^{(\rho_1)}(x, t) - \varepsilon^{(\rho_2)}(x, t) \geqslant h(0)(\rho_1 - \rho_2)$, and that $\varepsilon^{(\rho)}$ depends continuously on ρ.

2.4. *Asymptotic stability*

This subsection is devoted to the proof of Theorem 2.1.6. For $\lambda_1, \lambda_2 \in C([0,1]; \Lambda_{\underline{K}})$, $(v_1, \sigma_1), (v_2, \sigma_2) \in B$, we define the functional

$$
\begin{aligned}
&V(\lambda_1, \lambda_2, v_1, v_2, \sigma_1, \sigma_2)(t)\\
&\quad = \int_0^1 \big(h(0)(\sigma_1 - \sigma_2)^2 + (v_1 - v_2)^2\big)\,\mathrm{d}x\\
&\qquad + \int_0^1 \int_0^\infty \big(\mathfrak{p}_r[\lambda_1, \sigma_1] - \mathfrak{p}_r[\lambda_2, \sigma_2]\big)^2\,\mathrm{d}h(r)\,\mathrm{d}x.
\end{aligned} \tag{2.4.1}
$$

Using (3.3.2) we check that whenever (v_i, σ_i) for $i = 1, 2$ are solutions of (1.2.1) with the respective choice of $\lambda = \lambda_i$, then

$$
\frac{\mathrm{d}}{\mathrm{d}t} V(\lambda_1, \lambda_2, v_1, v_2, \sigma_1, \sigma_2)(t) \leqslant 0 \quad \text{a.e.} \tag{2.4.2}
$$

For $n \in \mathbb{N}$ and $x \in [0,1]$, $t \geqslant 0$, $r \geqslant 0$, we define the sequences

$$
\begin{aligned}
&v_n(x,t) = \underline{v}(x, t + nT),\\
&\sigma_n(x,t) = \underline{\sigma}(x, t + nT),\\
&\lambda_n(x,r) = \mathfrak{p}_r\big[\underline{\lambda}(\cdot,\cdot), \underline{\sigma}(\cdot,\cdot)\big](nT).
\end{aligned} \tag{2.4.3}
$$

By Lemma 3.1.2 and Proposition 3.1.1, we have $\lambda_n \in C([0,1]; \Lambda_{\underline{K}})$ for all n, and putting $\varepsilon_n(x,t) = \mathcal{F}[\,\underline{\lambda}, \underline{\sigma}\,](x, t + nT)$ we obtain, for all $x \in [0,1]$ and $t \geqslant 0$, that

$$
\varepsilon_n(x,t) = \mathcal{F}[\lambda_n, \sigma_n](x,t). \tag{2.4.4}
$$

The sequence $\{(v_n, \sigma_n); n \in \mathbb{N}\}$ is equibounded in B; there exists therefore a subsequence $\{n_k\}$ in $\mathbb{N}$ and an element $(v, \sigma) \in B$ such that

$$
\begin{aligned}
&(\partial_t v_{n_k}, \partial_x v_{n_k}, \partial_t \sigma_{n_k}, \partial_x \sigma_{n_k}) \to (\partial_t v, \partial_x v, \partial_t \sigma, \partial_x \sigma)\\
&\quad \text{weakly-star in } L^\infty\big(0,\infty; L^2(0,1)\big),\\
&(v_{n_k}, \sigma_{n_k}) \to (v, \sigma)\\
&\quad \text{locally uniformly in } [0,1] \times [0,\infty[.
\end{aligned} \tag{2.4.5}
$$

From (3.1.14) it follows that $\{\lambda_n\}$ is an equibounded and equicontinuous sequence in $C([0,1]; \Lambda_{\underline{K}})$. Since $\Lambda_{\underline{K}}$ is a compact subset of $C[0, \underline{K}\,]$, we may use the Arzelà–Ascoli theorem and assume that the subsequence $\{n_k\}$ is such that

$$
\lambda_{n_k} \to \lambda \in C\big([0,1]; \Lambda_{\underline{K}}\big) \quad \text{uniformly in } [0,1] \times [0, \underline{K}\,]. \tag{2.4.6}
$$

All elements (v_n, σ_n) are solutions to (1.2.1) with ε_n given by (2.4.4). Passing to the limit as $n_k \to \infty$ we conclude that (v, σ) is a solution to (1.2.1). For all $k \in \mathbb{N}$ we have by (2.4.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(t) \leqslant 0 \quad \text{a.e.,} \tag{2.4.7}$$

hence

$$\sup_{t \geqslant 0} V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(t) \leqslant V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(0). \tag{2.4.8}$$

The right-hand side of (2.4.8) tends to 0 as $k \to \infty$, and we conclude that

$$\lim_{k \to \infty} \sup_{t \geqslant 0} \int_0^1 \big(|v_{n_k} - v|^2 + |\sigma_{n_k} - \sigma|^2\big)(x, t)\, \mathrm{d}x = 0. \tag{2.4.9}$$

We now prove that both v and σ are T-periodic. Put $v_+(x, t) = v(x, t + T)$, $\sigma_+(x, t) = \sigma(x, t + T)$, $\lambda_+(x, r) = \mathfrak{p}_r[\lambda(\cdot, \cdot), \sigma(\cdot, \cdot)](T)$, and

$$\beta = \lim_{t \to \infty} V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma)(t) \geqslant 0. \tag{2.4.10}$$

For all $t \geqslant 0$ we have

$$\begin{aligned} \beta &= \lim_{k \to \infty} V(\lambda_{n_k+1}, \lambda_{n_k}, v_{n_k+1}, v_{n_k}, \sigma_{n_k+1}, \sigma_{n_k})(t) \\ &= V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma)(t), \end{aligned} \tag{2.4.11}$$

hence $\frac{\mathrm{d}}{\mathrm{d}t} V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma) = 0$ a.e. in $[0, \infty[$. By construction we have $\lambda_+(x, 0) = \sigma_+(x, 0)$, $\lambda(x, 0) = \sigma(x, 0)$ for all $x \in [0, 1]$. From Proposition 3.3.2 it follows that there exists a function $R(x, t)$ such that $R(x, \cdot)$ is nondecreasing for every x and $\sigma_+(x, t) - \sigma(x, t) = \lambda_+(x, R(x, t)) - \lambda(x, R(x, t))$. For every x there exists therefore the limit $\sigma_\infty(x) = \lim_{t \to \infty}(\sigma_+(x, t) - \sigma(x, t)) = \lim_{t \to \infty}(\sigma(x, t + T) - \sigma(x, t))$. Since σ is bounded, we have

$$\lim_{t \to \infty} \big(\sigma_+(x, t) - \sigma(x, t)\big) = 0 \quad \forall x \in [0, 1]. \tag{2.4.12}$$

Using again Proposition 3.3.2, we similarly obtain

$$\lim_{t \to \infty} \big(\mathfrak{p}_r[\lambda_+, \sigma_+](x, t) - \mathfrak{p}_r[\lambda, \sigma](x, t)\big) = 0 \quad \forall x \in [0, 1], \forall r > 0. \tag{2.4.13}$$

Let $\delta > 0$ be arbitrarily given. We fix some $m \in \mathbb{N}$ and $t_m > 0$ such that for all $t \geqslant t_m$ and all $j = 1, \ldots, m$, we have

$$\left|\sigma\left(\frac{j}{m}, t + T\right) - \sigma\left(\frac{j}{m}, t\right)\right| < \frac{\delta}{2}. \tag{2.4.14}$$

For each $y \in [(j-1)/m, j/m]$ and $t \geqslant 0$, we have

$$\left|\sigma(y,t) - \sigma\left(\frac{j}{m}, t\right)\right| \leqslant \frac{1}{\sqrt{m}}\left(\int_{(j-1)/m}^{j/m} \left|\partial_x \sigma(x,t)\right|^2 \mathrm{d}x\right)^{1/2} \leqslant \frac{C}{\sqrt{m}} \tag{2.4.15}$$

with some constant C independent of t and m. We thus can find $t^* > 0$ such that

$$\left|\sigma(\cdot, t+T) - \sigma(\cdot, t)\right|_{L^\infty(0,1)} < \delta \quad \text{for } t \geqslant t^*. \tag{2.4.16}$$

Let $\ell \in \mathbb{N}$ be such that, by virtue of (2.4.9), we have

$$\left|\int_0^1 \left|\sigma(x,\cdot) - \sigma_{n_k}(x,\cdot)\right|^2 \mathrm{d}x\right|_{L^\infty(0,\infty)} < \delta^2 \quad \text{for } k \geqslant \ell. \tag{2.4.17}$$

Put $t^{**} = t^* + n_\ell T$. For $s \geqslant T^{**}$, we have $s - n_\ell T \geqslant t^*$, hence

$$\begin{aligned}\left|\underline{\sigma}(\cdot, s+T) - \underline{\sigma}(\cdot, s)\right|_{L^2(0,1)} &\leqslant \left|\underline{\sigma}(\cdot, s+T) - \sigma(\cdot, s - n_\ell T + T)\right|_{L^2(0,1)} \\ &\quad + \left|\sigma(\cdot, s - n_\ell T + T) - \sigma(\cdot, s - n_\ell T)\right|_{L^2(0,1)} \\ &\quad + \left|\sigma(\cdot, s - n_\ell T) - \underline{\sigma}(\cdot, s)\right|_{L^2(0,1)} \\ &\leqslant 3\delta. \end{aligned} \tag{2.4.18}$$

Let now $t \geqslant 0$ be arbitrary. We fix $k \geqslant \ell$ such that $t + n_k T \geqslant t^{**}$. Then

$$\begin{aligned}\left|\sigma(\cdot, t+T) - \sigma(\cdot, t)\right|_{L^2(0,1)} &\leqslant \left|\sigma(\cdot, t+T) - \sigma_{n_k}(\cdot, t+T)\right|_{L^2(0,1)} \\ &\quad + \left|\underline{\sigma}(\cdot, t + n_k T + T) - \underline{\sigma}(\cdot, t + n_k T)\right|_{L^2(0,1)} \\ &\quad + \left|\sigma(\cdot, t) - \sigma_{n_k}(\cdot, t)\right|_{L^2(0,1)} \\ &\leqslant 5\delta. \end{aligned} \tag{2.4.19}$$

Since $\delta > 0$ was arbitrary, we obtain $\sigma(x, t+T) = \sigma(x,t)$ for all x and t, and from the fact that (v, σ) is a solution to (1.2.1) we obtain also $v(x, t+T) = v(x,t)$. Using (2.4.11) and (2.4.13) we obtain $\beta = 0$ and $\lambda_+ = \lambda$.

To conclude the proof, consider the sequence

$$d_n := V(\lambda_n, \lambda, v_n, v, \sigma_n, \sigma)(0). \tag{2.4.20}$$

By (2.4.2), we have $d_{n+1} \leqslant d_n$ for all $n \in \mathbb{N}$. As $d_{n_k} \to 0$, the whole sequence $\{d_n\}$ converges to 0 and using (2.4.2) again, we obtain

$$\lim_{n\to\infty} \sup_{t\geqslant 0} \int_0^1 \left(|v_n - v|^2 + |\sigma_n - \sigma|^2\right)(x,t)\,\mathrm{d}x = 0. \tag{2.4.21}$$

Combining (2.4.21) with the elementary interpolation inequality

$$|w|_{L^\infty(0,1)} \leqslant |w|_{L^2(0,1)} + 2|w|^{1/2}_{L^2(0,1)}|\partial_x w|^{1/2}_{L^2(0,1)},$$

we see that the whole sequence $\{(v_n, \sigma_n)\}$ converges uniformly to (v, σ) in $[0,1] \times [0, \infty[$. It remains to prove that (2.1.13) holds. To this end, we consider again any $\delta > 0$ and find n_0 such that $|v_n(x,t) - v(x,t)| + |\sigma_n(x,t) - \sigma(x,t)| < \delta$ for all (x,t) and all $n \geqslant n_0$. For $t \geqslant n_0 T$ we find $n \geqslant n_0$ such that $t - nT \in [0, T]$. Then

$$\begin{aligned}
&\left|\underline{v}(\cdot,t) - v(\cdot,t)\right|_{L^\infty(0,1)} + \left|\underline{\sigma}(\cdot,t) - \sigma(\cdot,t)\right|_{L^\infty(0,1)} \\
&\quad = \left|v_n(\cdot, t-nT) - v(\cdot, t-nT)\right|_{L^\infty(0,1)} \\
&\qquad + \left|\sigma_n(\cdot, t-nT) - \sigma(\cdot, t-nT)\right|_{L^\infty(0,1)} < \delta,
\end{aligned} \tag{2.4.22}$$

and Theorem 2.1.6 is proved.

3. Hysteresis operators

The first axiomatic approach to hysteresis was proposed by Madelung in [19], and a basic mathematical theory of hysteresis operators has been developed by Krasnosel'skii and his collaborators. The results of this group are summarized in the monograph [10] which constitutes until now the main source of reference on hysteresis. Our presentation here is based on more recent results from [14] which are needed here, in particular the energy inequalities in Section 3.4. The so-called *play operator* introduced in [10] is the main building block of the theory.

3.1. *The play operator*

For our purposes, it is convenient to work in the space $G_R(\mathbb{R}_+)$ of right-continuous regulated functions of time $t \in \mathbb{R}_+$, that is, functions $w : \mathbb{R}_+ \to \mathbb{R}$ which admit the left limit $w(t-)$ at each point $t > 0$, and the right limit $w(t+)$ exists and coincides with $w(t)$ for each $t \geqslant 0$. More information about regulated functions can be found, e.g., in [1,2,7, 16,24].

We endow the space $G_R(\mathbb{R}_+)$ with the system of seminorms

$$\|w\|_{[0,t]} = \sup\left\{\left|w(\tau)\right|; \tau \in [0,t]\right\} \quad \text{for } w \in G_R(\mathbb{R}_+) \text{ and } t \in \mathbb{R}_+. \tag{3.1.1}$$

With the metric

$$\Delta(u,v) = \sup_{T>0} \frac{\|u-v\|_{[0,T]}}{1 + \|u-v\|_{[0,T]}} \quad \text{for } u, v \in G_R(\mathbb{R}_+), \tag{3.1.2}$$

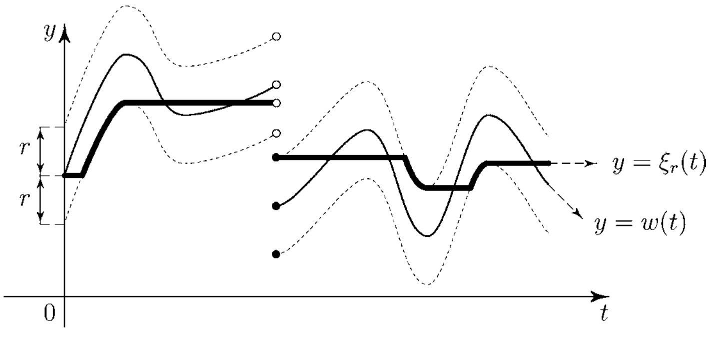

Fig. 1. Optimal *BV*-approximation.

the set $G_R(\mathbb{R}_+)$ becomes a Fréchet space. Similarly, $BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ will denote the space of right-continuous functions of bounded variation on each interval $[0, T]$ for any $T > 0$ and $C(\mathbb{R}_+)$ is the space of continuous functions on $\mathbb{R}_+$. We have $BV_R^{\mathrm{loc}}(\mathbb{R}_+) \subset G_R(\mathbb{R}_+)$ and the embedding is dense, while $C(\mathbb{R}_+)$ is a closed subspace of $G_R(\mathbb{R}_+)$.

The uniform approximation problem for real-valued regulated functions by functions of bounded variation has actually an interesting solution. For each $w \in G_R(\mathbb{R}_+)$, a parameter $r > 0$ and an initial condition $\xi_r^0 \in [w(0) - r, w(0) + r]$, there exists a unique $\xi_r \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ in the r-neighborhood of w with minimal total variation, that is (see Figure 1 for $\xi_r^0 = w(0)$),

$$\left|w(t) - \xi_r(t)\right| \leqslant r \quad \forall t \geqslant 0, \tag{3.1.3}$$

$$\xi_r(0) = \xi_r^0, \tag{3.1.4}$$

$$\operatorname*{Var}_{[0,t]} \xi_r = \min\left\{\operatorname*{Var}_{[0,t]} \eta;\ \eta \in BV_R^{\mathrm{loc}}(\mathbb{R}_+), = \xi_r^0,\ \|w - \eta\|_{[0,t]} \leqslant r\right\} \quad \forall t > 0. \tag{3.1.5}$$

This result goes back to A. Vladimirov and V. Chernorutskii for the case of continuous functions w; for a proof see [23]. An extension to $L^\infty(\mathbb{R}_+)$ has been done in [17]. The function ξ_r can also be characterized as the unique solution of the variational inequality

$$\left|w(t) - \xi_r(t)\right| \leqslant r \quad \forall t \geqslant 0, \tag{3.1.6}$$

$$\xi_r(0) = \xi_r^0, \tag{3.1.7}$$

$$\int_0^t \big(w(\tau) - \xi_r(\tau) - y(\tau)\big)\, \mathrm{d}\xi_r(\tau) \geqslant 0 \quad \forall t \geqslant 0, \forall y \in G_R(\mathbb{R}_+),\ \|y\|_{[0,t]} \leqslant r, \tag{3.1.8}$$

where the integration in (3.1.8) is understood in the Young or Kurzweil sense, see [16,17]. If moreover, w is continuous, then ξ_r is continuous, we can restrict ourselves to continuous test functions y, and (3.1.8) can be interpreted as the usual Stieltjes integral.

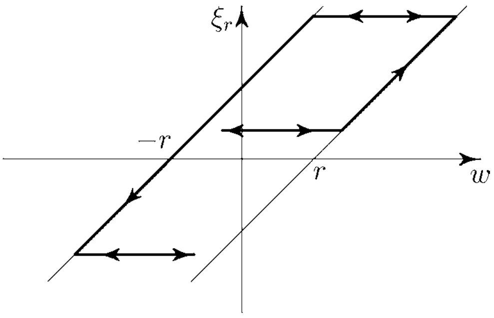

Fig. 2. A diagram of the play.

Let $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ denote the space of absolutely continuous functions on $\mathbb{R}_+$. It is an easy exercise to show that if $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then the solution ξ_r to (3.1.6)–(3.1.8) belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and fulfills the variational inequality

$$\dot{\xi}_r(t)\big(w(t) - \xi_r(t) - y\big) \geqslant 0 \quad \text{a.e. } \forall y \in [-r, r]. \tag{3.1.9}$$

Let us consider the mapping $\hat{\mathfrak{p}}_r : \mathbb{R} \times G_R(\mathbb{R}_+) \to BV^{\mathrm{loc}}_R(\mathbb{R}_+)$ which with each $\hat{\xi}^0_r \in \mathbb{R}$ and $w \in G_R(\mathbb{R}_+)$ associates the solution ξ_r of (3.1.6)–(3.1.8) with

$$\xi^0_r = \max\big\{w(0) - r, \min\big\{w(0) + r, \hat{\xi}^0_r\big\}\big\}. \tag{3.1.10}$$

Then $\hat{\mathfrak{p}}_r$ is a hysteresis operator called the *play*, and alternative equivalent definitions of the play can be found in [3,10,25]. Figure 2 shows a typical w–ξ_r diagram. The horizontal parts of the graph are reversible, motions along the lines $\xi_r = w \pm r$ are irreversible.

More complex hysteresis behavior can be modelled by considering the whole family $\{\xi_r\}_{r>0}$ corresponding to a given $w \in G_R(\mathbb{R}_+)$. In fact, [3], Theorem 2.7.7 shows that a very large class of hysteresis operators admits a representation by means the one-parametric play system which accounts for the *hysteresis memory* and the parameter r plays the role of *memory variable*. We introduce the *hysteresis state space*

$$\Lambda = \Big\{\lambda : \mathbb{R}_+ \to \mathbb{R};\ \big|\lambda(r) - \lambda(s)\big| \leqslant |r - s|\ \forall r, s \in \mathbb{R}_+,\ \lim_{r\to+\infty} \lambda(r) = 0\Big\}, \tag{3.1.11}$$

and choose the initial condition $\{\hat{\xi}^0_r\}_{r>0}$ in the form

$$\hat{\xi}^0_r = \lambda(r) \quad \text{for } r > 0, \tag{3.1.12}$$

where $\lambda \in \Lambda$ is given. We define the operators $\mathfrak{p}_r : \Lambda \times G_R(\mathbb{R}_+) \to BV^{\mathrm{loc}}_R(\mathbb{R}_+)$ for $r > 0$ by the formula

$$\mathfrak{p}_r[\lambda, w] = \hat{\mathfrak{p}}_r\big[\lambda(r), w\big] \tag{3.1.13}$$

for $\lambda \in \Lambda$ and $w \in G_R(\mathbb{R}_+)$. Consistently with the definition, we set $\mathfrak{p}_0[\lambda, w](t) = w(t)$ for all $t \geqslant 0$.

The following result was proved in [14,17].

PROPOSITION 3.1.1. *For every $\lambda \in \Lambda$, $w \in G_R(\mathbb{R}_+)$ and $t \geqslant 0$, the mapping $r \mapsto \lambda_t(r) = \mathfrak{p}_r[\lambda, w](t)$ belongs to Λ, and for all $\lambda_1, \lambda_2 \in \Lambda$, $w_1, w_2 \in G_R(\mathbb{R}_+)$ and $t \geqslant 0$, we have*

$$\begin{aligned} &\left|\mathfrak{p}_r[\lambda_1, w_1](t) - \mathfrak{p}_r[\lambda_2, w_2](t)\right| \\ &\quad \leqslant \max\left\{\left|\lambda_1(r) - \lambda_2(r)\right|, \|w_1 - w_2\|_{[0,t]}\right\} \quad \forall r \geqslant 0. \end{aligned} \tag{3.1.14}$$

The play operator thus generates for every $t \geqslant 0$ a continuous *state mapping* $\Pi_t : \Lambda \times G_R(\mathbb{R}_+) \to \Lambda$ which with each $(\lambda, w) \in \Lambda \times G_R(\mathbb{R}_+)$ associates the state $\lambda_t \in \Lambda$ at time t.

In order to study further properties of the play, we first derive an explicit formula for $\mathfrak{p}_r[\lambda, w]$ if w is a step function of the form

$$w(t) = \sum_{k=1}^{m} w_{k-1} \chi_{[t_{k-1}, t_k[}(t) \quad \text{for } t \geqslant 0 \tag{3.1.15}$$

with some given $w_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$, where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = +\infty$ is a given sequence and χ_A for $A \subset \mathbb{R}$ is the characteristic function of the set A, that is, $\chi_A(t) = 1$ for $t \in A$, $\chi_A(t) = 0$ otherwise. We define analogously to (3.1.10) for $\lambda \in \Lambda$ and $v \in \mathbb{R}$ the function $P[\lambda, v] : \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$P[\lambda, v](r) = \max\left\{v - r, \min\left\{v + r, \lambda(r)\right\}\right\}, \tag{3.1.16}$$

see Figure 3. In particular, P can be considered as a mapping from $\Lambda \times \mathbb{R}$ to Λ. One can directly check as a one-dimensional counterpart of [16], Proposition 4.3, using the Young or Kurzweil integral calculus and the inequality

$$\left(P[\lambda, v](r) - \lambda(r)\right)\left(v - P[\lambda, v](r) - z\right) \geqslant 0 \quad \forall |z| \leqslant r \tag{3.1.17}$$

that we have

$$\xi_r(t) = \sum_{k=1}^{m} \xi_{k-1}^{(r)} \chi_{[t_{k-1}, t_k[}(t) \quad \text{for } t \geqslant 0, \tag{3.1.18}$$

with

$$\xi_k^{(r)} = \lambda_k(r), \quad \lambda_k = P[\lambda_{k-1}, w_k], \lambda_{-1} = \lambda, \tag{3.1.19}$$

for $k = 0, \ldots, m-1$, see Figure 3.

Every function $w \in G_R(\mathbb{R}_+)$ can be approximated uniformly on every compact interval by step functions of the form (3.1.15). Proposition 3.1.1 enables us to extend (3.1.19)

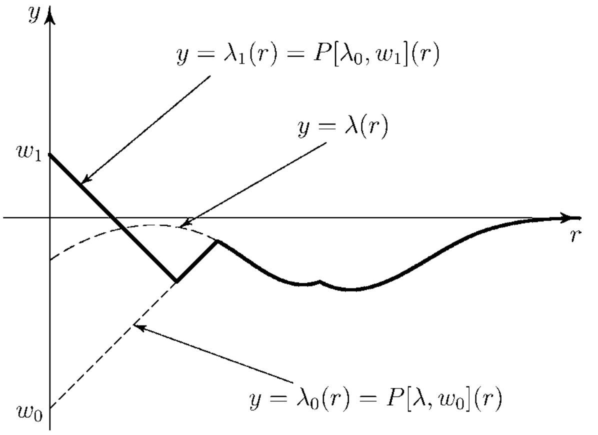

Fig. 3. Distribution of play operators in consecutive times.

to the whole space $G_R(\mathbb{R}_+)$ and obtain for a function $w \in G_R(\mathbb{R}_+)$ which is monotone (nondecreasing or nonincreasing) in an interval $[t_0, t_1]$ the representation formula

$$\begin{aligned}\mathfrak{p}_r[\lambda, w](t) &= P\big[\lambda_{t_0}, w(t)\big](r)\\ &= \max\big\{w(t) - r, \min\big\{w(t) + r, \lambda_{t_0}(r)\big\}\big\}\end{aligned} \tag{3.1.20}$$

for $t \in [t_0, t_1]$, see Figure 2. It is perhaps interesting to note that (3.1.20) has originally been used in [10] as alternative definition of the play on continuous piecewise monotone inputs, extended afterwards by density and continuity to the whole space of continuous functions.

More generally, the play possesses the *semigroup property* as a time-continuous version of (3.1.19), namely

$$\mathfrak{p}_r[\lambda, w](t + s) = \mathfrak{p}_r\big[\lambda_s, w(s + \cdot)\big](t) \tag{3.1.21}$$

for all $w \in G_R(\mathbb{R}_+)$, $\lambda \in \Lambda$ and $s, t \geqslant 0$.

The choice (3.1.11) of the state space is justified by the fact that it consists of elements which are *asymptotically reachable from the reference initial state* $\lambda \equiv 0$, that is,

$$\begin{aligned}&\forall \lambda \in \Lambda, \exists w \in G_R(\mathbb{R}_+), \forall \varepsilon > 0, \exists T > 0,\\ &\quad \sup_{r>0}\big|\lambda(r) - \mathfrak{p}_r[0, w](T)\big| < \varepsilon.\end{aligned} \tag{3.1.22}$$

Instead of a formal proof of this statement, we rather illustrate the construction of w on Figure 4. We set for instance $T_k = 2k$ for $k = 0, 1, 2, \ldots$ and fix a sequence $\varepsilon_k \to 0$ as $k \to \infty$. The function w will be defined as a step function successively in $[T_k, T_{k+1}]$ with a maximum absolute value at $T_k + 1$ and with jumps of decreasing amplitude at points $T_k + 1 < t_1 < t_2 < \cdots < T_{m_k} < T_{k+1}$. The graph of the function $\lambda_k(r) = \mathfrak{p}_r[0, w](T_{k+1})$ is piecewise affine with alternating slopes $+1$ and -1 for $0 \leqslant r \leqslant |w(T_k + 1)|$, and is chosen so as $|\lambda(r) - \lambda_k(r)| < \varepsilon_k$ for $k = 0, 1, 2, \ldots$.

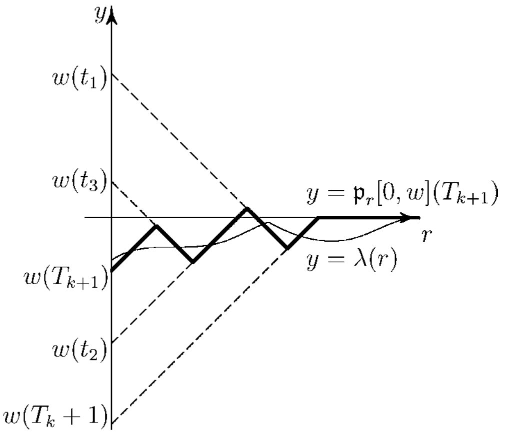

Fig. 4. Construction of w in (3.1.22) at times $T_k + 1 < t_1 < t_2 < t_3 < T_{k+1}$.

Consider now a subset Λ_K of the state space Λ defined as

$$\Lambda_K = \left\{\lambda \in \Lambda; \lambda(r) = 0 \text{ for } r \geqslant K\right\} \tag{3.1.23}$$

for any $K > 0$. We now prove another property of the play which is used several times throughout the text.

LEMMA 3.1.2. *Let $w \in G_R(\mathbb{R}_+)$ and $t \geqslant 0$ be given. Set*

$$w_{\max}(t) = \sup_{\tau \in [0,t]} w(\tau), \qquad w_{\min}(t) = \inf_{\tau \in [0,t]} w(\tau). \tag{3.1.24}$$

Then for all $\lambda \in \Lambda$ and $r > 0$, we have

$$\mathfrak{p}_r[\lambda, w](\tau) \leqslant \max\left\{\lambda(r), w_{\max}(t) - r\right\} \quad \forall \tau \in [0, t], \tag{3.1.25}$$

$$\mathfrak{p}_r[\lambda, w](\tau) \geqslant \min\left\{\lambda(r), w_{\min}(t) + r\right\} \quad \forall \tau \in [0, t], \tag{3.1.26}$$

$$\mathfrak{p}_r[\lambda, w](t) = \lambda(r) \quad \textit{for } r > \left\| m_\lambda\big(w(\cdot)\big)\right\|_{[0,t]}, \tag{3.1.27}$$

where for $v \in \mathbb{R}$ we put $m_\lambda(v) = \inf\{r \geqslant 0; |\lambda(r) - v| = r\}$. In particular, for $K > 0$, $\lambda \in \Lambda_K$ we have $\lambda_t \in \Lambda_{K_t}$ for all $t \geqslant 0$, where $K_t = \max\{K, \|w\|_{[0,t]}\}$.

PROOF. By density and continuity, it suffices to prove the assertion for step functions w of the form (3.1.15) using the recurrent formula (3.1.19). We show by induction that

$$\lambda_{k-1}(r) \leqslant \max\left\{\lambda(r), w_{\max}(t) - r\right\} \quad \forall r \geqslant 0, \tag{3.1.28}$$

for every $t_{k-1} \leqslant t$. Indeed, (3.1.28) holds for $k = 0$. Assume now that for some $k > 0$, $t_{k-1} \leqslant t$ and $r > 0$, we have $\lambda_{k-1}(r) > w_{\max}(t) - r$. By virtue of (3.1.16), (3.1.19) and of the induction hypothesis, we have $\lambda(r) \geqslant \lambda_{k-2}(r) > w_{\max}(t) - r$, hence $\lambda_{k-1}(r) \leqslant \lambda_{k-2}(r) \leqslant \lambda(r)$, and (3.1.25) follows. The proof of (3.1.26) is similar. To check (3.1.27), we notice that the function $r \mapsto r - |v - \lambda(r)|$ is nondecreasing for every $v \in \mathbb{R}$, hence for $r > \|m_\lambda(w(\cdot))\|_{[0,t]}$ and for all $\tau \in [0,t]$ we have $|\lambda(r) - w(\tau)| \leqslant r$, that is, $\lambda(r) - r \leqslant w(\tau) \leqslant \lambda(r) + r$. Then (3.1.16), (3.1.19) yield immediately that $\lambda_{k-1}(r) = \lambda(r)$. For $\lambda \in \Lambda_K$ and $v \in \mathbb{R}$ we have $\max\{K, |v|\} \geqslant m_\lambda(v)$, and using (3.1.27) we easily complete the proof. □

Let us derive some consequences from Lemma 3.1.2. Assume that $m_\lambda(w(\cdot))$ attains at a point $\bar{t} \geqslant 0$ its maximum over $[0, \bar{t}\,]$, that is,

$$\bar{r} := m_\lambda\bigl(w(\bar{t}\,)\bigr) = \bigl\|m_\lambda\bigl(w(\cdot)\bigr)\bigr\|_{[0,\bar{t}]}. \tag{3.1.29}$$

The case $\bar{r} = 0$ is trivial, as it implies $w(t) = \lambda(0)$ for all $t \in [0, \bar{t}\,]$. For $\bar{r} > 0$ we distinguish the cases

(i) $w(\bar{t}\,) = \lambda(\bar{r}) + \bar{r}$,

(ii) $w(\bar{t}\,) = \lambda(\bar{r}) - \bar{r}$.

If (i) holds and $w(t) > w(\bar{t}\,)$ for some $t \in [0, \bar{t}\,]$, then $\lambda(\bar{r}) + \bar{r} < w(t)$, hence $m_\lambda(w(t)) > \bar{r}$ which contradicts (3.1.29). We thus have $w(\bar{t}\,) = w_{\max}(\bar{t}\,)$, and Lemma 3.1.2 together with (3.1.6) yield

$$\mathfrak{p}_r[\lambda, w]\bigl(\bar{t}\,\bigr) = \max\bigl\{\lambda(r), w\bigl(\bar{t}\,\bigr) - r\bigr\}. \tag{3.1.30}$$

Similarly, in the case (ii) we have $w(\bar{t}\,) = w_{\min}(\bar{t}\,)$ and

$$\mathfrak{p}_r[\lambda, w]\bigl(\bar{t}\,\bigr) = \min\bigl\{\lambda(r), w\bigl(\bar{t}\,\bigr) + r\bigr\}. \tag{3.1.31}$$

From the above considerations we conclude the following corollary.

COROLLARY 3.1.3. *Let $w \in G_R(\mathbb{R}_+)$ be T-periodic, that is, $w(t+T) = w(t)$ for all $t \geqslant 0$, with a fixed period $T > 0$. Then $\mathfrak{p}_r[\lambda, w]$ is T-periodic for $t \geqslant T$, for all $\lambda \in \Lambda$.*

PROOF. We may again consider only step functions w and then pass to the uniform limit, if necessary. The function $m_\lambda(w(\cdot))$ is T-periodic and attains its maximum at some point $\bar{t} \in [0, T]$, hence also at all points $\bar{t} + kT$, $k \in \mathbb{N}$. From (3.1.30)–(3.1.31) and the semigroup property (3.1.21) we obtain the assertion. □

3.2. *Prandtl–Ishlinskii operator*

We describe here a construction which has been suggested in [8,21] as a model for elasto-plastic hysteresis. Each individual play represents a rigid–plastic element with kinematic hardening, and their linear superposition corresponds to a combination in series of such elements. A passage to the whole one-parametric continuum of plays can be done by homogenization, see, e.g., [6].

Given a *distribution function* $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$, we define the value of the *Prandtl–Ishlinskii operator* $\mathcal{F} : \Lambda \times G_R(\mathbb{R}_+) \to G_R(\mathbb{R}_+)$ *generated by* h for an initial state $\lambda \in \Lambda$ an input $w \in G_R(\mathbb{R}_+)$ by the formula

$$\mathcal{F}[\lambda, w](t) = h(0)w(t) + \int_0^\infty \mathfrak{p}_r[\lambda, w](t)\,\mathrm{d}h(r). \tag{3.2.1}$$

By (3.1.27), the definition is meaningful if and only if

$$\int_0^\infty \lambda(r)\,\mathrm{d}h(r) < \infty. \tag{3.2.2}$$

This is always true if, for instance, $\lambda \in \Lambda_K$ for some $K > 0$. The function

$$H(s) = \int_0^s h(r)\,\mathrm{d}r \tag{3.2.3}$$

is the so-called *initial loading curve* which depicts the reaction of a hysteresis system with no previous memory to an input which monotonically increasing from zero. Indeed, assuming $\lambda \equiv 0$, $w(0) = 0$ and w increasing in $[0, T]$, we obtain from (3.1.20) that $\mathfrak{p}_r[\lambda, w](t) = P[0, w(t)](r) = \max\{w(t) - r, 0\}$, hence

$$\mathcal{F}[\lambda, w](t) = h(0)w(t) + \int_0^{w(t)} \big(w(t) - r\big)\,\mathrm{d}h(r) = H\big(w(t)\big). \tag{3.2.4}$$

Let us have a short look at the hysteresis branches starting from the initial loading curve at time t_0. Assume that $w(t_0) > 0$, $\lambda_{t_0} = \max\{w(t_0) - r, 0\}$ and that w decreases in $[t_0, t_1]$, $t_1 > t_0$, $w(t_1) > -w(t_0)$. By (3.1.20) we have

$$\mathfrak{p}_r[\lambda, w](t) = \begin{cases} w(t) + r & \text{for } 0 < r < \frac{1}{2}\big(w(t_0) - w(t)\big), \\ w(t_0) - r & \text{for } \frac{1}{2}\big(w(t_0) - w(t)\big) \leqslant r < w(t_0), \\ 0 & \text{for } w(t_0) \leqslant r, \end{cases} \tag{3.2.5}$$

hence

$$\mathcal{F}[\lambda, w](t) = H\big(w(t_0)\big) - 2H\left(\frac{1}{2}\big(w(t_0) - w(t)\big)\right). \tag{3.2.6}$$

A similar computation in the case $w(t_0) < 0$, $\lambda_{t_0} = \min\{w(t_0) + r, 0\}$, w increases in $[t_0, t_1]$, $t_1 > t_0$, $w(t_1) < -w(t_0)$, yields

$$\mathcal{F}[\lambda, w](t) = H\big(w(t_0)\big) + 2H\left(\frac{1}{2}\big(w(t) - w(t_0)\big)\right). \tag{3.2.7}$$

We see that the hysteresis branches are homothetic copies with factor 2 of the initial loading curve, reversed if w decreases. This phenomenon is known in plasticity as the "Masing law". Figure 5 shows two typical situations, where $h(r) \geqslant 0$ and

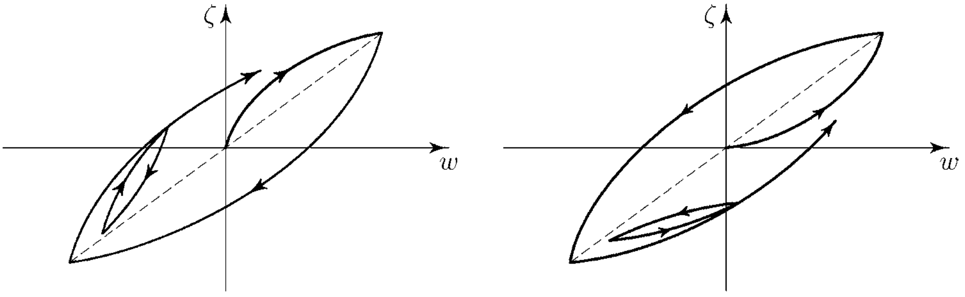

Fig. 5. Clockwise and counterclockwise hysteresis in $\zeta(t) = \mathcal{F}[0, w](t)$.

- either h is nondecreasing and the loops are oriented counterclockwise,
- or h is nonincreasing and the loops are oriented clockwise.

We will see below that the orientation of the loops is important for the energy dissipation properties of the model. Furthermore, it was shown in [11] that the two cases correspond to mutually inverse operators associated with mutually inverse initial loading curves (note that if H is convex, then H^{-1} is concave and vice versa). The following result is a variant of [14], Corollary II.3.4.

PROPOSITION 3.2.1. *Let $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ be such that $h(r) > 0$ for all $r > 0$, and let H given by* (3.2.3) *be unbounded. Let H^{-1} be the inverse function to H, let $\widehat{\mathcal{F}}$ be the Prandtl–Ishlinskii operator of the form* (3.2.1) *generated by $\hat{h} = \mathrm{d}H^{-1}/\mathrm{d}r$. Then for all $w \in G_R(\mathbb{R}_+)$, $K > 0$, $\lambda \in \Lambda_K$ and $t \geqslant 0$, we have*

$$\widehat{\mathcal{F}}\big[\mu, \mathcal{F}[\lambda, w]\big](t) = w(t), \tag{3.2.8}$$

where $\mu \in \Lambda_{H(K)}$ is given for $s \geqslant 0$ by the formula

$$\mu(s) = -\int_{H^{-1}(s)}^{\infty} \lambda'(r)h(r)\,\mathrm{d}r. \tag{3.2.9}$$

The local Lipschitz continuity of $\mathcal{F}$ follows immediately from Proposition 3.1.1 and Lemma 3.1.2, and we state the result explicitly as follows.

PROPOSITION 3.2.2. *Let $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ and $K > 0$ be given, and let $\mathcal{F}$ be the operator* (3.2.1). *Then for all $w_1, w_2 \in G_R(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda_K$ and $t \geqslant 0$, we have*

$$\begin{aligned} &\big|\mathcal{F}[\lambda_1, w_1](t) - \mathcal{F}[\lambda_2, w_2](t)\big| \\ &\quad\leqslant \big|h(0)\big|\big|w_1(t) - w_2(t)\big| \\ &\qquad + \Big(\operatorname*{Var}_{[0,R(t)]} h\Big)\max\big\{\|\lambda_1 - \lambda_2\|_{[0,K]}, \|w_1 - w_2\|_{[0,t]}\big\}, \end{aligned} \tag{3.2.10}$$

where $R(t) = \max\{K, \|w_1\|_{[0,t]}, \|w_2\|_{[0,t]}\}$ and Var *denotes the total variation.*

We will not consider here the question of continuous dependence of $\mathcal{F}$ on the distribution function h, and an interested reader may find more information on this subject in [6].

Let $w \in G_R(\mathbb{R}_+)$ and $0 \leqslant t_1 < t_2$ be arbitrarily chosen. Putting in (3.2.16) $\lambda_1 = \lambda_2 =: \lambda$ and $w_1 = w$, $w_2(t) = w(t)$ for $t \in [0, t_1[$, $w_2(t) = w(t_1)$ for $t \in [t_1, t_2]$, $\zeta = \mathcal{F}[\lambda, w]$, we obtain that

$$\left|\zeta(t_2) - \zeta(t_1)\right| \leqslant \Bigl(\left|h(0)\right| + \operatorname*{Var}_{[0,R(t_2)]} h\Bigr) \left\|w - w(t_1)\right\|_{[t_1,t_2]}. \tag{3.2.11}$$

In particular, if $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ then $\zeta \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, and we have

$$\left|\dot\zeta(t)\right| \leqslant \Bigl(\left|h(0)\right| + \operatorname*{Var}_{[0,R(t)]} h\Bigr)\left|\dot w(t)\right| \quad \text{a.e., with } R(t) = \max\bigl\{K, \|w\|_{[0,t]}\bigr\}. \tag{3.2.12}$$

Moreover, if $\dot w(t) = 0$, then $\dot\zeta(t)$ exists and equals 0. Assume now that w increases in an interval $[t_0, t_1]$. From (3.1.20) it follows for $t \in [t_0, t_1]$ that

$$\begin{aligned}
&\mathcal{F}[\lambda, w](t) - \mathcal{F}[\lambda, w](t_0) \\
&\quad = h(0)\bigl(w(t) - w(t_0)\bigr) + \int_0^{m_{\lambda_{t_0}}(w(t))} \bigl(w(t) - r - \lambda_{t_0}(r)\bigr)\,\mathrm{d}h(r) \\
&\quad = \int_0^{m_{\lambda_{t_0}}(w(t))} h(r)\bigl(1 + \lambda'_{t_0}(r)\bigr)\,\mathrm{d}r \\
&\quad = \int_{w(t_0)}^{w(t)} h\bigl(m_{\lambda_{t_0}}(u)\bigr)\,\mathrm{d}u.
\end{aligned} \tag{3.2.13}$$

Similarly, if w decreases in $[t_0, t_1]$ then

$$\mathcal{F}[\lambda, w](t) - \mathcal{F}[\lambda, w](t_0) = -\int_{w(t)}^{w(t_0)} h\bigl(m_{\lambda_{t_0}}(u)\bigr)\,\mathrm{d}u \quad \text{for } t \in [t_0, t_1]. \tag{3.2.14}$$

From now on, we restrict ourselves to counterclockwise Prandtl–Ishlinskii operators and assume that

$$\text{the function } h \text{ is positive and nondecreasing in } [0, \infty[. \tag{3.2.15}$$

Then (3.2.12) reads

$$\left|\dot\zeta(t)\right| \leqslant h\bigl(R(t)\bigr)\left|\dot w(t)\right| \quad \text{a.e.} \tag{3.2.16}$$

If w is monotone in a neighborhood of t, $\dot w(t) \neq 0$, and $\dot\zeta(t)$ exist at some point t, then and we may conclude using (3.2.13) that $\dot\zeta(t)$ and $\dot w(t)$ have the same sign, and

$$\dot\zeta(t)\dot w(t) \geqslant h(0)\dot w^2(t). \tag{3.2.17}$$

Inequality (3.2.17) therefore holds a.e. if w is continuously differentiable. By [14], Proposition II.4.2, the Prandtl–Ishlinskii operator is locally Lipschitz continuous in $W^{1,1}(0,T)$ for every $T>0$, hence (3.2.17) can be a.e. extended to any $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$.

REMARK 3.2.3. The Prandtl–Ishlinskii operator (3.2.1) can be considered as a special case of the *Preisach operator*

$$\mathcal{P}[\lambda, w](t) = a w(t) + \int_0^\infty \psi\bigl(r, \mathfrak{p}_r[\lambda, w](t)\bigr)\,\mathrm{d}r, \tag{3.2.18}$$

where $a \in \mathbb{R}$ is a constant and ψ is a given function of two variables. The original construction based on the concept proposed in [22] and based on the concept of *two-parametric relays*, used systematically in [20,25], is shown in [12] to be equivalent to (3.2.18). More about the relationship between the operators (3.2.1) and (3.2.18) can be found in [13,14].

3.3. *Monotonicity*

The variational character of the Prandtl–Ishlinskii operator induces natural monotonicity for absolutely continuous inputs. Assume that h satisfies (3.2.15), $w_1, w_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, and $\lambda_1, \lambda_2 \in \Lambda$ are given, and set $\xi_r^{(i)} = \mathfrak{p}_r[\lambda_i, w_i]$, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1,2$, where $\mathcal{F}$ is given by (3.2.1). From (3.1.9) it follows that $(\dot\xi_r^{(1)} - \dot\xi_r^{(2)})(w_1 - w_2 - \xi_r^{(1)} + \xi_r^{(2)}) \geqslant 0$ a.e., hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\xi_r^{(1)} - \xi_r^{(2)}\bigr)^2 \leqslant \bigl(\dot\xi_r^{(1)} - \dot\xi_r^{(2)}\bigr)(w_1 - w_2) \quad \text{a.e.}, \tag{3.3.1}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(h(0)(w_1 - w_2)^2 + \int_0^\infty \bigl(\xi_r^{(1)} - \xi_r^{(2)}\bigr)^2\,\mathrm{d}h(r)\right)\\ &\quad\leqslant (\dot\zeta_1 - \dot\zeta_2)(w_1 - w_2) \quad \text{a.e.}\end{aligned} \tag{3.3.2}$$

Let $W^{1,1}_T(\mathbb{R}_+)$ denote the space of T-periodic absolutely continuous functions defined on $\mathbb{R}_+$. In view of Corollary 3.1.3, we obtain for all $w_1, w_2 \in W^{1,1}_T(\mathbb{R}_+)$ and ζ_i, λ_i as above that ζ_1, ζ_2 are T-periodic for $t \geqslant T$ and

$$\int_T^{2T} \bigl(\dot\zeta_1(t) - \dot\zeta_2(t)\bigr)\bigl(w_1(t) - w_2(t)\bigr)\,\mathrm{d}t \geqslant 0. \tag{3.3.3}$$

We obviously have equality in (3.3.3) provided $w_1 - w_2 = \text{const}$, but in this case we actually can easily prove more, namely the following proposition.

PROPOSITION 3.3.1. *Let $\lambda_1, \lambda_2 \in \Lambda$, $w_1 \in W^{1,1}_T(\mathbb{R}_+)$ and $c \in \mathbb{R}$ be given, and put $w_2(t) = w_1(t) + c$, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1,2$, with $\mathcal{F}$ given by* (3.2.1). *Then there exists $\tilde c \in \mathbb{R}$ such that $\zeta_2(t) = \zeta_1(t) + \tilde c$ for $t \geqslant T$.*

PROOF. For $r > 0$ and $i = 1, 2$ set $\xi_r^{(i)} = \mathfrak{p}_r[\lambda_i, w_i]$. By (3.3.1) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\xi_r^{(1)} - \xi_r^{(2)} + c\big)^2(t) \leqslant 0 \quad \text{a.e.}$$

From Corollary 3.1.3 we obtain that $\xi_r^{(1)}(t) - \xi_r^{(2)}(t) = c_r = \text{const}$ for $t \geqslant T$, and the assertion follows. □

The converse of Proposition 3.3.1 holds if h in (3.2.1) is strictly monotone, so that inequalities (3.3.2) and (3.3.3) are in fact "almost" strict. This fact is less obvious and we state it in the form given in [14], Theorem II.4.10, Corollary II.4.11 and Proposition II.4.12.

PROPOSITION 3.3.2. *Let the function h in* (3.2.15) *be increasing and let $w_1, w_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda$ be given, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1, 2$, with $\mathcal{F}$ given by* (3.2.1). *Assume that* (3.3.2) *holds with equality sign a.e. Then there exists a nondecreasing function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\begin{aligned} &\mathfrak{p}_r[\lambda_1, w_1](t) - \mathfrak{p}_r[\lambda_2, w_2](t) \\ &\quad = \begin{cases} \lambda_1^0(r) - \lambda_2^0(r) & \text{for } r \geqslant R(t), \\ \lambda_1^0\big(R(t)\big) - \lambda_2^0\big(R(t)\big) & \text{for } 0 \leqslant r < R(t), \end{cases} \end{aligned} \tag{3.3.4}$$

where $\lambda_i^0(r) = \mathfrak{p}_r[\lambda_i, w_i](0)$ for $i = 1, 2$. In particular, $w_1(t) - w_2(t) = \lambda_1^0(R(t)) - \lambda_2^0(R(t))$ for all $t \geqslant 0$. If $w_1, w_2 \in W_T^{1,1}(\mathbb{R}_+)$ and

$$\int_T^{2T} \big(\dot\zeta_1(t) - \dot\zeta_2(t)\big)\big(w_1(t) - w_2(t)\big)\,\mathrm{d}t \leqslant 0, \tag{3.3.5}$$

then $w_1(t) - w_2(t) \equiv \text{const}$. If moreover, $\lambda_1 = \lambda_2 = \lambda \in \Lambda_K$ then

$$\begin{aligned} h(0)\big(w_1(t) - w_2(t)\big)^2 &\leqslant \big(\zeta_1(t) - \zeta_2(t)\big)\big(w_1(t) - w_2(t)\big) \\ &\leqslant h\big(k(t)\big)\big(w_1(t) - w_2(t)\big)^2 \end{aligned} \tag{3.3.6}$$

with $k(t) = \max\{K, \|w_1\|_{[0,t]}, \|w_2\|_{[0,t]}\}$.

We see that Prandtl–Ishlinskii operators possess some sort of "two-level monotonicity" which may be used in a Minty-type argument, see [14], Section III.3.

3.4. *Energy dissipation*

We still assume that (3.2.15) holds. With the operator $\mathcal{F}$, we associate the *potential energy operator* $\mathcal{U}$ of the form

$$\mathcal{U}[\lambda, w](t) = \frac{1}{2}h(0)\big|w(t)\big|^2 + \frac{1}{2}\int_0^\infty \big|\mathfrak{p}_r[\lambda, w](t)\big|^2\,\mathrm{d}h(r). \tag{3.4.1}$$

If we interpret w as stress, $\zeta = \mathcal{F}[\lambda, w]$ as strain and $U = \mathcal{U}[\lambda, w]$ as potential energy, then, putting $\xi_r(t) = \mathfrak{p}_r[\lambda, w](t)$ and assuming that the input w is absolutely continuous, we obtain for the dissipation rate $d(t)$ the integral expression

$$d(t) := \dot{\zeta}(t)w(t) - \dot{U}(t) = \int_0^\infty \dot{\xi}_r(t)\big(w(t) - \xi_r(t)\big)\,\mathrm{d}h(r) \quad \text{a.e.} \tag{3.4.2}$$

Let us examine this formula in more detail. By (3.1.9), we can have $\dot{\xi}_r(t) > 0$ only if $w(t) - \xi_r(t) = r$ and $\dot{\xi}_r(t) < 0$ only if $w(t) - \xi_r(t) = -r$. Consequently, we have

$$d(t) = \int_0^\infty \big|\dot{\xi}_r(t)\big| r\,\mathrm{d}h(r) \geqslant 0 \quad \text{a.e.} \tag{3.4.3}$$

in agreement with the second principle of thermodynamics. The integral of $d(t)$ over a closed cycle yields the area of the corresponding hysteresis loop, indeed.

For the sake of completeness, we derive a formula for the potential energy operator associated with the inverse operator

$$\begin{aligned} w(t) = \widehat{\mathcal{F}}[\mu, \zeta](t) &= \hat{h}(0)\zeta(t) + \int_0^\infty \mathfrak{p}_r[\mu, \zeta](t)\,\mathrm{d}\hat{h}(r) \\ &= \hat{h}(\infty)\zeta(t) - \int_0^\infty \big(\zeta(t) - \mathfrak{p}_r[\mu, \zeta](t)\big)\,\mathrm{d}\hat{h}(r) \end{aligned} \tag{3.4.4}$$

generated by $\hat{h}$ as in Proposition 3.2.1. The function $\hat{h}$ is nonincreasing and positive, hence $\hat{h}(\infty) = \lim_{r\to\infty} \hat{h}(r)$ is well defined. Putting

$$\begin{aligned} e(t) &= \widehat{\mathcal{U}}[\mu, \zeta](t) \\ &= \frac{1}{2}\hat{h}(\infty)\big|\zeta(t)\big|^2 - \frac{1}{2}\int_0^\infty \big|\zeta(t) - \mathfrak{p}_r[\mu, \zeta](t)\big|^2\,\mathrm{d}\hat{h}(r), \end{aligned} \tag{3.4.5}$$

we easily check that the positive sign in (3.4.3) is preserved.

Besides the "physical energy inequality" (3.4.2), the Prandtl–Ishlinskii operator (3.2.1), (3.2.15) (as well as other hysteresis operators with convex/concave branches, for a detailed discussion on this subject see [14]) admits a "higher-order energy inequality"

$$\ddot{\zeta}(t)\dot{w}(t) - \dot{V}(t) \geqslant 0 \tag{3.4.6}$$

in the sense of distributions (see (3.4.9)), where we set

$$V(t) = \frac{1}{2}\dot{\zeta}(t)\dot{w}(t) \quad \text{for a.e. } t > 0. \tag{3.4.7}$$

This observation has been made for the first time in [11] in the context of periodic functions, and later on several different proofs have been published. Since this result plays a central role in our analysis, we state it precisely and give a sketch of the proof. As time differentiation is involved, we restrict ourselves to regular inputs and outputs.

THEOREM 3.4.1. *Let hypothesis* (3.2.15) *hold, and for* $p > 0$ *set*

$$\kappa(p) = \inf\left\{\frac{h(r) - h(s)}{r - s}; 0 \leqslant s < r \leqslant p\right\}. \tag{3.4.8}$$

Then for every $K > 0$, $\lambda \in \Lambda_K$ *and* $w \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ *such that* $\zeta = \mathcal{F}[\lambda, w]$ *with* $\mathcal{F}$ *given by* (3.2.1) *belongs to* $W^{2,1}_{\mathrm{loc}}(\mathbb{R}_+)$, *the function* $V(t)$ *given by* (3.4.7) *equals almost everywhere to a function of bounded variation. Moreover, for every* $T > 0$, $p \geqslant \max\{K, \|w\|_{[0,T]}\}$ *and every* $0 \leqslant t_0 < t_1 < T$, *we have*

$$\int_{t_0}^{t_1} \ddot{\zeta}(t)\dot{w}(t)\,\mathrm{d}t - V(t_1-) + V(t_0+) \geqslant \frac{1}{4}\kappa(p)\int_{t_0}^{t_1} \left|\dot{w}(t)\right|^3 \mathrm{d}t. \tag{3.4.9}$$

In particular, if w *is* T*-periodic then*

$$\int_{T}^{2T} \ddot{\zeta}(t)\dot{w}(t)\,\mathrm{d}t \geqslant \frac{1}{4}\kappa(p)\int_{T}^{2T} \left|\dot{w}(t)\right|^3 \mathrm{d}t. \tag{3.4.10}$$

REMARK 3.4.2. As noticed in Remark 1.2.4, the function κ is a measure for the curvature of the initial loading curve H given by (3.2.3); in particular, H is strictly convex if κ is positive. The "dissipation term" on the right-hand side of (3.4.9) is thus proportional to the minimal curvature of H. Inequality (3.4.9) would be in fact a trivial application of the integration by parts formula if the hysteresis branches $\eta(t) = g(w(t))$ were smooth enough. Indeed, in this case we would have

$$\ddot{\zeta}(t)\dot{w}(t) - \dot{V}(t) = \frac{1}{2}g''\big(w(t)\big)\dot{w}^3(t). \tag{3.4.11}$$

The right-hand side of (3.4.11) is formally positive because g is convex if w increases and concave if w decreases, cf. the counterclockwise case on Figure 5. However, the "second-order potential energy" $V(t)$ (which is indeed positive by virtue of (3.2.17)) is typically discontinuous in time *even if h is smooth*, e.g., on the transition from a minor loop to the major loop, and this fact makes the rigorous argument technically complicated.

The proof of Theorem 3.4.1 is based on a series of the following lemmas.

LEMMA 3.4.3. *Let* $\varphi :]a, b[\to \mathbb{R}$ *and* $c \geqslant 0$ *be such that* $\varphi(a+) > 0$ *and the function* $v \mapsto \varphi(v) - cv$ *is nondecreasing in* $]a, b[$. *Then the function*

$$\psi(v) := \frac{1}{\varphi(v)} + c\int_a^v \frac{\mathrm{d}s}{\varphi^2(s)}$$

is nonincreasing in $]a, b[$.

PROOF. The assertion is obvious if φ is absolutely continuous; otherwise we approximate φ by piecewise linear interpolates and pass to the limit in continuity points of φ. Discontinuity points can be handled directly. $\square$

LEMMA 3.4.4. *Let $w \in W^{1,\infty}(T_0, T_1)$ be an increasing function, and let $c \geqslant 0$ and $g : [w(T_0), w(T_1)] \to \mathbb{R}$ be such that the function $v \mapsto g(v) - \frac{c}{2}v^2$ is convex in $[w(T_0), w(T_1)]$, $g'(w(T_0)+) > 0$. Assume that $\zeta = g(w) \in W^{2,1}(T_0, T_1)$, and for $t \in]T_0, T_1[$ put $V(t) = \frac{1}{2}\dot{\zeta}(t)\dot{w}(t)$. Then V coincides a.e. with a function of bounded variation in $[T_0, T_1]$, and for every $T_0 \leqslant t_0 < t_1 \leqslant T_1$ we have*

$$\int_{t_0}^{t_1} \ddot{\zeta}(t)\dot{w}(t)\,\mathrm{d}t - V(t_1-) + V(t_0+) \geqslant \frac{c}{2}\int_{t_0}^{t_1} \left|\dot{w}(t)\right|^3 \mathrm{d}t. \tag{3.4.12}$$

PROOF. We first choose $t_0 < t_1$ such that $w(t_0), w(t_1)$ are continuity points of g'. By Lemma 3.4.3, the function

$$\eta(t) = \frac{1}{g'(w(t))} + c\int_{t_0}^{t} \frac{\dot{w}(\tau)}{(g'(w(\tau)))^2}\,\mathrm{d}\tau \tag{3.4.13}$$

is nonincreasing in $[t_0, t_1]$. Integrating by parts we obtain

$$\begin{aligned}
\int_{t_0}^{t_1} \ddot{\zeta}(t)\dot{w}(t)\,\mathrm{d}t &= \int_{t_0}^{t_1} \frac{1}{g'(w(t))}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\zeta}^2(t)\big)\,\mathrm{d}t \\
&= V(t_1) - V(t_0) - \int_{t_0}^{t_1} \frac{1}{2}\dot{\zeta}^2(t)\,\mathrm{d}\left(\frac{1}{g'(w)}\right)(t) \\
&= V(t_1) - V(t_0) - \int_{t_0}^{t_1} \frac{1}{2}\dot{\zeta}^2(t)\,\mathrm{d}\eta(t) + \frac{c}{2}\int_{t_0}^{t_1} \frac{\dot{\zeta}^2(t)\dot{w}(t)}{(g'(w(t)))^2}\,\mathrm{d}t \\
&\geqslant V(t_1) - V(t_0) + \frac{c}{2}\int_{t_0}^{t_1} \left|\dot{w}(t)\right|^3 \mathrm{d}t.
\end{aligned} \tag{3.4.14}$$

Consequently, the function

$$t \mapsto \int_{t_0}^{t} \ddot{\zeta}(\tau)\dot{w}(\tau)\,\mathrm{d}\tau - \frac{c}{2}\int_{t_0}^{t} \left|\dot{w}(\tau)\right|^3 \mathrm{d}\tau - V(t)$$

is a.e. nondecreasing, hence V has (up to a set of measure zero) bounded variation, and (3.4.12) is obtained by passing to the limit. □

We do not repeat the same proof for the following "decreasing" counterpart to Lemma 3.4.4.

LEMMA 3.4.5. *Let $w \in W^{1,\infty}(T_0, T_1)$ be a decreasing function, and let $c \geqslant 0$ and $g : [w(T_1), w(T_0)] \to \mathbb{R}$ be such that the function $v \mapsto g(v) + \frac{c}{2}v^2$ is concave in $[w(T_1), w(T_0)]$, $g'(w(T_0)-) > 0$. Let $\zeta(t)$ and $V(t)$ be as in Lemma* 3.4.4. *Then* (3.4.12) *holds for all $T_0 \leqslant t_0 < t_1 \leqslant T_1$.*

We are now ready to pass to the proof of Theorem 3.4.1.

PROOF OF THEOREM 3.4.1. Let $t_0 < t_1$ be fixed, and set

$$N = \big\{t \in [t_0, t_1]; \dot{\zeta}(t) = 0\big\}.$$

The function ζ is continuously differentiable, hence N is closed, and there exist pairwise disjoint intervals $]\tau_j, \tau^j[$, j belonging to an at most countable index set J, such that

$$]t_0, t_1[\, \setminus N = \bigcup_{j \in J}]\tau_j, \tau^j[. \tag{3.4.15}$$

Let us now fix some $j \in J$. The function ζ (and also w by virtue of (3.2.16)) are strictly monotone in $]\tau_j, \tau^j[$, hence we are in the situation of either Lemma 3.4.4 or Lemma 3.4.5 with g defined by (3.2.13) or (3.2.14). To be more precise, we distinguish between the two cases in order to determine the constant c.

1. Let $\dot{\zeta} > 0$ in $]\tau_j, \tau^j[$, and put

$$g_j(v) = \mathcal{F}[\lambda, w](\tau_j) + \int_{w(\tau_j)}^{v} h\big(m_{\lambda_{\tau_j}}(u)\big)\, \mathrm{d}u. \tag{3.4.16}$$

For a.e. $w(\tau_j) < v_1 < v_2 < w(\tau^j)$ we have

$$\frac{g_j'(v_2) - g_j'(v_1)}{v_2 - v_1} = \frac{h(m_{\lambda_{\tau_j}}(v_2)) - h(m_{\lambda_{\tau_j}}(v_1))}{m_{\lambda_{\tau_j}}(v_2) - m_{\lambda_{\tau_j}}(v_1)} \frac{m_{\lambda_{\tau_j}}(v_2) - m_{\lambda_{\tau_j}}(v_1)}{v_2 - v_1}. \tag{3.4.17}$$

Set $r_i = m_{\lambda_{\tau_j}}(v_i)$ for $i = 1, 2$. Then

$$v_1 - v_2 = \big(r_1 + \lambda_{\tau_j}(r_1)\big) - \big(r_2 + \lambda_{\tau_j}(r_2)\big) \leqslant 2(r_1 - r_2).$$

From (3.4.17), (3.4.8) and Lemma 3.1.2 it follows that

$$\frac{g_j'(v_2) - g_j'(v_1)}{v_2 - v_1} \geqslant \frac{\kappa(p)}{2}. \tag{3.4.18}$$

The function $v \mapsto g_j'(v) - \frac{\kappa(p)}{2} v$ is nonincreasing, hence $v \mapsto g_j(v) - \frac{\kappa(p)}{4} v^2$ is convex, and we may use Lemma 3.4.4 to obtain that

$$\int_{\tau_j}^{\tau^j} \ddot{\zeta}(t)\dot{w}(t)\, \mathrm{d}t - V\big(\tau^j -\big) + V(\tau_j +) \geqslant \frac{1}{4}\kappa(p) \int_{\tau_j}^{\tau^j} \big|\dot{w}(t)\big|^3\, \mathrm{d}t. \tag{3.4.19}$$

2. Let $\dot{\zeta} < 0$ in $]\tau_j, \tau^j[$, and put

$$g_j(v) = \mathcal{F}[\lambda, w](\tau_j) - \int_{v}^{w(\tau_j)} h\big(m_{\lambda_{\tau_j}}(u)\big)\, \mathrm{d}u. \tag{3.4.20}$$

Repeating the above procedure we show that the function $v \mapsto g_j(v) + \frac{\kappa(p)}{4} v^2$ is concave, and Lemma 3.4.4 yields again that (3.4.19) holds.

At all points τ_j, τ^j except possibly the cases $\tau_j = t_0$ or $\tau^j = t_1$, we have $\dot{\zeta}(\tau_j) = \dot{\zeta}(\tau^j) = 0$, hence $V(\tau^j -) = V(\tau_j +) = 0$. Furthermore, almost everywhere in N we have $\dot{w}(t) = 0$, hence we may sum all inequalities (3.4.19) over $j \in J$ and obtain the assertion. The periodic case follows from Corollary 3.1.3 which enables us to consider in (3.4.9) any integration domain of length T in $[T, \infty[$. □

3.5. *Parameter dependent hysteresis*

We now extend the Prandtl–Ishlinskii construction to functions depending also on a spatial variable x by assuming that each point x has its own memory. In our situation, we only consider the one-dimensional case $x \in [0, 1]$ and input functions continuous in t.

Let an initial memory distribution $\lambda \in L^1(0, 1; \Lambda_K)$ be given for some $K > 0$. For inputs w defined in $[0, 1] \times \mathbb{R}_+$ and such that $w \in L^1(0, 1; C[0, T]) \cap L^\infty(]0, 1[\times]0, T[)$ we define similarly as in (3.1.13) and (3.2.1),

$$\mathfrak{p}_r[\lambda, w](x, t) = \hat{\mathfrak{p}}_r\big[\lambda(x, r), w(x, \cdot)\big](t), \tag{3.5.1}$$

$$\mathcal{F}[\lambda, w](x, t) = h(0) w(x, t) + \int_0^\infty \mathfrak{p}_r[\lambda, w](x, t)\, \mathrm{d}h(r) \tag{3.5.2}$$

for $(x, t) \in [0, 1] \times \mathbb{R}_+$, where h is a function satisfying (3.2.15). In fact, we may have considered h which depends also on x, and a detailed discussion on this subject can be found in [6]. Here, for the sake of simplicity, we restrict ourselves to the *spatially homogeneous case*.

Assume first that both λ and w are continuous in x. Then for all $x, y \in [0, 1]$ and $t \in [0, T]$, we have by virtue of (3.2.10) that

$$\begin{aligned} &\big|\mathcal{F}[\lambda, w](x, t) - \mathcal{F}[\lambda, w](y, t)\big| \\ &\quad \leqslant h\big(R(T)\big) \max\big\{\big\|\lambda(x, \cdot) - \lambda(y, \cdot)\big\|_{[0,K]}, \big\|w(x, \cdot) - w(y, \cdot)\big\|_{[0,t]}\big\}, \end{aligned} \tag{3.5.3}$$

where $R(T) = \max\{K, \sup\{|w(z, t)| (z, t) \in [0, 1] \times [0, T]\}\}$, hence $\mathcal{F}[\lambda, w]$ is continuous on $[0, 1] \times [0, T]$. Using (3.2.10) again for sequences $\lambda^{(n)}$ and $w^{(n)}$, we derive the implications

$$\left.\begin{array}{ll} \lambda^{(n)} \to \lambda & \text{uniformly} \\ w^{(n)} \to w & \text{uniformly} \end{array}\right\} \implies \mathcal{F}\big[\lambda^{(n)}, w^{(n)}\big] \to \mathcal{F}[\lambda, w] \quad \text{uniformly}, \tag{3.5.4}$$

and

$$\left.\begin{array}{ll} \lambda^{(n)} \to \lambda & \text{strongly in } L^1(0,1;\Lambda_K) \\ w^{(n)} \to w & \text{strongly in } L^1\big(0,1;C[0,T]\big) \\ w^{(n)} & \text{bounded in } L^\infty\big(]0,1[\times]0,T[\big) \end{array}\right\}$$
$$\Longrightarrow \quad \left\{\begin{array}{ll} \mathcal{F}\big[\lambda^{(n)},w^{(n)}\big] \to \mathcal{F}[\lambda,w] & \text{strongly in } L^1\big(0,1;C[0,T]\big), \\ \mathcal{F}\big[\lambda^{(n)},w^{(n)}\big] & \text{bounded in } L^\infty\big(]0,1[\times]0,T[\big). \end{array}\right. \tag{3.5.5}$$

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CHAPTER 5

Mathematical Issues Concerning the Navier–Stokes Equations and Some of Its Generalizations

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In Memory of

Olga Alexandrovna Ladyzhenskaya

March 7, 1922–January 12, 2004

Jindřich Nečas

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Abstract

This chapter primarily deals with internal, isothermal, unsteady flows of a class of incompressible fluids with both constant and shear or pressure dependent viscosity that includes the Navier–Stokes fluid as a special subclass.

We begin with a description of fluids within the framework of a continuum. We then discuss various ways in which the response of a fluid can depart from that of a Navier–Stokes fluid. Next, we introduce a general thermodynamic framework that has been successful in describing the disparate response of continua that includes those of inelasticity, solid-to-solid transformation, viscoelasticity, granular materials, blood and asphalt rheology, etc. Here, it leads to a novel derivation of the constitutive equation for the Cauchy stress for fluids with constant, or shear and/or pressure, or density dependent viscosity within a full thermomechanical setting. One advantage of this approach consists in a transparent treatment of the constraint of incompressibility.

We then concentrate on the mathematical analysis of three-dimensional unsteady flows of fluids with shear dependent viscosity that includes the Navier–Stokes model and Ladyzhenskaya's model as special cases.

We are interested in the issues connected with mathematical self-consistency of the models, i.e., we are interested in knowing whether (1) flows exist for reasonable, but arbitrary initial data and all instants of time, (2) flows are uniquely determined, (3) the velocity is bounded and (4) the long-time behavior of all possible flows can be captured by a finite-dimensional, small (compact) set attracting all flow trajectories exponentially.

For simplicity, we eliminate the choice of boundary conditions and their influence on the flows by assuming that all functions are spatially periodic with zero mean value over a periodic cell. All these results can however be extended to internal flows wherein the tangential component of the velocity satisfies Navier's slip at the boundary. Most of the results also hold for the no-slip boundary condition.

While the mathematical consistency understood in the above sense for the Navier–Stokes model in three dimensions has not been established as yet, we will show that Ladyzhenskaya's model and some of its generalization enjoy all above characteristics for a certain range of parameters. We also discuss briefly further results related to generalizations of the Navier–Stokes equations.

Keywords: Incompressible fluid, Mathematical analysis, Navier–Stokes fluid, Non-Newtonian fluid, Rheology

Part A. Incompressible fluids with shear, pressure and density dependent viscosity from the point of view of continuum physics

1. Introduction

1.1. *What is a fluid?*

The meaning of words provided in even the most advanced of dictionaries, say *The Oxford English Dictionary* [134], will rarely serve the needs of a scientist or technologist adequately and this is never more evident than in the case of the meaning assigned to the word "fluid" in its substantive form: "A substance whose particles move freely among themselves, so as to give way before the slightest pressure". The inadequacy, in the present case, stems from the latter part of the sentence which states that fluids cannot resist pressure; more so as the above definition is immediately followed by the classification: "Fluids are divided into liquids which are incompletely elastic, and gases, which are completely so". With regard to the first definition, as "fluids" obviously include liquids such as water, which under normal ranges of pressure are essentially incompressible and can support a purely spherical state of stress without flowing the definition offered in the dictionary is, if not totally wrong,[1] at the very least confounding. Much, if not all of hydrostatics is based on the premise that most liquids are incompressible.

What then does one mean by a fluid? When we encounter the word "fluid" for the first time in a physics course at school, we are told that a "fluid" is a body that takes the shape of a container. This meaning assigned to a fluid, can after due care, be used to conclude that a fluid is a body whose symmetry group is the unimodular group.[2] Such a definition is also not without difficulty. While a liquid takes the shape of the container partially if its volume is less than that of the container, a gas expands to always fill a container. The definition via symmetry groups can handle this difficulty in the sense that it requires densities to be constant while determining the symmetry group. However, this places an unnecessary restriction with regard to defining gases, as this is akin to defining a body on only a small subclass of processes that the body can undergo. We shall not get into a detailed discussion of these subtle issues here.

Another definition for a fluid that is quite common, specially with those conversant with the notion of stress, is that a fluid is a body that cannot support a shear stress, as opposed to pressure as required by the definition in [134]. A natural question that immediately arises is that of time scales. How long can a fluid body not support a shear stress? How does one measure this inability to support a shear stress? Is it with the naked eye or is it to be inferred with the aid of sophisticated instruments? Is the assessment to be made in one

[1] One could take the point of view that no body is perfectly incompressible and thus the body does deform, ever so slightly, due to the application of pressure. The definition however cannot be developed thusly as the intent of the dictionary definition referred to above is to convey the impression that the body suffers significant deformation due to the slightest application of the pressure.

[2] This statement is not strictly correct. A special subclass of fluids, those that are referred to as "simple fluids" admit such an interpretation (see [102,147]). However, it is possible that there are anisotropic fluids whose symmetry group is not the unimodular group (see [116]).

second, one day, one month or one year? These questions are not being raised merely from the philosophical standpoint. They have very pragmatic underpinnings. It is possible, say in the time scale of one hour, that one might be unable to discern the flow or deformation that a body undergoes, with the naked eye. This is indeed the case with regard to the experiment on asphalt that has been going on for over seventy years (see [64] for a description of the experiment). The earlier definition for the fluid cannot escape the issue of time scale either. One has to contend with how long it takes to attain the shape of the container.

The importance of the notion of time scales was recognized by Maxwell. He observes [90]: "In the case of a viscous fluid it is time which is required, and if enough time is given, the very smallest force will produce a sensible effect, such as would require a very large force if suddenly applied. Thus a block of pitch may be so hard that you cannot make a dent in it by striking it with your knuckles; and yet it will in the course of time flatten itself by its weight, and glide downhill like a stream of water". The keywords in the above remarks of Maxwell are "if enough time is given". Thus, what we can infer at best is whether a body is more or less fluid-like, i.e., within the time scales of the observation of our interest does a small shear stress produce a sensible deformation or does it not. Let us then accept to "understand" a "fluid" as a body that, in the time scale of observation of interest, undergoes discernible deformation due to the application of a sufficiently small shear stress.[3]

1.2. *Navier–Stokes fluid model*

The popular Navier–Stokes model traces its origin to the seminal work of Newton [100] followed by the penetrating studies by Navier [93], Poisson [105] and Saint-Venant [122], culminating in the definitive study of Stokes [139].[4] In his immortal Principia, Newton [100] states: "The resistance arising from the want of lubricity in parts of the fluid, other things being equal, is proportional to the velocity with which the parts of the fluid are separated from one another". What is now popularly referred to as the Navier–Stokes model implies a linear relationship between the shear stress and the shear rate. However, it was recognized over a century ago that this want of lubricity need not be proportional to the shear stress. Trouton [145] observes "the rate of flow of the material under shearing stress cannot be in simple proportion to shear rate". However, the popular view persisted namely that the rate of flow was proportional to the shear stress as evidenced by the following remarks of Bingham [11]: "When viscous substance, either a liquid or a gas, is subjected to a shearing stress, a continuous deformation results which is, within certain restrictions directly proportional to the shearing stress. This fundamental law of viscous

[3] We assume we can agree on what we mean by the time scale of observation of interest. It is also important to recognize that if the shear stress is too small, its effect, the flow, might not be discernible. Thus, we also have to contend with the notion of a spatial scale for discerning movement and a force scale for discerning forces.

[4] It is interesting to observe what Stokes [139] has to say concerning the development of the fluid model that is referred to as the Navier–Stokes model. Stokes remarks: "I afterward found that Poisson had written a memoir on the same subject, and on referring to it found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the later before this society. ... The same equations have been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI, p. 389), but his principles differ from mine still more than do Poisson's".

flow...”. Though Bingham offers a caveat “within certain restrictions”, his use of the terms “fundamental law of viscous flow” clearly indicates how well the notion of the proportional relations between a kinematical measure of flow and the shear stress was ingrained in the fluid dynamics of those times.

We will record below, for the sake of discussion, the classical fluid models that bear the names of Euler, and Navier and Stokes.

Homogeneous compressible Euler fluid

$$\mathbf{T} = -p(\rho)\mathbf{I}. \tag{A.1.1}$$

Homogeneous incompressible Euler fluid

$$\mathbf{T} = -p\mathbf{I}, \qquad \operatorname{tr}\mathbf{D} = 0. \tag{A.1.2}$$

Homogeneous compressible Navier–Stokes fluid

$$\mathbf{T} = -p(\rho)\mathbf{I} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu(\rho)\mathbf{D}. \tag{A.1.3}$$

Homogeneous incompressible Navier–Stokes fluid

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \qquad \operatorname{tr}\mathbf{D} = 0. \tag{A.1.4}$$

In the above definitions, $\mathbf{T}$ denotes the Cauchy stress, ρ is the density, λ and μ are the bulk and shear moduli of viscosity and $\mathbf{D}$ is the symmetric part of the velocity gradient. In (A.1.1) and (A.1.3) the pressure is defined through an equation of state, while in (A.1.2) and (A.1.4), it is the reaction force due to the constraint that the fluid be incompressible.

Within the course of this chapter we will confine our mathematical discussion mainly to the incompressible Navier–Stokes fluid model (A.1.4) and many of its generalizations.

A model that is not of the form (A.1.3) and (A.1.4) falls into the category of (compressible and incompressible) non-Newtonian fluids.[5] This exclusionary definition leads to innumerable fluid models and choices among them have to be based on the observed response of real fluids that cannot be adequately captured by the above models. This leads us to a discussion of these observed departures from Newtonian behavior.

1.3. *Departures from Newtonian behavior*

We briefly list several typical characteristics of non-Newtonian response. We shall provide a detailed characterization of those phenomena, and the corresponding models, whose mathematical properties will be discussed in this chapter. A reader interested in a more details concerning non-Newtonian fluids is referred, for example, to the monographs [127,147] and [55], the article by Burgers [15], and the review article by Rajagopal [108].

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[5] Navier–Stokes fluids are usually referred to in the fluid mechanics literature as Newtonian fluids. The equations of motions for Newtonian fluids are referred to as the Navier–Stokes equations.

Shear-thinning/shear-thickening. Let us consider an unsteady simple shear flow in which the velocity field $\mathbf{v}$ is given by

$$\mathbf{v} = u(y,t)\mathbf{i}, \tag{A.1.5}$$

in a Cartesian coordinate system (x, y, z) with base vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, respectively, t denoting time. We notice that (A.1.5) automatically meets

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0, \tag{A.1.6}$$

and the only nonzero component for the shear stress corresponding to (A.1.3) or (A.1.4) is given by

$$T_{xy}(y,t) = \mu u_{,y}(y,t), \quad \text{where } u_{,y} := \frac{\mathrm{d}u}{\mathrm{d}y}, \tag{A.1.7}$$

i.e., the shear stress varies proportionally with respect to the gradient of the velocity, the constant of proportionality being the viscosity. Thus, the graph of the shear stress versus the velocity gradient (in this case the shear rate) is a straight line (see curve 3 in Figure 1).

Let us consider a steady shearing flow, i.e., a flow wherein $u = u(y)$ and $\kappa := u_{,y} =$ const at each point of the domain occupied by the fluid. It is observed that in many fluids there is a considerable departure from the above relationship (A.1.7) between the shear stress and the shear rate. In some fluids it is observed that the relationship is as depicted by the curve 1 in Figure 1, i.e., the generalized viscosity, which is defined through

$$\mu_g(\kappa) := \frac{T_{xy}}{\kappa}, \tag{A.1.8}$$

is monotonically increasing (cf. curve 1 in Figure 2). Thus, in such fluids, the viscosity increases with the shear rate and they are referred to as shear-thickening fluids. On the other hand, there are fluids whose relationship between the shear stress and the shear rate is as depicted by curve 2 in Figure 1. In such fluids, the generalized viscosity decreases

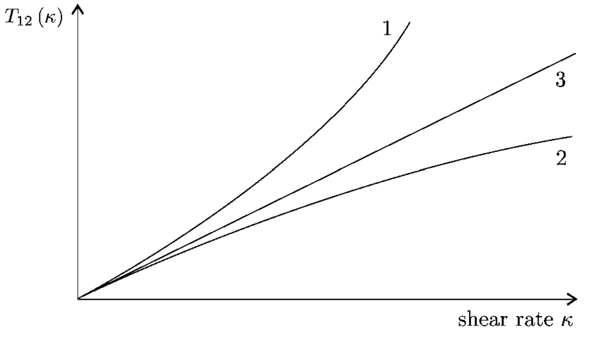

Fig. 1. Shear-thinning/shear-thickening.

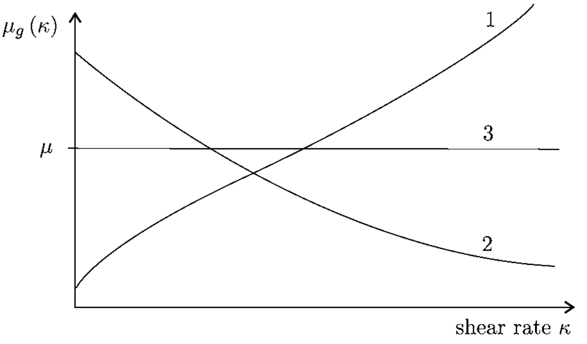

Fig. 2. Generalized viscosity.

with increasing shear rate and for this reason such fluids are called shear-thinning fluids. The Newtonian fluid is thus a very special fluid. It neither shear thins nor shear thickens.

Models for fluids with shear dependent viscosity are used in many areas of engineering science such as geophysics, glaciology, colloid mechanics, polymer mechanics, blood and food rheology, etc. An illustrative list of references for such models and their applications is given in [89].

Normal stress differences in simple shear flows. Next, let us compute the normal stresses along the x, y and z directions for the simple shear flow (A.1.5). A trivial calculation leads to, in the case of models (A.1.3) and (A.1.4),

$$T_{xx} = T_{yy} = T_{zz} = -p,$$

and thus

$$T_{xx} - T_{yy} = T_{xx} - T_{zz} = T_{yy} - T_{zz} = 0.$$

That is the normal stress differences are zero for a Navier–Stokes fluid. However, it can be shown that some of the phenomena that are observed during the flows of fluids such as die-swell, rod-climbing, secondary flows in cylindrical pipes of noncircular cross-section, etc., have as their basis nonzero differences between these normal stresses.

Stress-relaxation. When subject to a step change in strain ε (see Figure 3(a)) that results in a simple shear flow (A.1.5), the strain rate $\dot{\varepsilon}$ is zero except at $t = 0$ as portrayed in Figure 3(b). The stress $\sigma := T_{xy}$ in bodies modeled by (A.1.3) and (A.1.4) suffers an abrupt change that is undefined at the instant the strain has suffered a change and is zero at all other instants (see Figure 4(b)). On the other hand, there are many bodies that respond in the manner shown in Figure 5. The graph at the right depicts fluid-like behavior as no stress is necessary to maintain a fixed strain, in the long run. Figure 5(a) represents solid-like response. The Newtonian fluid model is incapable of describing stress-relaxation,

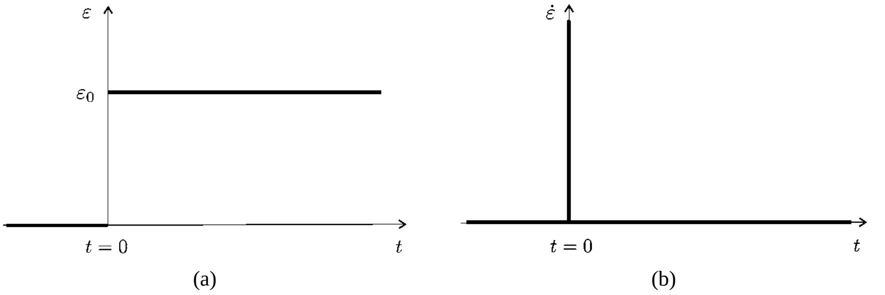

Fig. 3. Stress-relaxation test: response to a step change in strain (a). Its derivative is sketched in (b).

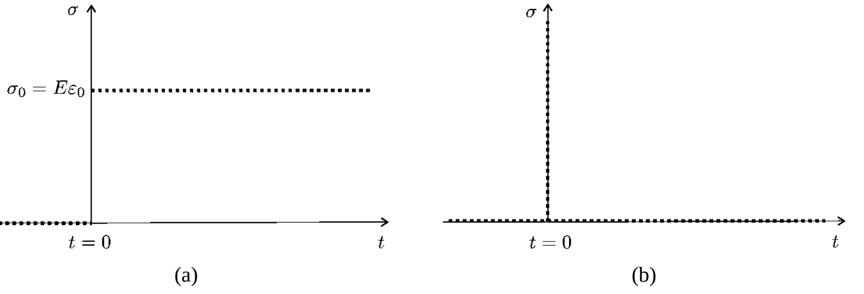

Fig. 4. Shear stress response to a step change in strain for the linear spring (a) and the Navier–Stokes fluid (b).

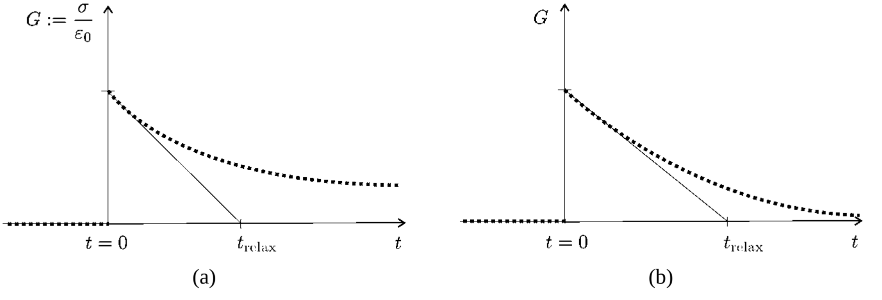

Fig. 5. Stress-relaxation for more realistic materials.

a phenomenon exhibited by many real bodies. It is important to recognize the fact that a Newtonian fluid stress relaxes instantaneously (see Figure 4(b)).[6]

Creep. Next, let us consider a body that is subject to a step change in the stress (see Figure 6). In the case of a Newtonian fluid the strain will increase linearly with time (see Figure 7(b)). However, there are many bodies whose strain will vary as depicted in Figure 8. The curve at the left depicts solid-like behavior while the curve at the right depicts

[6]This does not mean that it has instantaneous elasticity.

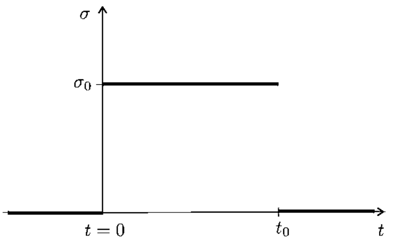

Fig. 6. Creep test.

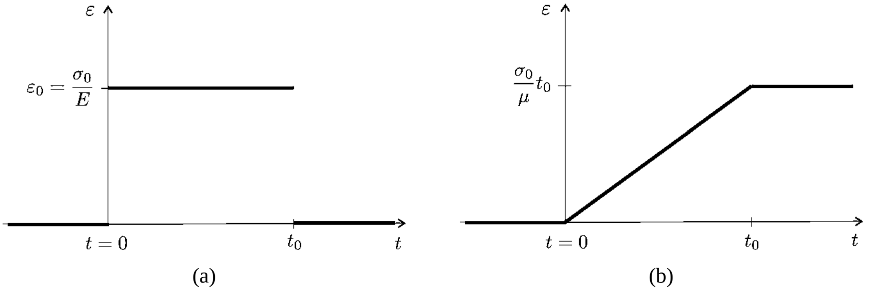

Fig. 7. Deformation response to step change of shear stress for the linear spring (a) and a Newtonian fluid (b).

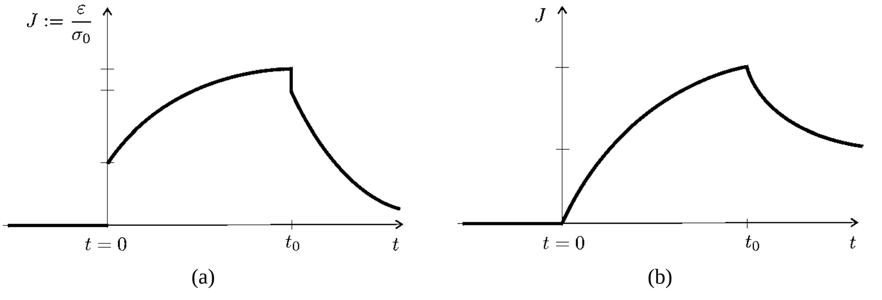

Fig. 8. Creep of solid-like and fluid-like materials.

fluid-like behavior. The response is referred to as “creep” as the body “flows” while the stress is held constant. A Newtonian fluid creeps linearly with time. Many real fluids creep nonlinearly with time.

Jump discontinuities in stresses

Yield stress. Bodies that have a threshold value for the stress before they can flow are supposed to exhibit the phenomenon of “yielding”, see Figure 9. However, if one takes the point of view that a fluid is a body that cannot sustain shear, then by definition there can be no “shear stress threshold” to the flow, which is the basic premise of the notion of a “yield stress”. This is yet another example where the importance of time scales comes into play.

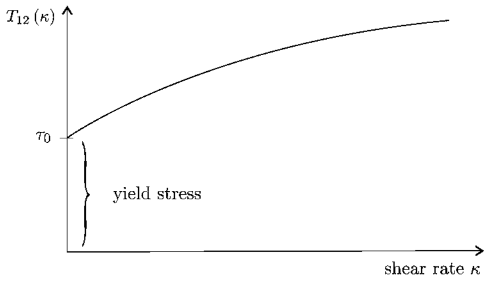

Fig. 9. Yield stress.

It might seem, with respect to by some time scale of observation, that the flow of a fluid is not discernible until a sufficiently large stress is applied. This does not mean that the body in question can support small values of shear stresses, indefinitely, without undergoing any deformation. It merely means that the flow that is induced is not significant. A Newtonian fluid has no threshold before it can start flowing. A material responding as a Newtonian fluid after a “yield stress” is reached is called a Bingham fluid.

Activation criterion. It is possible that in some fluids, the response characteristics can change when a certain criterion, that could depend on the stress, strain rate or other kinematical quantities, is met. An interesting example of the same is the phenomena of coagulation or dissolution of blood. Of course, here issues are more complicated as complex chemical reactions are taking place in the fluid[7] of interest, blood.

Platelet activation is followed by their interactions with a variety of proteins that leads to the aggregation of platelets which in turn leads to coagulation, i.e., the formation of clots. The activated platelets also serve as sites for enzyme complexes that play an important role in the formation of clots. These clots, as well as the original blood, are viscoelastic fluids, the clot being significantly more viscous than regular blood. In many situations the viscoelasticity is inconsequential and can be ignored and the fluid can be approximated as a generalized Newtonian fluid. While the formation of the clot takes a finite length of time, we can neglect this with respect to a time scale of interest associated with the flowing blood. As the viscosity has increased considerably over a sufficiently short time, in a simple shear flow, the fluid could be regarded as suffering a jump discontinuity as depicted in Figure 10(a). On deforming the clot further, we notice a most interesting phenomenon. At a sufficiently high stress, dissolution of the clot takes place and the viscosity decreases significantly returning close to its original value as depicted in Figure 10(b). Thus, in general “activation” can lead to either an increase or decrease in viscosity over a very short space of time whereby we can think of it as a jump. See [2] for more details.

Pressure-thickening fluids – fluids with pressure dependent viscosities. Except for our discussion above concerning the behavior of fluids due to activation as a consequence

[7]Blood is not a single constituent fluid, though it is often modeled as one. It is a complex mixture that consists in a variety of cells, proteins, plasma and other biological matter. Plasma exhibits Newtonian behavior, while blood, when modeled as a single constituent fluid, behaves like a Newtonian fluid in large arteries but exhibits pronounced non-Newtonian behavior in small blood vessels.

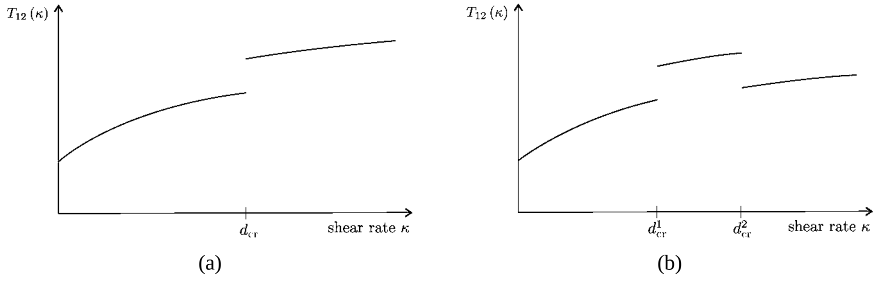

Fig. 10. Activation and deactivation of fluids with shear dependent viscosity modelled as jump discontinuities in stress.

of chemical reactions, the above departures from Newtonian response are at the heart of what is usually referred to as non-Newtonian fluid mechanics. We now turn to a somewhat different departure from the classical Newtonian model. Notice that the model (A.1.3) is an explicit expression for the stress, in terms of kinematical variable $\mathbf{D}$, and the density ρ, while (A.1.4) provides an explicit relationship $\mathbf{D}$ and $\mathbf{T}$ as $p = -\frac{1}{3}\operatorname{tr}\mathbf{T}$. If the equation of state relating the "thermodynamic pressure" p and the density ρ is invertible, then we could express λ and μ as functions of the pressure. Thus, in the case of a compressible Navier–Stokes fluid the viscosity μ clearly depends on the pressure. The question to ask is if, in fluids that are usually considered as incompressible liquids such as water under normal operating conditions, the viscosity could be a function of the pressure? The answer to this question is an unequivocal yes in virtue of the fact that when the range of pressures to which the fluid is subject to is sufficiently large, while the density may vary by a few percent, the viscosity could vary by several orders of magnitude, in fact by as much a factor of 10^8! Thus, it is reasonable to suppose a liquid to be incompressible while at the same time to assume that the viscosity is pressure dependent.

In the case of an incompressible fluid whose viscosity depends on both the pressure (mean normal stress) and the symmetric part of the velocity gradient, i.e., when the stress is given by the representation

$$\mathbf{T} = -p\mathbf{I} + \mu(p, \mathbf{D})\mathbf{D}, \tag{A.1.9}$$

as $p = -\frac{1}{3}\operatorname{tr}\mathbf{T}$, it becomes obvious that we have an implicit relationship between $\mathbf{T}$ and $\mathbf{D}$, and the constitutive relation is of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}, \tag{A.1.10}$$

i.e., we have an implicit constitutive equation.

It immediately follows from (A.1.10) that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{T}}\dot{\mathbf{T}} + \frac{\partial \mathbf{f}}{\partial \mathbf{D}}\dot{\mathbf{D}} = \mathbf{0}, \tag{A.1.11}$$

which can be expressed as

$$\big[\mathbf{A}(\mathbf{T},\mathbf{D})\big]\dot{\mathbf{T}} + \big[\mathbf{B}(\mathbf{T},\mathbf{D})\big]\dot{\mathbf{D}} = \mathbf{0}. \tag{A.1.12}$$

The constitutive relation (A.1.12) is on the one hand more general than (A.1.10) in that an implicit equation of the form (A.1.12) may not be integrable to yield an equation of the form (A.1.10); on the other hand (A.1.12) presupposes the differentiality of $\mathbf{T}$ and $\mathbf{D}$ with respect to time, an assumption that is more demanding than that required of $\mathbf{T}$ and $\mathbf{D}$ in (A.1.10).

A further generalization within the context of implicit constitutive relations for compressible bodies is the equation

$$\mathbf{g}(\rho,\mathbf{T},\mathbf{D}) = \mathbf{0}. \tag{A.1.13}$$

Before we get into a more detailed discussion of implicit models for fluids let us consider a brief history of fluids with pressure dependent viscosity. Stokes [139] recognized that in general the viscosity of a fluid could depend upon the pressure. It is clear from his discussion that he is considering liquids such as water. Having recognized the dependence of the viscosity on the pressure, he makes the simplifying assumption “If we suppose μ to be independent of the pressure also, and substitute…”. Having made the assumption that the viscosity is independent of the pressure, he feels the need to substantiate that such is indeed the case for a restricted class of flows, those in pipes and channels (he bases his rationale on the experiments of Du Buat [27]). He remarks: “Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Du Buat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. … I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure”.

While the range of pressures attained in Du Buat’s experiment might justify the assumption made by Stokes for a certain class of problems, one cannot in general make such an assumption. There are many technologically significant problems such as elastohydrodynamics (see [140]) wherein the fluid is subject to such a range of pressures that the viscosity changes by several orders of magnitude. There is a considerable amount of literature concerning the variation of viscosity with pressure and an exhaustive discussion of the literature before 1931 can be found in the authoritative treatise on the physics of high pressure by Bridgman [13].

Andrade [4] suggested that the viscosity depends on the pressure, density and temperature in the following manner

$$\mu(p,\rho,\theta) = A\rho^{1/2}\exp\left(\frac{B}{\theta}\big(p + D\rho^2\big)\right), \tag{A.1.14}$$

where A, B and D are constants. In processes where the temperature is uniformly constant, in the case of many liquids, it would be reasonable to assume that the liquid is incompressible and the viscosity varies exponentially with the pressure. This is precisely the assumption that is made in studies in elastohydrodynamics.

One can carry out a formal analysis based on standard representation theorems for isotropic functions (see [137]) that requires that (A.1.10) satisfies, for all orthogonal tensors $\mathbf{Q}$,

$$\mathbf{g}\big(\rho, \mathbf{QTQ}^{\mathrm{T}}, \mathbf{QDQ}^{\mathrm{T}}\big) = \mathbf{Qg}(\rho, \mathbf{T}, \mathbf{D})\mathbf{Q}^{\mathrm{T}}.$$

It then follows that one obtains an implicit constitutive relation of the form

$$\begin{aligned}&\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD} + \mathbf{DT})\\ &\quad + \alpha_6\big(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2\big) + \alpha_7\big(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}\big) + \alpha_8\big(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2\big) = \mathbf{0},\end{aligned} \tag{A.1.15}$$

where the material moduli α_i, $i = 0, \dots, 8$, depend on

$$\begin{aligned}&\rho, \quad \operatorname{tr}\mathbf{T}, \quad \operatorname{tr}\mathbf{D}, \quad \operatorname{tr}\mathbf{T}^2, \quad \operatorname{tr}\mathbf{D}^2, \quad \operatorname{tr}\mathbf{T}^3, \quad \operatorname{tr}\mathbf{D}^3,\\ &\operatorname{tr}(\mathbf{TD}), \quad \operatorname{tr}\big(\mathbf{T}^2\mathbf{D}\big), \quad \operatorname{tr}\big(\mathbf{D}^2\mathbf{T}\big), \quad \operatorname{tr}\big(\mathbf{T}^2\mathbf{D}^2\big).\end{aligned}$$

The model

$$\mathbf{T} = -p(\rho)\mathbf{I} + \beta\big(\rho, \operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}^2\big)\mathbf{D}$$

is a special subclass of models of the form (A.1.15). The counterpart in the case of an incompressible fluid would be

$$\mathbf{T} = -p\mathbf{I} + \mu\big(p, \operatorname{tr}\mathbf{D}^2\big)\mathbf{D}, \qquad \operatorname{tr}\mathbf{D} = 0. \tag{A.1.16}$$

We shall later provide a thermodynamic basis for the development of the model (A.1.16).

2. Balance equations

2.1. *Kinematics*

We shall keep our discussion of kinematics to a bare minimum. Let $\mathcal{B}$ denote the abstract body and let $\kappa : \mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is a three-dimensional Euclidean space, be a placer and $\kappa(\mathcal{B})$ the configuration (placement) of the body. We shall assume that the placer is one to one. By a motion we mean a one parameter family of placers (see [101]). It follows that if $\kappa_{\mathrm{R}}(\mathcal{B})$ is some reference configuration, and $\kappa_t(\mathcal{B})$ a configuration at time t, then we can identify the motion with a mapping $\chi_{\kappa_{\mathrm{R}}} : \kappa_{\mathrm{R}}(\mathcal{B}) \times \mathbb{R} \to \kappa_t(\mathcal{B})$ such that[8]

$$x = \chi_{\kappa_{\mathrm{R}}}(X, t). \tag{A.2.1}$$

[8]It is customary to denote x and X which are points in a Euclidean space in bold face. We however choose not to do so. On the other hand, all vectors and higher-order tensors are indicated by bold face.

We shall suppose that $\chi_{\kappa_{\mathrm{R}}}$ is sufficiently smooth to render the operations defined on it meaningful. Since $\chi_{\kappa_{\mathrm{R}}}$ is one to one, we can define its inverse so that

$$X = \chi_{\kappa_{\mathrm{R}}}^{-1}(x,t). \tag{A.2.2}$$

Thus, any (scalar) property φ associated with an abstract body $\mathcal{B}$ can be expressed as (analogously we proceed for vectors or tensors)

$$\varphi = \varphi(P,t) = \hat{\varphi}(X,t) = \tilde{\varphi}(x,t). \tag{A.2.3}$$

We define the following Lagrangian and Eulerian temporal and spatial derivatives:

$$\dot{\varphi} := \frac{\partial\hat{\varphi}}{\partial t}, \qquad \varphi_{,t} := \frac{\partial\tilde{\varphi}}{\partial t}, \qquad \nabla_X\varphi = \frac{\partial\hat{\varphi}}{\partial X}, \qquad \nabla_x\varphi := \frac{\partial\tilde{\varphi}}{\partial x}. \tag{A.2.4}$$

The Lagrangian and Eulerian divergence operators will be expressed as Div and div, respectively. We shall dispense with "˜" and "ˆ", the use of the Lagrangian or Eulerian representation being obvious from the context.

The velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are defined through

$$\mathbf{v} = \frac{\partial\chi_{\kappa_{\mathrm{R}}}}{\partial t}, \qquad \mathbf{a} = \frac{\partial^2\chi_{\kappa_{\mathrm{R}}}}{\partial t^2}, \tag{A.2.5}$$

and the deformation gradient $\mathbf{F}_{\kappa_{\mathrm{R}}}$ is defined through

$$\mathbf{F}_{\kappa_{\mathrm{R}}} = \frac{\partial\chi_{\kappa_{\mathrm{R}}}}{\partial X}. \tag{A.2.6}$$

The velocity gradient $\mathbf{L}$ and its symmetric part $\mathbf{D}$ are defined through

$$\mathbf{L} = \nabla_x\mathbf{v}, \qquad \mathbf{D} = \frac{1}{2}\big(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\big). \tag{A.2.7}$$

It immediately follows that

$$\mathbf{L} = \dot{\mathbf{F}}_{\kappa_{\mathrm{R}}}\mathbf{F}_{\kappa_{\mathrm{R}}}^{-1}. \tag{A.2.8}$$

It also follows from the notation and definitions given above, in particular from (A.2.4) and (A.2.5), that

$$\dot{\varphi} = \varphi_{,t} + \nabla_x\varphi\cdot\mathbf{v}. \tag{A.2.9}$$

2.2. *Balance of mass – incompressibility – inhomogeneity*

The balance of mass in its Lagrangian form states that

$$\int_{\mathcal{P}_{\mathrm{R}}}\rho_{\mathrm{R}}(X)\,\mathrm{d}X = \int_{\mathcal{P}_t}\rho(x,t)\,\mathrm{d}x \quad \text{for all } \mathcal{P}_{\mathrm{R}}\subset\kappa_{\mathrm{R}}(\mathcal{B}) \text{ with } \mathcal{P}_t := \chi_{\kappa_{\mathrm{R}}}(\mathcal{P}_{\mathrm{R}},t), \tag{A.2.10}$$

which immediately leads to, using the substitution theorem,

$$\rho(x,t)\det \mathbf{F}_{\kappa_{\mathrm{R}}}(X,t) = \rho_{\mathrm{R}}(X). \tag{A.2.11}$$

A body is *incompressible* if

$$\int_{\mathcal{P}_{\mathrm{R}}} \mathrm{d}X = \int_{\mathcal{P}_t} \mathrm{d}x \quad \text{for all } \mathcal{P}_{\mathrm{R}} \subset \kappa_{\mathrm{R}}(\mathcal{B}),$$

which leads to

$$\det \mathbf{F}_{\kappa_{\mathrm{R}}}(X,t) = 1 \quad \text{for all } X \in \kappa_{\mathrm{R}}(\mathcal{B}). \tag{A.2.12}$$

If $\det \mathbf{F}_{\kappa_{\mathrm{R}}}$ is continuously differentiable with respect to time, then by virtue of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\det \mathbf{F}_{\kappa_{\mathrm{R}}} = \operatorname{div}\mathbf{v}\det \mathbf{F}_{\kappa_{\mathrm{R}}},$$

we conclude, since $\det \mathbf{F}_{\kappa_{\mathrm{R}}} \neq 0$ that

$$\operatorname{div}\mathbf{v}(x,t) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \kappa_t(\mathcal{B}). \tag{A.2.13}$$

It is usually in the above form that the constraint of incompressibility is enforced in fluid mechanics.

From the Eulerian perspective, the balance of mass takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}_t} \rho\,\mathrm{d}x = 0 \quad \text{for all } \mathcal{P}_t \subset \kappa_t(\mathcal{B}). \tag{A.2.14}$$

It immediately follows that

$$\rho_{,t} + (\nabla_x \rho)\cdot\mathbf{v} + \rho\operatorname{div}\mathbf{v} = 0 \quad \Longleftrightarrow \quad \rho_{,t} + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{A.2.15}$$

If the fluid is incompressible, it immediately follows from (A.2.15) that

$$\begin{aligned} \rho_{,t} + (\nabla_x \rho)\cdot\mathbf{v} = 0 \quad &\Longleftrightarrow \quad \dot{\rho} = 0 \\ &\Longleftrightarrow \quad \rho(t,x) = \rho(0,X) = \rho(0,P) = \rho_{\mathrm{R}}(X). \end{aligned} \tag{A.2.16}$$

That is, for a fixed particle, the density is constant, as a function of time. However, the density of a particle may vary from one particle to another. The fact that the density varies over a certain region of space does not imply that the fluid is not incompressible. This variation is due to the fact that the fluid is inhomogeneous, a concept that has not been grasped clearly in many studies devoted to Mechanics of Fluids (see [3] for a discussion).

2.3. *Balance of linear momentum*

The balance of linear momentum for a continuum is a generalization of the second law of Newton in classical mechanics, and when applied to each subset $\mathcal{P}_t = \chi_{\kappa_R}(\mathcal{P}_R, t)$ of the current configuration takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}_t} \rho\mathbf{v}\,\mathrm{d}x = \int_{\mathcal{P}_t} \rho\mathbf{b}\,\mathrm{d}x + \int_{\partial\mathcal{P}_t} \mathbf{T}^{\mathrm{T}}\mathbf{n}\,\mathrm{d}S, \tag{A.2.17}$$

where $\mathbf{T}$ denotes the Cauchy stress that is related to the surface traction $\mathbf{t}$ through $\mathbf{t} = \mathbf{T}^{\mathrm{T}}\mathbf{n}$ and $\mathbf{b}$ denotes the specific body force. It then leads to the balance of linear momentum in its local Eulerian form

$$\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T}^{\mathrm{T}} + \rho\mathbf{b}. \tag{A.2.18}$$

Two comments are in order.

First, when considering the case $\kappa_t(\mathcal{B}) = \kappa_R(\mathcal{B})$ for all $t \geqslant 0$, on setting $\Omega := \kappa_R(\mathcal{B})$, it is not difficult to conclude for incompressible fluids, that (A.2.17) and (A.2.14) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{O} \rho\mathbf{v}\,\mathrm{d}x + \int_{\partial O}\big[(\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n}) - \mathbf{T}^{\mathrm{T}}\mathbf{n}\big]\,\mathrm{d}S = \int_{O}\rho\mathbf{b}\,\mathrm{d}x \tag{A.2.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{O} \rho\,\mathrm{d}x + \int_{\partial O}\rho(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}S = 0 \tag{A.2.20}$$

valid for all (fixed) subsets O of Ω.

When compared to (A.2.17), this formulation is more suitable for further consideration in those problems where the velocity field $\mathbf{v}$ is taken as a primitive field defined on $\Omega \times \langle 0, \infty)$ (i.e., it is not defined through (A.2.5)).

To illustrate this convenience, we give a simple analogy from classical mechanics: consider a motion of a mass-spring system described by the second-order ordinary differential equations for displacement of the mass from its equilibrium position and compare it with a free fall of the mass captured by the first-order ordinary differential equations for the velocity.

Second, the derivation of (A.2.18) from (A.2.17) and similarly (A.2.15) from (A.2.14) requires certain smoothness of particular terms. In analysis, the classical formulations of the balance equations (A.2.18) and (A.2.15) are usually starting points for definition of various kinds of solutions. Following Oseen [103] (see also [35,36]), we want to emphasize that the notion of a weak solution (or suitable weak solution) is very natural for the equations of continuum mechanics, since their weak formulation can be directly obtained from the original formulations of the balance laws (A.2.14) and (A.2.17) or better still (A.2.19) and (A.2.20). This observation is equally applicable to the other balance equations of continuum physics as well.

2.4. *Balance of angular momentum*

In the absence of internal couples, the balance of angular momentum implies that the Cauchy stress is symmetric, i.e.,

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}. \tag{A.2.21}$$

2.5. *Balance of energy*

We shall merely record the local form of the balance of energy which is

$$\rho\dot{\epsilon} = \mathbf{T} \cdot \nabla_x \mathbf{v} - \operatorname{div} \mathbf{q} + \rho r, \tag{A.2.22}$$

where ϵ denotes the specific internal energy, $\mathbf{q}$ denotes the heat flux vector and r denotes the specific radiant heating.

2.6. *Further thermodynamic considerations (the second law). Reduced dissipation equation*

To know how a body is constituted and to distinguish one body from another, we need to know how bodies store energy. How, and how much of, this energy that is stored in a body can be recovered from the body? How much of the working on a body is converted to energy in thermal form (heat)? What is the nature of the latent energy that is associated with the changes in phase that the body undergoes? What is the nature of the latent energy (which is different in general from latent heat, see Rajagopal and Srinivasa [114])? By what different means does a body produce the entropy? These are but few of the pieces of information that one needs to know in order to describe the response of the body. Merely knowing this information is insufficient to describe how the body will respond to external stimuli. One needs to know the exact role these quantities play in determining the response of the body. A body's response has to meet the basic balance laws of mass, linear and angular momentum, energy and the second law of thermodynamics.

Various forms for the second law of thermodynamics have been proposed and are associated with the names of Kelvin, Plank, Claussius, Duhem, Carathéodory and others. Essentially, the second law states that the rate of entropy production has to be nonnegative.[9] A special form of the second law, the Claussius–Duhem inequality, has been used, within the context of a continua, to obtain restrictions on allowable constitutive relations (see [21]). This is enforced by allowing the body to undergo arbitrary processes in which the second law is required to hold. The problem with such an approach is that the constitutive structure that we ascribe to a body is only meant to hold for a certain class of processes. The body might behave quite differently outside this class of processes. For instance, while rubber may behave like an elastic material in the sense that the stored energy depends only

[9]There is a disagreement as to whether this inequality ought to be enforced locally at every point in the body, or only globally, even from the point of view of statistical thermodynamics.

on the deformation gradient and this energy can be completely recovered in processes that are reasonably slow in some sense, the same rubber if deformed at exceedingly high strain rates crystallizes and not only does the energy that is stored not depend purely on the deformation gradient, all the energy that is supplied to the body cannot be recovered. Thus, the models for rubber depend on the process class one has in mind and this would not allow one to subject the body to arbitrary processes. We thus find it more reasonable to assume the constitutive structures for the rate of entropy production, based on physical grounds, that are automatically nonnegative, for the process class of interest.

Let us first introduce the second law of thermodynamics in the form

$$\rho\theta\dot{\eta} \geqslant -\operatorname{div}\mathbf{q} + \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} + \rho r, \tag{A.2.23}$$

where η denotes the specific entropy.

On introducing the specific Helmholtz potential ψ through

$$\psi := \epsilon - \theta\eta,$$

and using the balance of energy (A.2.22), we can express (A.2.23) as

$$\mathbf{T}\cdot\mathbf{L} - \rho\dot{\psi} - \rho\dot{\theta}\eta - \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} \geqslant 0. \tag{A.2.24}$$

The above inequality is usually referred to as the rate of dissipation inequality. This inequality is commonly used in continuum mechanics to obtain restrictions on the constitutive relations. A serious difficulty with regard to such an approach becomes immediately apparent. No restrictions whatsoever can be placed on the radiant heating. More importantly, the radiant heating is treated as a quantity that adjusts itself to meet the balance of energy. But this is clearly unacceptable as the radiant heating has to be a constitutive specification. How a body responds to radiant heating is critical, especially in view of the fact that all the energy that our world receives is in the form of electromagnetic radiation which is converted to energy in its thermal form (see [118] for a discussion of these issues). As we shall be primarily interested in the mechanical response of fluids, we shall ignore the radiant heating altogether, but we should bear in mind the above observation when we consider more general processes.

We shall define the specific rate of entropy production ξ through

$$\xi := \mathbf{T}\cdot\mathbf{L} - \rho\dot{\psi} - \rho\dot{\theta}\eta - \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta}. \tag{A.2.25}$$

We shall make constitutive assumptions for the rate of entropy production ξ and require that (A.2.25) hold in all admissible processes (see [49]). Thus, (A.2.25) will be used as a constraint that is to be met in all admissible processes. We shall choose ξ so that it is nonnegative and thus the second law is automatically met.

We now come to a crucial step in our thermodynamic considerations. From among a class of admissible nonnegative rate of entropy productions, we choose that which is maximal. This is asking a great deal more than the second law of thermodynamics. The rationale for the same is the following. Let us consider an isolated system. For such a system, it is well accepted that its entropy becomes a maximum and the system would reach equilibrium. The assumption that the rate of entropy production is a maximum ensures that the body attains its equilibrium as quickly as possible. Thus, this assumption can be viewed as an assumption of economy or an assumption of laziness, the system tries to get to the equilibrium state as quickly as possible, i.e., in the most economic manner. It is important to recognize that this is merely an assumption and not some deep principle of physics. The efficacy of the assumption has to be borne out by its predictions and to date the assumption has led to meaningful results in predicting the response of a wide variety of materials (see results pertinent to viscoelasticity [115,116], classical plasticity [112,113], twinning [110,111], solid to solid phase transition [114]), crystallization in polymers [119,120], response of single crystal supper alloys [106], etc.).

2.7. *Isothermal flows at uniform temperature*

Here, we shall restrict ourselves to flows that take place at constant temperature for the whole period of interest at all points of the body. Consequently, the equations governing such flows for a compressible fluid are

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v}, \qquad \rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{A.2.26}$$

while for an incompressible fluid they take the form

$$\operatorname{div} \mathbf{v} = 0, \qquad \dot{\rho} = 0, \qquad \rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}. \tag{A.2.27}$$

Note also that (A.2.24) and (A.2.25) reduce to

$$\mathbf{T} \cdot \mathbf{D} - \rho \dot{\psi} = \xi \quad \text{and} \quad \xi \geqslant 0, \tag{A.2.28}$$

where the symmetry of $\mathbf{T}$, see (A.2.21), is used.

In order to obtain a feel for the structure of the constitutive quantities appearing in (A.2.28), we consider first the Cauchy stress for the incompressible and the compressible Euler fluid, and then for the incompressible and the compressible Navier–Stokes fluid. Note that Euler fluids are ideal fluids in that there is no dissipation in any process undergone by the fluid, i.e., $\xi \equiv 0$ in all processes.

Compressible Euler fluid. Since $\xi \equiv 0$ and (A.1.1) implies

$$\mathbf{T} \cdot \mathbf{L} = -p(\rho)\mathbf{I} \cdot \mathbf{L} = -p(\rho) \operatorname{tr} \mathbf{L} = -p(\rho) \operatorname{tr} \mathbf{D} = -p(\rho) \operatorname{div} \mathbf{v},$$

the reduced thermomechanical equation (A.2.28) simplifies to

$$\rho \dot{\psi} = -p(\rho) \operatorname{div} \mathbf{v}. \tag{A.2.29}$$

This suggests that it might be appropriate to consider ψ of the form

$$\psi = \Psi(\rho). \tag{A.2.30}$$

In fact, since an ideal fluid is an elastic fluid, it follows that its specific Helmholtz free energy ψ depends only on the deformation gradient $\mathbf{F}$. If we suppose that the symmetry group of a fluid is the unimodular group, then the balance of mass could lead to the conclusion that ψ depends on the density ρ.

Using $(\text{A.2.26})_1$, we then have from (A.2.30)

$$\dot{\psi} = \Psi_{,\rho}(\rho)\dot{\rho} = -\rho\Psi_{,\rho}(\rho)\,\mathrm{div}\,\mathbf{v}, \tag{A.2.31}$$

and we conclude from (A.2.29) and (A.2.31) that

$$p(\rho) = \rho^2\Psi_{,\rho}(\rho). \tag{A.2.32}$$

Incompressible Euler fluid. Since we are dealing with a homogeneous fluid we have $\rho \equiv \rho^*$, where ρ^* is a positive constant. We also have

$$\dot{\Psi}\left(\rho^*\right) = 0, \qquad \mathbf{T}\cdot\mathbf{L} = -p\mathbf{I}\cdot\mathbf{L} = -p(\rho)\,\mathrm{div}\,\mathbf{v} = 0 \quad \text{and} \quad \xi \equiv 0.$$

Thus, each term in (A.2.28) vanishes and (A.2.28) clearly holds.

Compressible Navier–Stokes fluid. Consider $\mathbf{T}$ of the form (A.1.3) and ψ of the form (A.2.30) fulfilling (A.2.32). Denoting by $\mathbf{C}^\delta$ the deviatoric (traceless) part of any tensor $\mathbf{C}$, i.e., $\mathbf{C}^\delta = \mathbf{C} - \frac{1}{3}(\mathrm{tr}\,\mathbf{C})\mathbf{I}$, we then have

$$\begin{aligned}
\xi &= \mathbf{T}\cdot\mathbf{L} - \rho\dot{\psi} \\
&= -p(\rho)\,\mathrm{div}\,\mathbf{v} + 2\mu(\rho)\mathbf{D}\cdot\mathbf{D} + \lambda(\rho)(\mathrm{tr}\,\mathbf{D})^2 + \rho^2\Psi_{,\rho}(\rho) \\
&= 2\mu(\rho)\mathbf{D}\cdot\mathbf{D} + \lambda(\rho)(\mathrm{tr}\,\mathbf{D})^2 \\
&= 2\mu(\rho)\mathbf{D}^\delta\cdot\mathbf{D}^\delta + \left(\lambda(\rho) + \frac{2}{3}\mu(\rho)\right)(\mathrm{tr}\,\mathbf{D})^2.
\end{aligned}$$

Note that the nonnegativity of the rate of dissipation is met if $\mu(\rho) \geqslant 0$ and $\lambda(\rho) + \frac{2}{3}\mu(\rho) \geqslant 0$.

Incompressible Navier–Stokes fluid. Similar considerations as those for the case of a compressible Navier–Stokes fluid imply

$$\xi = 2\mu\mathbf{D}\cdot\mathbf{D} = 2\mu|\mathbf{D}|^2.$$

Note that for both the incompressible Euler and Navier–Stokes fluid we have

$$p = -\frac{1}{3}\,\mathrm{tr}\,\mathbf{T}.$$

2.8. *Natural configurations*

Most bodies can exist stress free in more than one configuration and such configurations are referred to as "natural configurations" (see [29,109]). Given a current configuration of a homogeneously deformed body, the stress-free configuration that the body takes on upon the removal of all external stimuli is the underlying "natural configuration" corresponding to the current configuration of the body. As a body undergoes a thermodynamic process, in general, the underlying natural configuration evolves. The evolution of this underlying natural configuration is determined by the maximization of entropy production (see how this methodology is used in viscoelasticity [115,116], classical plasticity [112,113], twinning [110,111], solid to solid phase transition [114], crystallization in polymers [119,120], single crystal super alloys [106]). In the case of both incompressible and compressible Navier–Stokes fluids and the generalizations discussed here, the current configuration $\kappa_t(\mathcal{B})$ itself serves as the natural configuration.

3. The constitutive models for compressible and incompressible Navier–Stokes fluids and some of their generalizations

3.1. *Standard approach*

The starting point for the development of the model for a homogeneous compressible Navier–Stokes fluid is the assumption that the Cauchy stress depends on the density and the velocity gradient, i.e.,

$$\mathbf{T} = \mathbf{f}(\rho, \mathbf{L}). \tag{A.3.1}$$

It follows from the assumption of frame-indifference that the stress can depend on the velocity gradient only through its symmetric part, i.e.,

$$\mathbf{T} = \mathbf{f}(\rho, \mathbf{D}). \tag{A.3.2}$$

The requirement that the fluid be isotropic then implies that (in fact, frame-indifference is itself sufficient to obtain the following result)

$$\mathbf{f}(\rho, \mathbf{D}) = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{D} + \alpha_3 \mathbf{D}^2, \tag{A.3.3}$$

where $\alpha_i = \alpha_i(\rho, I_{\mathbf{D}}, II_{\mathbf{D}}, III_{\mathbf{D}})$, and

$$I_{\mathbf{D}} = \operatorname{tr}\mathbf{D}, \qquad II_{\mathbf{D}} = \frac{1}{2}\left[(\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}\mathbf{D}^2\right], \qquad III_{\mathbf{D}} = \det\mathbf{D}.$$

If we require that the stress be linear in $\mathbf{D}$, then we immediately obtain

$$\mathbf{T} = -p(\rho)\mathbf{I} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu(\rho)\mathbf{D}, \tag{A.3.4}$$

which is the classical homogeneous compressible Navier–Stokes fluid.

Starting with the assumption that the fluid is incompressible and homogeneous, and

$$\mathbf{T} = \mathbf{g}(\mathbf{L}), \tag{A.3.5}$$

a similar procedure leads to (see [147])

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}. \tag{A.3.6}$$

The standard procedure for dealing with constraints such as incompressibility, namely that the constraint reactions do no work (see [146]) is fraught with several tacit assumptions (we shall not discuss them here) that restrict the class of models that are possible. For instance, it will not allow the material modulus μ to depend on the Lagrange multiplier p. The alternate approach presented below attempts to avoid such drawbacks. Another general alternative procedure has been recently developed in [117] that establishes the result within a purely mechanical context.

3.2. *Alternate approach*

We provide below an alternate approach for deriving the constitutive relation for homogeneous compressible and incompressible Navier–Stokes fluids. Instead of assuming a constitutive equation for the stress as the starting point, we shall start assuming forms for the Helmholtz potential and the rate of dissipation, namely two scalars.

We first focus on the derivation of the constitutive equation for the Cauchy stress for a compressible Navier–Stokes fluid supposing that

$$\psi(x,t) = \Psi\big(\rho(x,t)\big) \tag{A.3.7}$$

and

$$\begin{aligned}\xi = \Xi(\mathbf{D}) &= 2\mu(\rho)\mathbf{D}\cdot\mathbf{D} + \lambda(\rho)(\operatorname{tr}\mathbf{D})^2\\ &= 2\mu(\rho)\big|\mathbf{D}^\delta\big|^2 + \left(\lambda(\rho) + \frac{2}{3}\mu(\rho)\right)(\operatorname{tr}\mathbf{D})^2,\\ &\text{where } \mu(\rho) \geqslant 0, \lambda(\rho) + \frac{2}{3}\mu(\rho) \geqslant 0.\end{aligned} \tag{A.3.8}$$

With such a choice of ξ the second law is automatically met, and (A.2.28) takes the form (cf. (A.2.31))

$$\xi = \big(\mathbf{T} + \rho^2\Psi_{,\rho}(\rho)\mathbf{I}\big)\cdot\mathbf{D}. \tag{A.3.9}$$

For a fixed $\mathbf{T}$ there are plenty of $\mathbf{D}$'s that satisfy (A.3.8) and (A.3.9). We pick a $\mathbf{D}$ such that $\mathbf{D}$ maximizes (i.e., a process that maximizes ξ subject to the constraint (A.3.9)) (A.3.8) and fulfills (A.3.9). This leads to a constrained maximization that gives the following necessary condition

$$\frac{\partial \Xi}{\partial \mathbf{D}} - \lambda_1\left(\mathbf{T} + \rho^2\Psi_{,\rho}(\rho)\mathbf{I} - \frac{\partial \Xi}{\partial \mathbf{D}}\right) = \mathbf{0},$$

or equivalently,

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial \Xi}{\partial \mathbf{D}} = \big(\mathbf{T} + \rho^2\Psi_{,\rho}(\rho)\mathbf{I}\big). \tag{A.3.10}$$

To determine the Lagrange multiplier that is associated with the constraint we take the scalar product of (A.3.10) with $\mathbf{D}$. Using (A.3.9), (A.3.10) and the fact that

$$\frac{\partial \Xi}{\partial \mathbf{D}} = 2\big(2\mu(\rho)\mathbf{D} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{I}\big), \tag{A.3.11}$$

we find that

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\frac{\partial \Xi}{\partial \mathbf{D}}\cdot\mathbf{D}} = \frac{1}{2}. \tag{A.3.12}$$

Inserting (A.3.11) and (A.3.12) into (A.3.10) we obtain

$$\mathbf{T} = -\rho^2\Psi_{,\rho}(\rho)\mathbf{I} + 2\mu(\rho)\mathbf{D} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{I}. \tag{A.3.13}$$

Finally, setting $p(\rho) = \rho^2\Psi_{,\rho}(\rho)$ we obtain the Cauchy stress for compressible Navier–Stokes fluid, cf. (A.1.3).

Next, we provide a derivation for an hierarchy of incompressible fluid models that generalize the incompressible Navier–Stokes fluid in the following sense: the viscosity may not only be a constant, but it can be a function that may depend on the density, the symmetric part of the velocity gradient $\mathbf{D}$ specifically through $\mathbf{D}\cdot\mathbf{D}$, or the mean normal stress, i.e., the pressure $p := -\frac{1}{3}\operatorname{tr}\mathbf{T}$, or it can depend on any or all of them. We shall consider the most general case within this setting by assuming that

$$\xi = \Xi(p, \rho, \mathbf{D}) = 2\nu(p, \rho, \mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}. \tag{A.3.14}$$

Clearly, if $\nu \geqslant 0$ then automatically $\xi \geqslant 0$, ensuring that the second law is complied with.

We assume that the specific Helmholtz potential ψ is of the form (A.3.7). By virtue of the fact that the fluid is incompressible, i.e.,

$$\operatorname{tr}\mathbf{D} = 0, \tag{A.3.15}$$

we obtain $\dot{\rho} = 0$, $\dot{\psi}$ vanishes in (A.2.28) and we have from (A.2.28)

$$\mathbf{T}\cdot\mathbf{D} = \Xi. \tag{A.3.16}$$

Following the same procedure as that presented above, in case of a compressible fluid, we maximize Ξ with respect to $\mathbf{D}$ that is subject to the constraints (A.3.15) and (A.3.16). As the necessary condition for the extremum we obtain the equation

$$(1+\lambda_1)\Xi_{,\mathbf{D}} - \lambda_1\mathbf{T} - \lambda_0\mathbf{I} = 0, \tag{A.3.17}$$

where λ_0 and λ_1 are the Lagrange multipliers due to the constraints (A.3.15) and (A.3.16). We determine them as follows. Taking the scalar product of (A.3.17) with $\mathbf{D}$, and using (A.3.15) and (A.3.16) we obtain

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\Xi_{,\mathbf{D}}\cdot\mathbf{D}}. \tag{A.3.18}$$

Note that

$$\Xi_{,\mathbf{D}} = 4\big(\nu(p,\rho,\mathbf{D}\cdot\mathbf{D}) + \nu_{,\mathbf{D}}(p,\rho,\mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}\big)\mathbf{D}. \tag{A.3.19}$$

Consequently, $\operatorname{tr}\Xi_{,\mathbf{D}} = 0$ by virtue of (A.3.15). Thus, taking the trace of (A.3.17) we have

$$-\frac{\lambda_0}{\lambda_1} = -p \quad \text{with } p = -\frac{1}{3}\operatorname{tr}\mathbf{T}. \tag{A.3.20}$$

Using (A.3.17)–(A.3.20), we finally find that (A.3.17) takes the form

$$\mathbf{T} = -p\mathbf{I} + 2\nu(p,\rho,\mathbf{D}\cdot\mathbf{D})\mathbf{D}. \tag{A.3.21}$$

Mathematical issues related to the system (A.2.27) with the constitutive equation (A.3.21) will be discussed in the second part of this treatise. The fluid given by (A.3.21) has the ability to shear thin, shear thicken and pressure thicken. After adding the yield stress or activation criterion, the model could capture phenomena connected with the development of discontinuous stresses. On the other hand, the model (A.2.27) together with (A.3.21) cannot stress relax or creep in a nonlinear way, nor can it exhibit nonzero normal stress differences in a simple shear flow.

4. Boundary conditions

No aspect of mathematical modeling has been neglected as that of determining appropriate boundary conditions. Mathematicians seem especially oblivious to the fact that boundary conditions are constitutive specifications. In fact, boundary conditions require an understanding of the nature of the bodies that are divided by the boundary. Boundaries are rarely sharp, with the constituents that abut either side of the boundary invariably exchanging molecules. In the case of the boundary between two liquids or a gas and a liquid this molecular exchange is quite obvious, it is not so in the case of a reasonably impervious solid boundary and a liquid. The ever popular "no-slip" (adherence) boundary condition is supposed to have had the imprimatur of Stokes behind it, but Stokes' opinions concerning the status of the “no-slip” condition are nowhere close to unequivocal as many investigators lead one to believe. A variety of suggestions were put forward by the pioneers of the field, Bernoulli, Du Buat, Navier, Poisson, Girard, Stokes and others, as to the condition that ought to be applied on the boundary between an impervious solid and a liquid. One fact that was obvious to all of them was that boundary conditions ought to be derived, just as constitutive relations are developed for the material in the bulk, even more so. This is made evident by Stokes [139] who remarks: “Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at the boundary”. After emphasizing the need to derive the equations that ought to be applied at a boundary, Stokes [139] goes on to derive a variety of such boundary conditions.

That Stokes [139] was in two minds about the appropriateness of the “no-slip” boundary condition is evident from his following remarks: “Du Buat found by experiment that when the mean velocity of water flowing through a pipe is less than one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me...”, but he goes on to add: “I have said that when the velocity is not small the tangential force called into action by the sliding of water over the inner surface of the pipe varies nearly as the square of the velocity...”. The keywords that demand our attention are “the sliding of water over the inner surface”. Sliding implies that Stokes believed that the fluid is slipping at the boundary. That he was far from convinced concerning the applicability of the “no-slip” condition is made crystal clear when he remarks: “The most interesting questions concerning the subject require for their solution a knowledge of the conditions which must be satisfied at the surface of solid in contact with the fluid, which, except in the case of very small motions, are unknown”. To Stokes the determination of appropriate boundary conditions was an open problem.

An excellent concise history concerning boundary conditions for fluids can be found in [48]. We discuss briefly some of the boundary conditions that have been proposed for a fluid flowing past a solid impervious boundary.

Navier [93] derived a slip condition which can be duly generalized to the condition

$$\mathbf{v}_{\boldsymbol{\tau}} = -K(\mathbf{Tn})_{\boldsymbol{\tau}}, \quad K \geqslant 0, \tag{A.4.1}$$

where $\mathbf{n}$ is the unit outward normal vector and $\mathbf{z}_{\boldsymbol{\tau}}$ stands for $\mathbf{z} - (\mathbf{z} \cdot \mathbf{n})\mathbf{n}$ at the boundary point (note that for internal flows fulfilling $\mathbf{v} \cdot \mathbf{n} = 0$ at the boundary we have $\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v}$); K is usually assumed to be a constant but it could however be assumed to be a function of the normal stresses and the shear rate, i.e.,

$$K = K\big(\mathbf{Tn} \cdot \mathbf{n}, |\mathbf{D}|^2\big). \tag{A.4.2}$$

The above boundary conditions, when $K > 0$, is referred to as the slip boundary condition. If $K = 0$, we obtain the classical “no-slip” boundary condition.

Another boundary condition that is sometimes used, especially when dealing with non-Newtonian fluids, is the "threshold-slip" condition. This takes the form

$$\begin{aligned} &\big|(\mathbf{Tn})_{\boldsymbol\tau}\big| \leqslant \alpha|\mathbf{Tn}\cdot\mathbf{n}| \quad \Longrightarrow \quad \mathbf{v}_{\boldsymbol\tau} = 0, \\ &\big|(\mathbf{Tn})_{\boldsymbol\tau}\big| > \alpha|\mathbf{Tn}\cdot\mathbf{n}| \quad \Longrightarrow \quad \mathbf{v}_{\boldsymbol\tau} \neq 0 \quad \text{and} \quad -\gamma\frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{v}\cdot\mathbf{n}|} = (\mathbf{Tn})_{\boldsymbol\tau}, \end{aligned} \tag{A.4.3}$$

where $\gamma = \gamma(\mathbf{Tn}\cdot\mathbf{n}, \mathbf{v}_{\boldsymbol\tau})$.

The above condition implies that the fluid will not slip until the ratio of the magnitude of the shear stress and the magnitude of the normal stress exceeds a certain value. When it does exceed that value, it will slip and the slip velocity will depend on both the shear and normal stresses. It is also possible to require that γ depends on $|\mathbf{D}|^2$.

A much simpler condition that is commonly used is

$$\mathbf{v}_{\boldsymbol\tau} = \begin{cases} v_{0\boldsymbol\tau} & \text{if } |(\mathbf{Tn})_{\boldsymbol\tau}| > \beta, \\ 0 & \text{if } |(\mathbf{Tn})_{\boldsymbol\tau}| \leqslant \beta. \end{cases} \tag{A.4.4}$$

Thus the fluid will slip if the shear stress exceeds a certain value. Here, $v_{0\boldsymbol\tau}$ is a given function at the boundary.

If the boundary is permeable, then in addition to the possibility of $\mathbf{v}_{\boldsymbol\tau}$ not being equal to zero, we have to specify the normal component of the velocity $\mathbf{v}\cdot\mathbf{n}$. Several flows have been proposed for flows past porous media, however we shall not discuss them here.

In order to understand the characteristic features of the particular terms appearing in the system of PDEs it is convenient to eliminate the presence of the boundary (i.e., the effect of the boundary conditions on the flow). However, it is imperative to recognize that problems that have technological relevance are initial–boundary value problems, and primarily one is interested in understanding how information at the boundary affects the response of the fluid in the interior.

We can eliminate the effect of the boundary in two ways.

1. Assume that the fluid occupies the whole three-dimensional space with the velocity vanishing at $|x| \to +\infty$. Then starting with an initial condition

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 \in \mathbb{R}^3 \tag{A.4.5}$$

we are interested in knowing the properties of the velocity and the pressure at any later instant of the time $t > 0$ and any position $x \in \mathbb{R}^3$.

2. Assume that for $T, L \in (0,\infty)$,

$$\begin{aligned} &v_i, p : [0,T]\times\mathbb{R}^3 \to \mathbb{R} \text{ are } L\text{-periodic along each direction } x_i, \\ &\quad \text{with } \int_\Omega v_i \,\mathrm{d}x = 0, \int_\Omega p \,\mathrm{d}x = 0, i = 1,2,3. \end{aligned} \tag{A.4.6}$$

Here $\Omega = (0,L)\times(0,L)\times(0,L)$ is a periodic cell.

The advantage of the second case consists in the fact that we work on domain with a compact closure.

Part B. Mathematical analysis of flows of fluids with shear, pressure and density dependent viscosity

1. Introduction

1.1. *A taxonomy of models*

The objective of this part is to provide a survey of results regarding the mathematical analysis of the system of partial differential equations for the (unknown) density ρ, the velocity $\mathbf{v} = (v_1, v_2, v_3)$ and the pressure (mean normal stress) p, the partial differential equations being

$$\begin{aligned} &\rho_{,t} + \nabla\rho \cdot \mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v} = 0, \\ &\rho\big(\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v})\big) = -\nabla p + \operatorname{div}\big(2\nu\big(p, \rho, \left|\mathbf{D}(\mathbf{v})\right|^2\big)\mathbf{D}(\mathbf{v})\big) + \rho\mathbf{b}, \end{aligned} \tag{B.1.1}$$

focusing however mostly on some of its simplifications specified below. The system (B.1.1) is exactly the system (A.2.27) with the constitutive equations (A.3.21) whose interpretation from the perspective of non-Newtonian fluid mechanics and the connection to compressible fluid models were discussed in Part A. In contrast to (A.2.27) and (A.3.21) we use a different notation in order to express the equations in the form (B.1.1). First of all, in virtue of the constraint of incompressibility, we have

$$\dot{\mathbf{v}} = \mathbf{v}_{,t} + [\nabla_x \mathbf{v}]\mathbf{v} = \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}),$$

where the tensor product $\mathbf{a}\otimes\mathbf{b}$ is the second-order tensor with components

$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j \quad \text{for any } \mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3).$$

Next note that in virtue of $(\text{B.1.1})_2$, we can rewrite $(\text{B.1.1})_1$ as $\rho_{,t} + \operatorname{div}(\rho\mathbf{v}) = 0$. We also explicitly use the notation $\mathbf{D}(\mathbf{v})$ instead of $\mathbf{D}$ in order to clearly identify our interest concerning the velocity field. As discussed in Part A, the model (B.1.1) includes a lot of special important cases particularly for homogeneous fluids.[10] Note that for the case of a homogeneous fluid, (B.1.1) reduces to

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) = -\nabla p + \operatorname{div}\big(2\nu\big(p, \left|\mathbf{D}(\mathbf{v})\right|^2\big)\mathbf{D}(\mathbf{v})\big) + \mathbf{b}, \tag{B.1.2}$$

obtained by multiplying $(\text{B.1.1})_2$ by $1/\rho_0$, and relabeling the dynamic pressure p/ρ_0 and the dynamic viscosity $\nu(p, \rho_0, |\mathbf{D}(\mathbf{v})|^2)/\rho_0$ again as p and $\nu(p, |\mathbf{D}(\mathbf{v})|^2)$, respectively. For later reference, we give a list of several models contained as a special subclasses of (B.1.2).

[10] Recall that in our setting a fluid is homogeneous if for some positive number ρ_0 the density fulfills $\rho(x,t) = \rho_0$ for all time instants $t \geqslant 0$ and all $x \in \kappa_t(\mathcal{B})$, as the viscosity does not otherwise depend on $P \in \mathcal{B}$.

(a) *Fluids with pressure dependent viscosity* where ν is independent of the shear rate, but depends on the pressure p,

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\big(\nu(p)\big[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\big]\big) = -\nabla p + \mathbf{b}. \tag{B.1.3}$$

(b) *Fluids with shear dependent viscosity* with the viscosity independent of the pressure,

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) = -\nabla p + \mathbf{b}; \tag{B.1.4}$$

here we introduce the notation

$$\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) := 2\nu\big(\big|\mathbf{D}(\mathbf{v})\big|^2\big)\mathbf{D}(\mathbf{v}). \tag{B.1.5}$$

This class of fluids includes:

(c) *Ladyzhenskaya's fluids*[11] with $\nu(|\mathbf{D}(\mathbf{v})|^2) = \nu_0 + \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$, where $r > 2$ is fixed, ν_0 and ν_1 are positive numbers,

$$\begin{aligned} &\operatorname{div}\mathbf{v} = 0,\\ &\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \nu_0\Delta\mathbf{v} - 2\nu_1\operatorname{div}\big(\big|\mathbf{D}(\mathbf{v})\big|^{r-2}\mathbf{D}(\mathbf{v})\big) = -\nabla p + \mathbf{b}; \end{aligned} \tag{B.1.6}$$

(d) *power-law fluids* with $\nu(|\mathbf{D}(\mathbf{v})|^2) = \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$ where $r \in (1,\infty)$ is fixed and ν_1 is a positive number,

$$\begin{aligned} &\operatorname{div}\mathbf{v} = 0,\\ &\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - 2\nu_1\operatorname{div}\big(\big|\mathbf{D}(\mathbf{v})\big|^{r-2}\mathbf{D}(\mathbf{v})\big) = -\nabla p + \mathbf{b}; \end{aligned} \tag{B.1.7}$$

(e) *Navier–Stokes fluids* with $\nu(p, |\mathbf{D}(\mathbf{v})|^2) = \nu_0$ (ν_0 being a positive number),

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \nu_0\Delta\mathbf{v} = -\nabla p + \mathbf{b}. \tag{B.1.8}$$

The equations of motions (B.1.8) for a Navier–Stokes fluid are referred to as the Navier–Stokes equations (NSEs), the equations (B.1.6) for Ladyzhenskaya's fluid will be referred to as Ladyzhenskaya's equations. A fluid captured by Ladyzhenskaya's equations reduces to NSEs (B.1.8) by taking $\nu_1 = 0$ in (B.1.6) and to power-law fluids by setting $\nu_0 = 0$. Note also that by setting $r = 2$ in Ladyzhenskaya's equations we again obtain NSEs with the constant viscosity $2(\nu_0 + \nu_1)$.

[11] For $r = 3$ this system of PDEs is frequently called Smagorinski's model of turbulence, see [135]. Then ν_0 is molecular viscosity and ν_1 is the turbulent viscosity.

1.2. *Mathematical self-consistency of the models*

An analyst concerned with the mathematical aspects of physics is primarily interested in questions concerning the *mathematical self-consistency*[12] of the equations that he or she is studying.

We say that a model is mathematically self-consistent if it exhibits at least the following properties:

(I) *Long-time and large-data existence*. When a reasonable set of boundary conditions are added to the governing equations for smooth, but arbitrary initial data, the model should admit a solution for all later instants of time.

(II) *Long-time and large-data uniqueness*. The motion should be fully determined by its initial, boundary and other data and depend on them continuously; particularly, such a motion should be unique for a given set of data.

(III) *Long-time and large-data regularity*. Physical quantities, such as the velocity in the case of fluids, should be bounded.

These three requirements thus form a minimal set of mathematical properties that one would like an *evolutionary* model of (classical) mechanics to exhibit, particularly the models (B.1.3)–(B.1.8).

A discussion of the current status of results with regard to the requirements (I)–(III) for the above models forms the backbone of the remaining part of this article. Towards this purpose, we eliminate the influence of the boundary by considering spatially periodic problem, cf. (A.4.6). On the other hand, we do not apply tools that are just suitable for periodic functions (such as Fourier series) but rather use tools and approaches that can be used under more general conditions for other boundary-value problems, as well.

1.3. *Weak solution: A natural notion of a solution for PDEs of the continuum physics*

The enforcement of the properties (I)–(III) requires a clear definition of what is meant by a solution. We obtain a hint concerning this from the balance of linear momentum for each (measurable) subset of the body (A.2.17), as recognized by Oseen [103]. Note that (A.2.19) requires some integrability of the first derivatives of the velocity and the integrability of the pressure, while the classical formulation[13] (B.1.8) is based on the knowledge of the second derivatives of $\mathbf{v}$ and the gradient of p. Oseen [103] not only observed this discrepancy between (A.2.19) and (B.1.8), but he also proposed and derived a notion of a *weak solution* directly from the integral formulation[14] of the balance of linear momentum (A.2.19).

To be more specific, following the procedure outlined by Oseen [103] (for other approaches see also [35], p. 55, and [36]) it is possible to conclude directly from

[12] See the video of Caffarelli's presentation of the third millennium problem "Navier–Stokes and smoothness" [16].

[13] Oseen in his monograph [103] only treats Navier–Stokes fluids and their linearizations.

[14] See also [35] and [36].

(A.2.19) and (A.2.20) that ρ, $\mathbf{v}$ and $\mathbf{T}$ fulfill, for all $t > 0$,

$$
\begin{aligned}
-\int_0^t\int_\Omega (\rho\mathbf{v})(\tau,x)\cdot\boldsymbol{\varphi}_{,\tau}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega (\rho\mathbf{v})(t,x)\cdot\boldsymbol{\varphi}(t,x)\,\mathrm{d}x& \\
-\int_\Omega (\rho\mathbf{v})(0,x)\cdot\boldsymbol{\varphi}(0,x)\,\mathrm{d}x - \int_0^t\int_\Omega (\rho\mathbf{v}\otimes\mathbf{v})\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau& \\
+\int_0^t\int_\Omega \mathbf{T}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau& \\
=\int_0^t\int_\Omega \rho\mathbf{b}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau& \qquad \text{(B.1.9)}
\end{aligned}
$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, +\infty; (\mathcal{C}^\infty_{\mathrm{per}})^3)$ and

$$
\begin{aligned}
-\int_0^t\int_\Omega \rho(\tau,x)\xi_{,\tau}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \rho(t,x)\xi(t,x)\,\mathrm{d}x& \\
-\int_\Omega \rho(0,x)\xi(0,x)\,\mathrm{d}x - \int_0^t\int_\Omega \rho\mathbf{v}\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}\tau = 0& \qquad \text{(B.1.10)}
\end{aligned}
$$

for all $\xi \in \mathcal{D}(-\infty, +\infty; \mathcal{C}^\infty_{\mathrm{per}})$.

Identities (B.1.9) and (B.1.10) are precisely the equations that are used to obtain solutions in the weak form to equations (B.1.1). Neither Oseen nor later on Leray [75] used the word "weak" in their interpretation of the solution, but both of them worked with it. While Oseen established the results concerning local-in-time existence, uniqueness and regularity for large data, Leray [75] proved long-time and large-data existence for weak solutions of the Navier–Stokes equations, thus verifying (I), but leaving open the determination of (II) and (III). These issues are still unresolved to our knowledge. The establishment of the properties (II) and (III) for the Navier–Stokes equation (B.1.8) represents the third millennium problem of the Clay Mathematical Institute [34].

The next issue concerns the function spaces where the solution satisfying (B.1.9) and (B.1.10) are to be found.

There is an interesting link between the constitutive theory via the maximization of entropy production presented in Part A and the choice of function spaces where the weak solutions are constructed. We showed earlier how the form of the constitutive equation for the Cauchy stress can be determined knowing the constitutive equations for the specific Helmholtz free energy ψ and for the rate of dissipation ξ by maximizing w.r.t. $\mathbf{D}$'s fulfilling the reduced thermomechanical equation and the divergenceless condition as the constraint. Here, we show that the form of ψ determines the appropriate function spaces for ρ, while the form of ξ determines those for $\mathbf{v}$. This link becomes even more transparent for more complex problems (see [88], for example).

Consider ψ and ξ of the form

$$\psi = \Psi(\rho) \quad \text{and} \quad \xi = 2\nu(p, \rho, \mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}. \qquad \text{(B.1.11)}$$

Assume that ρ fulfills

$$0 \leqslant \sup_{0\leqslant t\leqslant T} \int_\Omega \rho \Psi\big(\rho(t,x)\big)\,\mathrm{d}x < \infty \tag{B.1.12}$$

and

$$0 \leqslant \int_0^T \int_\Omega \nu\big(p, \rho, \mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})\big)\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{B.1.13}$$

If, for example, $\Psi(\rho) = \rho^\gamma$ with $\gamma > 1$ and $\nu(p, \rho, \mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})) = \nu_0$, then (B.1.12) and (B.1.13) imply that

$$\rho \in L^\infty\big(0, T; L^{\gamma+1}_{\mathrm{per}}\big) \quad \text{and} \quad \mathbf{D}(\mathbf{v}) \in L^2\big(0, T; L^2_{\mathrm{per}}\big). \tag{B.1.14}$$

In general, depending on the specific structure of Ψ, (B.1.12) implies that

$$\rho \in L^\infty(0, T; X_\Psi) \quad \text{for some space } X_\Psi. \tag{B.1.15}$$

If $\Psi(\rho) = \rho^\gamma$, then $X_\Psi = L^{\gamma+1}_{\mathrm{per}}$. Similarly, depending on the form of ν, one can conclude that

$$\mathbf{D}(\mathbf{v}) \in Y_{\mathrm{dis}} \quad \text{or} \quad \mathbf{v} \in X_{\mathrm{dis}}$$
$$\text{for certain function spaces } Y_{\mathrm{dis}} \text{ and } X_{\mathrm{dis}}, \text{ respectively.}$$

In case of the constant viscosity $Y_{\mathrm{dis}} = L^2(0, T; L^2_{\mathrm{per}})$ and $X_{\mathrm{dis}} = L^2(0, T; W^{1,2}_{\mathrm{per}}(\Omega))$.

Note that the reduced thermomechanical equation (A.2.28) requires that

$$\mathbf{T}\cdot\mathbf{D}(\mathbf{v}) = \xi + \rho\dot{\psi} = 2\nu\big(p, \rho, \mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})\big)\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}) + \rho\dot{\Psi}(\rho). \tag{B.1.16}$$

Now, if we formally set $\boldsymbol{\varphi} = \mathbf{v}$ in (B.1.9) we obtain with the help of $(\mathrm{B.1.1})_1$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\mathbf{v}|^2\,\mathrm{d}x + \int_\Omega \mathbf{T}\cdot\mathbf{D}(\mathbf{v})\,\mathrm{d}x = \int_\Omega \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}x \tag{B.1.17}$$

and using (B.1.16) we see that the second term in (B.1.17) can be expressed as

$$\begin{aligned}
&\int_\Omega \mathbf{T}\big(p, \rho, \mathbf{D}(\mathbf{v})\big)\cdot\mathbf{D}(\mathbf{v})\,\mathrm{d}x\\
&\quad = \int_\Omega \Xi\big(p, \rho, \mathbf{D}(\mathbf{v})\big)\,\mathrm{d}x + \int_\Omega \rho\dot{\psi}\,\mathrm{d}x\\
&\quad = \int_\Omega \Xi\big(p, \rho, \mathbf{D}(\mathbf{v})\big)\,\mathrm{d}x + \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(\rho\psi)\,\mathrm{d}x\\
&\quad = \int_\Omega \nu\big(p, \rho, \big|\mathbf{D}(\mathbf{v})\big|^2\big)\big|\mathbf{D}(\mathbf{v})\big|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\psi\,\mathrm{d}x,
\end{aligned} \tag{B.1.18}$$

where we have used the fact that $\dot{\rho} = 0$ (see (B.1.1)).

Assume that ρ_0 and $\mathbf{v}_0$ are Ω-periodic functions satisfying

$$\rho_0 \in X_\Psi \quad \text{and} \quad \alpha_1 \leqslant \rho_0 \leqslant \alpha_2, \tag{B.1.19}$$

$$\mathbf{v}_0 \in L^2(\Omega) \quad \text{and} \quad \operatorname{div} \mathbf{v}_0 = 0, \tag{B.1.20}$$

α_1, α_2 being positive constants. Then, the fact that ρ fulfills the transport equation implies that

$$\alpha_1 \leqslant \rho(x,t) \leqslant \alpha_2 \quad \text{for all } (x,t) \in \Omega \times (0, +\infty). \tag{B.1.21}$$

Consequently, it follows from (B.1.17)–(B.1.20) that (for all $T > 0$)

$$\mathbf{v} \in L^\infty\big(0, T; L^2_{\mathrm{per}}\big) \cap X_{\mathrm{dis}} \quad \text{and} \quad \rho \in L^\infty(0, T; X_\Psi). \tag{B.1.22}$$

The specific description depends on the behavior of the viscosity with respect to $\mathbf{D}$, p and ρ, respectively. See Section 7.3 for further details.

1.4. *Models and their invariance with respect to scaling*

Solutions of the equations for power-law fluids (B.1.7) considered for $r \in (1,3)$ are invariant with respect to the scaling

$$\begin{aligned} \mathbf{v}^\lambda(t,x) &:= \lambda^{(r-1)/(3-r)} \mathbf{v}\big(\lambda^{2/(3-r)} t, \lambda x\big), \\ p^\lambda(t,x) &:= \lambda^{2(r-1)/(3-r)} p\big(\lambda^{2/(3-r)} t, \lambda x\big). \end{aligned} \tag{B.1.23}$$

It means that if $(\mathbf{v}, p)$ solves (B.1.7) with $\mathbf{b} = \mathbf{0}$, then $(\mathbf{v}^\lambda, p^\lambda)$ solves (B.1.7) as well. Note that NSEs also satisfy the invariance with respect to the above scaling by setting $r = 2$ in (B.1.7).

By appealing to this scaling we can magnify the flow near the point of interest located inside the fluid domain. Studying the behavior of the averaged rate of dissipation $d(\mathbf{v})$ defined through

$$d(\mathbf{v}) := \int_{-1}^{0} \int_{\mathcal{B}_1(0)} \xi\big(\mathbf{D}(\mathbf{v})\big) \,\mathrm{d}x \,\mathrm{d}t = 2\nu_1 \int_{-1}^{0} \int_{\mathcal{B}_1(0)} \big|\mathbf{D}(\mathbf{v})\big|^r \,\mathrm{d}x \,\mathrm{d}t \tag{B.1.24}$$

for $d(\mathbf{v}^\lambda)$ as $\lambda \to \infty$, we can give the following classification of the problem:

$$\text{if} \quad d\big(\mathbf{v}^\lambda\big) \to \begin{cases} 0 \\ A \in (0,\infty) \\ \infty \end{cases} \text{as } \lambda \to \infty \quad \text{then the problem is} \begin{cases} \text{supercritical,} \\ \text{critical,} \\ \text{subcritical.} \end{cases}$$

Roughly speaking, we can say that for a subcritical problem the zooming (near a possible singularity) is penalized by $d(\mathbf{v}^\lambda)$ as $\lambda \to \infty$, while for supercritical case the energy dissipated out of the system is an insensitive measure of this magnification.

Because of this, standard regularity techniques based on difference quotient methods should in principle work for subcritical case, while supercritical problems are difficult to handle without any additional information. They are difficult to treat since weak formulations are not suitable for the application of finer regularity techniques that are in place. The Navier–Stokes equations in three spatial dimensions represents a supercritical problem.

In order to overcome this drawback presented by "supercritical" problems with regard to exploiting finer regularity techniques, Caffarelli, Kohn and Nirenberg [17] introduced the notion of a *suitable weak solution*, and established its existence. A key new property of the suitable form of a weak solution is the *local energy inequality*.

For (B.1.1) this is formally achieved by taking a sum of two identities: the first one is obtained by setting $\boldsymbol{\varphi} = \mathbf{v}\phi$ in (B.1.9) and the second one by setting $\xi = \frac{|\mathbf{v}|^2}{2}\phi$ in (B.1.10), with $\phi \in \mathcal{D}(-\infty, +\infty; \mathcal{C}^\infty_{\text{per}})$ satisfying $\phi(x,t) \geqslant 0$ for all t, x. The local energy inequality thus reads

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big(\rho|\mathbf{v}|^2\phi\big)(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega \big(\mathbf{T}\cdot\mathbf{D}(\mathbf{v})\phi\big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqslant \frac{1}{2}\int_\Omega \rho_0(x)\big|\mathbf{v}_0(x)\big|^2\phi(0,x)\,\mathrm{d}x + \frac{1}{2}\int_0^t\int_\Omega \rho|\mathbf{v}|^2(\mathbf{v}\cdot\nabla\phi + \phi_{,t})(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - \int_0^t\int_\Omega (\mathbf{T}\mathbf{v}\cdot\nabla\phi - \rho\mathbf{b}\cdot\mathbf{v}\phi)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \qquad \text{(B.1.25)}
\end{aligned}$$

Using this inequality, Caffarelli, Kohn and Nirenberg [17] were able to improve significantly the characterization of the structure of possible singularities for the Navier–Stokes equations in three dimensions. (Section 6 addresses this issue.)

Since in three spatial dimensions $d(\mathbf{v}^\lambda)$, defined through (B.1.24), fulfills

$$d\big(\mathbf{v}^\lambda\big) = 2\nu_1\lambda^{(5r-11)/(3-r)}\int_{-1/\lambda^{2/(3-r)}}^0\int_{\mathcal{B}_{1/\lambda}(0)}\big|\mathbf{D}\big(\mathbf{v}^1\big)\big|^r\,\mathrm{d}y\,\mathrm{d}\tau, \quad r \in (1,3), \tag{B.1.26}$$

we see that the evolutionary equations for power-law fluids represent a subcritical problem as $r > 11/5$. Thus, the power-law fluid model should be mathematically tractable for $r \geqslant 11/5$. The same is true for Ladyzhenskaya's equations (B.1.6) which has in comparison with the power-law fluid model an even better property: the viscosity $\nu(\mathbf{D}) = \nu_0 + \nu_1|\mathbf{D}|^{r-2}$ and consequently the corresponding nonlinear operator $-\operatorname{div}((\nu_0 + \nu_1|\mathbf{D}(\mathbf{v})|^{r-2})\mathbf{D}(\mathbf{v}))$ are *not* degenerate (while for power-law fluids, $\nu(\mathbf{D}) = \nu_1|\mathbf{D}|^{r-2}$ as $|\mathbf{D}| \to 0$ degenerates when $r > 2$, and becomes singular for $r < 2$).

Ladyzhenskaya's equations with $r \geqslant 11/5$ *are* mathematically self-consistent; long-time existence of weak solutions (property (I)) was proved by Ladyzhenskaya (see [65,67]), she also established long-time uniqueness (property (II)) for $r \geqslant 5/2$. Properties (II) and (III) for $r \geqslant 11/5$ were put into place by Bellout, Bloom and Nečas [9], Málek, Nečas and Růžička [83] and Málek, Nečas, Rokyta and Růžička [82]. While the boundedness of the velocity is implicit in the results presented in the earlier work, it has been explicitly stated for the first time in this chapter. The results in [83] however provide a resolution of the most difficult steps in this direction. Sections 3–5 focus on this topic.

Mathematical self-consistency of Ladyzhenskaya's equations (and some of its generalizations) is the central issue of this contribution. After introducing the notion of a weak solution and a suitable weak solution to equations for fluids with shear dependent viscosity (B.1.4) that includes the NSEs, Ladyzhenskaya's equations and power-law fluids as special cases, we deal with long-time existence of these models in Section 3 and using two methods we establish the existence of a suitable weak solution for $r > 9/5$, and the existence of a weak solution satisfying only the global energy inequality for $r > 8/5$.

Regularity of such a solution is studied in Section 4 and established for $r \geqslant 11/5$. Particularly, if $r \geqslant 11/5$ we conclude that the velocity is bounded. We also outline how higher differentiability techniques can be used as a tool in existence theory. Uniqueness and long-time behavior are addressed in Section 5.

The short Section 6 gives a survey of the results dealing with the structure of possible singularities of flows of a Navier–Stokes fluid.

The final Section 7 presents a brief survey of results concerning long-time and large-data existence for other models, namely homogeneous fluids with pressure dependent viscosity and inhomogeneous fluids with density or shear dependent viscosity.

2. Definitions of (suitable) weak solutions

Before giving a precise formulation of a (suitable) weak solution to the system of PDEs (B.1.4) and (B.1.5) describing unsteady flows of fluids with shear dependent viscosity, we need to specify the assumptions characterizing the structure of the tensor $\mathbf{S} = 2\nu(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$, and to define the function spaces we work with.

2.1. *Assumptions concerning the stress tensor*

Let us compute the expression

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) := \sum_{i,j,k,l=1}^{3} \frac{\partial \mathbf{S}_{ij}(\mathbf{A})}{\partial \mathbf{A}_{kl}} \mathbf{B}_{ij} \mathbf{B}_{kl}$$

for the Cauchy stress corresponding to Ladyzhenskaya's fluids and for power-law fluids. In the case of Ladyzhenskaya's fluid, i.e., when

$$\mathbf{S}(\mathbf{A}) = 2\big(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\big)\mathbf{A}, \quad r > 2, \tag{B.2.1}$$

we have

$$\begin{aligned}&\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \\ &\quad = 2\big(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2 + 2\nu_1 (r-2)|\mathbf{A}|^{r-4}(\mathbf{A} \cdot \mathbf{B})^2,\end{aligned} \tag{B.2.2}$$

which implies that

$$\begin{aligned}&\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \geqslant 2\big(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2 \geqslant C_1\big(1 + |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2 \\ &\quad \text{with } C_1 = 2\min(\nu_0, \nu_1)\end{aligned} \tag{B.2.3}$$

and

$$\begin{aligned}&\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leqslant C_2\big(1 + |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2 \\ &\quad \text{with } C_2 = 2\max\big(\nu_0, \nu_1(r-1)\big),\end{aligned} \tag{B.2.4}$$

while for power-law fluids (set $\nu_0 = 0$ in (B.2.2))

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \geqslant \begin{cases} 2\nu_1 |\mathbf{A}|^{r-2}|\mathbf{B}|^2 & \text{if } r \geqslant 2, \\ 2\nu_1 (r-1)|\mathbf{A}|^{r-2}|\mathbf{B}|^2 & \text{if } r < 2 \end{cases} \tag{B.2.5}$$

and

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leqslant \begin{cases} 2\nu_1 (r-1)|\mathbf{A}|^{r-2}|\mathbf{B}|^2 & \text{if } r \geqslant 2, \\ 2\nu_1 |\mathbf{A}|^{r-2}|\mathbf{B}|^2 & \text{if } r < 2. \end{cases} \tag{B.2.6}$$

Motivated by (B.2.3) and (B.2.4) for Ladyzhenskaya's fluid and (B.2.5) and (B.2.6) for the power-law fluid we make the following assumption concerning $\mathbf{S}$.

Let κ be either 0 or 1. We assume that

$$\mathbf{S} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \quad \text{with } \mathbf{S}(\mathbf{0}) = \mathbf{0} \quad \text{fulfills} \quad \mathbf{S} \in \big[\mathcal{C}^1\big(\mathbb{R}^{3\times 3}\big)\big]^{3\times 3}, \tag{B.2.7}$$

and that there are two positive constants C_1 and C_2 such that for a certain $r \in (1, \infty)$ and for all $\mathbf{0} \neq \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{\text{sym}}$,

$$C_1\big(\kappa + |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2 \leqslant \frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leqslant C_2\big(\kappa + |\mathbf{A}|^{r-2}\big)|\mathbf{B}|^2. \tag{B.2.8}$$

We also assume that $\kappa = 0$ if $r < 2$.

2.2. *Function spaces*

We primarily deal with functions defined on $\mathbb{R}^3$ that are periodic with the periodic cell $\Omega = (0, L)^3$. The space $\mathcal{C}^\infty_{\mathrm{per}}$ consists of smooth Ω-periodic functions.

Let r be such that $1 \leqslant r < \infty$. The Lebesgue space L^r_{per} is introduced as the closure of $\mathcal{C}^\infty_{\mathrm{per}}$-functions with $\int_\Omega f(x)\,\mathrm{d}x = 0$ where the closure is made w.r.t. the norm $\|\cdot\|_r$, where $\|f\|_r^r = \int_\Omega |f(x)|^r\,\mathrm{d}x$. The Sobolev space $W^{1,r}_{\mathrm{per}}$ is the space of Ω-periodic Lebesgue-measurable functions $f:\mathbb{R}^3 \to \mathbb{R}$ such that $\partial_{x_i} f$ exists in a weak sense and f and $\partial_{x_i} f$ belong to L^r_{per}. Both L^r_{per} and $W^{1,r}_{\mathrm{per}}$ are Banach spaces with the norms $\|f\|_r := (\int_\Omega |f(x)|^r\,\mathrm{d}x)^{1/r}$ and $\|f\|_{1,r} := (\int_\Omega |\nabla_x f(x)|^r\,\mathrm{d}x)^{1/r}$, respectively.

Let $(X, \|\cdot\|_X)$ be a Banach space of scalar functions defined on Ω. Then X^3 represents the space of vector-valued functions whose components belong to X. Also, X^* denotes the dual space to X and $\langle\cdot,\cdot\rangle$ the corresponding duality pairing. For $r' = r/(r-1)$, we usually write $W^{-1,r'}_{\mathrm{per}}$ instead of $(W^{1,r}_{\mathrm{per}})^*$.

We also introduce the space $W^{1,r}_{\mathrm{per,div}}$ being a closed subspace of $(W^{1,r}_{\mathrm{per}})^3$ defined as the closure (w.r.t. the norm $\|\cdot\|_{1,r}$) of all smooth Ω-periodic functions $\mathbf{v}$ with the zero mean value such that $\operatorname{div}\mathbf{v} = 0$. Note that $W^{1,r}_{\mathrm{per,div}} = \{\mathbf{v} \in W^{1,r}_{\mathrm{per}}, \operatorname{div}\mathbf{v} = 0\}$.

If Y is any Banach space, $T \in (0,\infty)$ and $1 \leqslant q \leqslant \infty$, then $L^q(0,T;Y)$ denotes the Bochner space formed by functions $g:(0,T) \to Y$ such that, for $1 \leqslant q < \infty$, $\|g\|_{L^q(0,T;Y)} := (\int_0^T \|g(t)\|_Y^q\,\mathrm{d}t)^{1/q}$ is finite. The norm in $L^\infty(0,T;Y)$ is defined as infimum of $\sup_{t\in[0,T]\setminus E} \|g(t)\|_Y$, where infimum is taken over all subsets E of $[0,T]$ having zero Lebesgue measure.

Also, if X is a reflexive Banach space, then X_{weak} denotes the space equipped with the weak topology. Thus, for example,

$$\mathcal{C}(0,T;X_{\mathrm{weak}}) \equiv \bigl\{\boldsymbol{\varphi} \in L^\infty(0,T;X); \bigl\langle\boldsymbol{\varphi}(\cdot), h\bigr\rangle \in \mathcal{C}\bigl(\langle 0,T\rangle\bigr) \text{ for all } h \in X^*\bigr\}.$$

Let $1 < \alpha, \beta < \infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces satisfying $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Then the *Aubin–Lions lemma* [77] says that the space

$$W := \bigl\{v \in L^\alpha(0,T;X_0); v_{,t} \in L^\beta(0,T;X_1)\bigr\}$$

is compactly embedded in $L^\alpha(0,T;X)$, i.e., $W \hookrightarrow\hookrightarrow L^\alpha(0,T;X)$.

2.3. *Definition of the problem* $(\mathcal{P})$ *and its (suitable) weak solutions*

Our main task is to study the mathematical properties of the solutions to the *problem* $(\mathcal{P})$ consisting of

- four partial differential equations (B.1.4) with $\mathbf{S}$ satisfying (B.2.7) and (B.2.8),

- the requirement spatial periodicity (A.4.6),
- the initial condition

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \quad \text{in } \mathbb{R}^3. \tag{B.2.9}$$

Let $T > 0$ be a fixed, but arbitrary number. We assume that the given functions $\mathbf{b}$ and $\mathbf{v}_0$ fulfill

$$\mathbf{b} \in \big(L^r\big(0,T;W^{1,r}_{\mathrm{per}}\big)\big)^* = L^{r'}\big(0,T;W^{-1,r'}_{\mathrm{per}}\big) \tag{B.2.10}$$

and

$$\operatorname{div}\mathbf{v}_0 = 0 \quad \text{in } \big(\mathcal{C}^\infty_{\mathrm{per}}\big)^* \quad \text{and} \quad \mathbf{v}_0 \in L^2_{\mathrm{per}}. \tag{B.2.11}$$

Let (B.2.7) and (B.2.8) hold.

We say that $(\mathbf{v}, p) = (v_1, v_2, v_3, p)$ is a *suitable weak solution* to the problem $(\mathcal{P})$ provided that

$$\mathbf{v} \in \mathcal{C}\big(0,T;L^2_{\mathrm{weak}}(\Omega)\big) \cap L^r\big(0,T;W^{1,r}_{\mathrm{per,div}}\big) \cap L^{5r/3}\big(0,T;L^{5r/3}\big); \tag{B.2.12}$$

$$\mathbf{v}_{,t} \in \begin{cases} L^{r'}\big(0,T;W^{-1,r'}_{\mathrm{per}}\big) \\ L^{5r/6}\big(0,T;W^{-1,5r/6}_{\mathrm{per}}\big) \end{cases} \quad \text{and} \quad p \in \begin{cases} L^{r'}\big(0,T;L^{r'}_{\mathrm{per}}\big) & \text{for } r \geqslant \frac{11}{5}, \\ L^{5r/6}\big(0,T;L^{5r/6}_{\mathrm{per}}\big) & \text{for } r < \frac{11}{5}; \end{cases} \tag{B.2.13}$$

$$\lim_{t\to 0_+} \big\|\mathbf{v}(t) - \mathbf{v}_0\big\|_2^2 = 0; \tag{B.2.14}$$

$$\begin{aligned} &\int_0^T \big\langle \mathbf{v}_{,t}(t), \boldsymbol{\varphi}(t)\big\rangle - \big((\mathbf{v}\otimes\mathbf{v})(t), \nabla\boldsymbol{\varphi}(t)\big) \\ &\qquad + \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}(t)\big)\big), \mathbf{D}\big(\boldsymbol{\varphi}(t)\big)\big) - \big(p(t), \operatorname{div}\boldsymbol{\varphi}(t)\big)\,\mathrm{d}t \\ &\quad = \int_0^T \big\langle \mathbf{b}(t), \boldsymbol{\varphi}(t)\big\rangle\,\mathrm{d}t \quad \text{for all } \boldsymbol{\varphi} \in L^s\big(0,T;W^{1,s}_{\mathrm{per}}\big) \text{ with} \\ &\quad s = r \quad \text{if } r \geqslant \frac{11}{5} \quad \text{and} \quad s = \frac{5r}{5r-6} \quad \text{if } \frac{6}{5} < r < \frac{11}{5}; \end{aligned} \tag{B.2.15}$$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big(|\mathbf{v}|^2\phi\big)(t,x)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\cdot\mathbf{D}(\mathbf{v})\phi\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad \leqslant \frac{1}{2}\int_\Omega \big|\mathbf{v}_0(x)\big|^2\phi(0,x)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \frac{|\mathbf{v}|^2}{2}\phi_{,t}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad + \int_0^t \langle \mathbf{b}, \mathbf{v}\phi\rangle\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \left(\frac{|\mathbf{v}|^2}{2}\mathbf{v} + p\mathbf{v} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\mathbf{v}\right)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned} \tag{B.2.16}$$

valid for all $\phi \in \mathcal{D}(-\infty,+\infty;\mathcal{C}^\infty_{\mathrm{per}})$, $\phi \geqslant 0$ and for almost all $t \in (0,T\rangle$.

- If $r \geqslant 3$, the integrability of $\mathbf{v}$ addressed in (B.2.12) can be improved. For example, if $r > 3$, $\mathbf{v}$ being in $L^r(0,T;W^{1,r})$ implies that $\mathbf{v} \in L^r(0,T;\mathcal{C}^{0,(r-3)/(3r)})$. Since our interest is focused on $r \in (1,3)$, we do not discuss the situation $r \geqslant 3$ in what follows.
- Note that (B.2.12) and (B.2.13) ensure that all terms in (B.2.15) make sense for $r > 6/5$, while all the terms in (B.2.16) are finite if $r > 9/5$.
- Note that taking $\phi \equiv 1$, one can conclude from (B.2.16) that the standard energy inequality

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\mathbf{v})\big)\,\mathrm{d}\tau \leqslant \frac{1}{2}\|\mathbf{v}_0\|_2^2 + \int_0^t \langle \mathbf{b}, \mathbf{v}\rangle\,\mathrm{d}\tau \tag{B.2.17}$$

makes sense provided that the right-hand side is finite.

We say that $(\mathbf{v}, p)$ is a *weak solution* to the problem $(\mathcal{P})$ if (B.2.12)–(B.2.15) and (B.2.17) hold.

- For the NSEs, the local energy inequality takes a slightly different form due to the linearity of $\mathbf{S}$ that allows one to perform the integration by parts once more. This gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big(|\mathbf{v}|^2\phi\big)(t,x)\,\mathrm{d}x + \nu_0\int_0^t\!\!\int_\Omega |\nabla\mathbf{v}|^2\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqslant \frac{1}{2}\int_\Omega \big|\mathbf{v}_0(x)\big|^2\phi(0,x)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega p\mathbf{v}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad + \int_0^t \langle \mathbf{b}, \mathbf{v}\phi\rangle\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \frac{|\mathbf{v}|^2}{2}(\phi_{,t} + \nu_0\Delta\phi + \mathbf{v}\cdot\nabla\phi)\,\mathrm{d}\tau
\end{aligned} \tag{B.2.18}$$

valid for all $\phi \in \mathcal{D}(-\infty,+\infty;\mathcal{C}^\infty_{\text{per}})$, $\phi \geqslant 0$, and for almost all $t \in (0,T\rangle$.
- Note that for $r \geqslant 11/5$ we can set $\boldsymbol{\varphi} = \mathbf{v}$ or $\boldsymbol{\varphi} = \mathbf{v}\phi$ in (B.2.15) and derive (B.2.16) and (B.2.17) in the form of an *equality*. Also, for $r \geqslant 11/5$, we have $\mathbf{v} \in \mathcal{C}(0,T;L^2_{\text{per}})$ that follows from the fact that $\mathbf{v} \in L^r(0,T;W^{1,r}_{\text{per}})$ and $\mathbf{v}_{,t} \in (L^r(0,T;W^{1,r}_{\text{per}}))^*$.
- Notice that the assumption (B.2.8) holds for Ladyzhenskaya's equations with $\kappa = 1$, and for power-law fluids with $\kappa = 0$.

2.4. *Useful inequalities*

We first obtain several inequalities that are consequences of (B.2.7) and (B.2.8). Since

$$\begin{aligned}
\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big)\cdot(\mathbf{A}-\mathbf{B}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{S}_{ij}\big(\mathbf{B} + s(\mathbf{A}-\mathbf{B})\big)\,\mathrm{d}s\,(\mathbf{A}-\mathbf{B})_{ij} \\
&= \int_0^1 \frac{\partial\mathbf{S}_{ij}}{\partial\mathbf{A}_{kl}}\big(\mathbf{B} + s(\mathbf{A}-\mathbf{B})\big)(\mathbf{A}-\mathbf{B})_{kl}(\mathbf{A}-\mathbf{B})_{ij}\,\mathrm{d}s,
\end{aligned}$$

it follows from the first inequality in (B.2.8) that

$$\begin{aligned}&\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot(\mathbf{A}-\mathbf{B})\\&\quad\geqslant C_1\int_0^1\big(\kappa+\big|\mathbf{B}+s(\mathbf{A}-\mathbf{B})\big|^{r-2}\big)\,\mathrm{d}s\,|\mathbf{A}-\mathbf{B}|^2\\&\quad\geqslant 0.\end{aligned}\tag{B.2.19}$$

If $r\geqslant 2$, (B.2.19) then implies (see Lemma 5.1.19 in [82] or [24]) that

$$\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot(\mathbf{A}-\mathbf{B})\geqslant C_3\begin{cases}|\mathbf{A}-\mathbf{B}|^r & \text{if } \kappa=0,\\ |\mathbf{A}-\mathbf{B}|^2+|\mathbf{A}-\mathbf{B}|^r & \text{if } \kappa=1.\end{cases}\tag{B.2.20}$$

Consequently, for $\mathbf{A}=\mathbf{D}(\mathbf{u})$ and $\mathbf{B}=\mathbf{D}(\mathbf{v})$, we have

$$\begin{aligned}&\big(\mathbf{S}\big(\mathbf{D}(\mathbf{u})\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big),\mathbf{D}(\mathbf{u}-\mathbf{v})\big)\\&\quad\geqslant C_3\begin{cases}\big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_r^r & \text{if } \kappa=0,\\ \big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_2^2+\big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_r^r & \text{if } \kappa=1.\end{cases}\end{aligned}\tag{B.2.21}$$

If $r<2$ (and $\kappa=0$), we show further that (B.2.19) implies that

$$\begin{aligned}&\big(\mathbf{S}\big(\mathbf{D}(\mathbf{u})\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big),\mathbf{D}(\mathbf{u}-\mathbf{v})\big)\big\|\mathbf{D}(\mathbf{u})+\mathbf{D}(\mathbf{v})\big\|_r^{2-r}\\&\quad\geqslant C_4\big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_r^2.\end{aligned}\tag{B.2.22}$$

Consequently, setting $\mathbf{v}\equiv\mathbf{0}$ in (B.2.21) and (B.2.22) we conclude that

$$\big(\mathbf{S}\big(\mathbf{D}(\mathbf{u})\big),\mathbf{D}(\mathbf{u})\big)\geqslant C_5\big\|\mathbf{D}(\mathbf{u})\big\|_r^r.\tag{B.2.23}$$

In fact, it follows directly from (B.2.19) that $\mathbf{S}$ is strictly monotone, i.e.,

$$\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot(\mathbf{A}-\mathbf{B})>0\quad\text{if } \mathbf{A}\neq\mathbf{B}.$$

Also, if $\mathbf{D}(\mathbf{u}),\mathbf{D}(\mathbf{v})\in L^r(0,T;L^r(\Omega)^{3\times 3})$, (B.2.22) and Hölder's inequality lead to

$$\int_0^T\big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_r^r\,\mathrm{d}t\leqslant C_6\bigg(\int_0^T\big(\mathbf{S}\big(\mathbf{D}(\mathbf{u})\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big),\mathbf{D}(\mathbf{u}-\mathbf{v})\big)\,\mathrm{d}t\bigg)^{r/2}.\tag{B.2.24}$$

To obtain (B.2.22) we use Hölder's inequality and the inequality $|\mathbf{B}+s(\mathbf{A}-\mathbf{B})|\leqslant|\mathbf{A}|+|\mathbf{B}|$ valid for all $s\in(0,1)$ in the following calculation, where $\mathbf{D}(s)$ abbreviates

$\mathbf{D}(\mathbf{v}) + s\mathbf{D}(\mathbf{u}-\mathbf{v})$,

$$\begin{aligned}
&\big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_r^r \\
&\quad = \int_\Omega \big|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})\big|^r \,\mathrm{d}x \\
&\quad = \int_\Omega \bigg[\bigg(\int_0^1 \big|\mathbf{D}(s)\big|^{r-2}\,\mathrm{d}s\bigg)\big|\mathbf{D}(\mathbf{u}-\mathbf{v})\big|^2\bigg]^{r/2} \bigg(\int_0^1 \big|\mathbf{D}(s)\big|^{r-2}\,\mathrm{d}s\bigg)^{-r/2} \mathrm{d}x \\
&\quad \leqslant c\bigg(\int_\Omega \big[\big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) - \mathbf{S}\big(\mathbf{D}(\mathbf{u})\big)\big)\cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\big]\bigg)^{r/2} \\
&\qquad \times \bigg(\int_\Omega \bigg(\int_0^1 \big|\mathbf{D}(s)\big|^{r-2}\,\mathrm{d}s\bigg)^{-r/(2-r)} \mathrm{d}x\bigg)^{(2-r)/2} \\
&\quad \leqslant \bigg(\int_\Omega \big[\big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) - \mathbf{S}\big(\mathbf{D}(\mathbf{u})\big)\big)\cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\big]\bigg)^{r/2} \big\|\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{v})\big\|_r^{(2-r)/2r},
\end{aligned}$$

which implies (B.2.22).

Analogously, we could check that for $r \in (1,2)$ and $\theta \in (\frac{1}{r},1)$ and for $\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}) \in L^r(0,T; L^r(\Omega)^{3\times 3})$,

$$\begin{aligned}
&\int_0^T \big\|\mathbf{D}(\mathbf{u}-\mathbf{v})\big\|_{r\theta}^{r\theta}\,\mathrm{d}t \\
&\quad \leqslant \widetilde{C}_6\bigg(\int_0^T\int_\Omega \big[\big(\mathbf{S}\big(\mathbf{D}(\mathbf{u})\big) - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big)\cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\big]^\theta\,\mathrm{d}x\,\mathrm{d}t\bigg)^{r/2}
\end{aligned} \tag{B.2.25}$$

holds, where $\widetilde{C}_6$ depends on $|\Omega|$ and T, and the L^r norms of $\mathbf{D}(\mathbf{u})$ and $\mathbf{D}(\mathbf{v})$.

It also follows from (B.2.7) and (B.2.8) that

$$\begin{aligned}
\mathbf{S}(\mathbf{A}) = \mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{0}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{S}(s\mathbf{A})\,\mathrm{d}s = \int_0^1 \frac{\partial\mathbf{S}(s\mathbf{A})}{\partial\mathbf{A}}\cdot\mathbf{A}\,\mathrm{d}s \\
&\leqslant C_2\int_0^1 \big(\kappa + s^{r-2}|\mathbf{A}|^{r-2}\big)\,\mathrm{d}s\,|\mathbf{A}| \leqslant C_2\kappa|\mathbf{A}| + C_2\frac{1}{r-1}|\mathbf{A}|^{r-1}.
\end{aligned}$$

Consequently, using the convention that $\kappa = 0$ if $r < 2$, we have

$$\big|\mathbf{S}(\mathbf{A})\big| \leqslant C_2\kappa|\mathbf{A}| + C_2\frac{1}{r-1}|\mathbf{A}|^{r-1} \leqslant C_0\big(\kappa + |\mathbf{A}|\big)^{r-1}. \tag{B.2.26}$$

If $r \in (1,+\infty)$, Korn's inequality states (see [94] or [92]) that there is a $C_N > 0$ such that

$$\|\nabla\mathbf{u}\|_r \leqslant C_N\big\|\mathbf{D}(\mathbf{u})\big\|_r \quad \text{for all } \mathbf{u}\in W^{1,r}_{\mathrm{per}}. \tag{B.2.27}$$

3. Existence of a (suitable) weak solution

3.1. *Formulation of the results and bibliographical notes*

The aim of this section is to establish the following result on long-time and large-data existence of a suitable weak solution to unsteady flows of fluids with shear dependent viscosity.

THEOREM 3.1. *Assume that* $\mathbf{S}: \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ *satisfies* (B.2.7) *and* (B.2.8) *with a fixed parameter* r. *Also, let* $\mathbf{b}$ *and* $\mathbf{v}_0$ *fulfill* (B.2.10) *and* (B.2.11), *respectively. If*

$$r > \frac{8}{5}, \tag{B.3.1}$$

then there is a weak solution to the problem $(\mathcal{P})$.

If in addition

$$r > \frac{9}{5}, \tag{B.3.2}$$

then there is a suitable weak solution to the problem $(\mathcal{P})$.

Finally, if

$$r \geqslant \frac{11}{5}, \tag{B.3.3}$$

then the local energy equality holds and $\mathbf{v} \in \mathcal{C}(0, T; L^2_{\mathrm{per}})$.

Mathematical analysis of (B.1.4) and (B.1.5) was initiated by Ladyzhenskaya (see [65,67]) who proved the existence of weak solutions for $r \geqslant 11/5$ treating homogeneous Dirichlet (i.e., no-slip) boundary condition. Her approach based on a combination of monotone operator theory together with compactness arguments works for easier boundary-value problems, such as the spatially periodic problem or Navier's slip. For the case when the viscosity depends on the full velocity gradient, i.e., $\nu = \nu(|\nabla \mathbf{v}|^2)$, the same results are presented in the book of Lions [77]. For a complementary reading, see [57].

Table 1 gives an overview of the results and methods available in two and more dimensions. We discuss the results in three dimensions in detail.

Note that Theorem 3.1 addresses, as a special case, the existence of a solution for long-time and large-data for the NSEs, the results obtained for the Cauchy problem in the fundamental article by Leray [75], and extended to bounded domains with no-slip boundary conditions by Hopf [53] and to the notion of suitable weak solutions by Caffarelli, Kohn and Nirenberg [17]. The technique of monotone operators [67] or [77] however does not pertain to these results (as (B.3.3) does not include $r = 2$).

Table 1

		$d=2$	$d \geqslant 3$	References
Spatially-periodic problem				
monotone operators + compactness	$\Longrightarrow$	$r \geqslant 2$	$r \geqslant 1 + \frac{2d}{d+2}$	[67,77]
monotone operators + L^∞-function	$\Longrightarrow$	$r > \frac{3}{2}$	$r \geqslant \frac{2(d+1)}{d+2}$	[44]
regularity technique (higher-differentiability)	$\Longrightarrow$	$r > 1$	$r > \frac{3d}{d+2}$	[82]
$\mathcal{C}^{1,\alpha}$-regularity	$\Longrightarrow$	$r \geqslant \frac{4}{3}$	?	[60]
Dirichlet problem (*no-slip boundary*)				
monotone operators + compactness	$\Longrightarrow$	$r \geqslant 2$	$r \geqslant 1 + \frac{2d}{d+2}$	[65,67,77]
regularity technique (higher-differentiability)	$\Longrightarrow$	$r \geqslant \frac{3}{2}$	$r \geqslant 2$ $(d=3)$	[58,84]
$\mathcal{C}^{1,\alpha}$-regularity	$\Longrightarrow$	$r \geqslant 2$	?	[59]

This gap in the existence theory was filled by the result presented by Málek, Nečas, Rokyta and Růžička [82], see [9] and [83] where the result is established for the first time. The method based on the regularity technique to obtain fractional higher differentiability gives, among other results, the existence of a weak solution for r fulfilling (B.3.2). This concerns the spatially periodic problem (A.4.6). For no-slip boundary conditions, existence for $r \geqslant 2$ is established in [84]. The idea behind this method will be explained in Section 4.

Later on in [44], using the fact that the nonlinear operator is strictly monotone and the fact that only L^∞-test functions are permitted in the weak formulation of the problem, Frehse, Málek and Steinhauer extended the existence theory for nonlinear parabolic systems with L^1 right-hand side (see [12]) and proved the existence of a weak solution for $r > 8/5$. In the following subsection the proof of Theorem 3.1 is established using the approach developed in [67] for $r \geqslant 11/5$ and that in [44] for $r \in (\frac{8}{5}, \frac{11}{5})$. Note that the last result for $r \in (\frac{8}{5}, 2)$ for no-slip boundary conditions is not completely solved as yet.

Frehse and Málek conjecture in [43] that one can exploit the restriction that only Lipschitz test functions are admissible and establish the existence of a weak solution for $r > 6/5$. See [45] for details concerning this technique for time independent problems.

3.2. *Definition of an approximate problem* $(\mathcal{P}^{\varepsilon,\eta})$ *and a priori estimates*

Let $\eta > 0$ and $\varepsilon > 0$ be fixed. If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, then $u * \omega^\eta := \frac{1}{\eta^3} \int_{\mathbb{R}^3} \omega(\frac{x-y}{\eta}) u(y)\,\mathrm{d}y$ with $\omega(\cdot) \in \mathcal{D}(B_1(0))$, $\omega \geqslant 0$, ω being radially symmetric, $\int_{B_1(0)} \omega = 1$, is the standard regularization of a function u.

We consider the *problem* $(\mathcal{P}^{\varepsilon,\eta})$ to find $(\mathbf{v}, p) := (\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ such that[15]

$$\operatorname{div}\mathbf{v} + \varepsilon|p|^\alpha p = 0 \quad \text{with} \begin{cases} \alpha = \frac{2-r}{r-1} & \text{for } r \geqslant \frac{11}{5}, \\ \alpha = \frac{5r-12}{6} & \text{for } r < \frac{11}{5}, \end{cases} \tag{B.3.10}$$

$$\mathbf{v}_{,t} + \operatorname{div}\big(\big(\mathbf{v}_{\text{div}} * \omega^\eta\big) \otimes \mathbf{v}\big) - \operatorname{div}\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) = -\nabla p + \mathbf{b}, \tag{B.3.11}$$

and $(\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ are Ω-periodic functions fulfilling (A.4.6) and each $\mathbf{v}^{\varepsilon,\eta}$ starts with the initial value specified in (B.2.9). In (B.3.11), $\mathbf{v}_{\text{div}} = \mathbf{v}^{\varepsilon,\eta}_{\text{div}}$ denotes the projection of $\mathbf{v} = \mathbf{v}^{\varepsilon,\eta}$ into the space of divergenceless functions, i.e., $\mathbf{v}$ and $\mathbf{v}_{\text{div}}$ are related through the Helmholtz decomposition

$$\mathbf{v}^{\varepsilon,\eta} = \mathbf{v}^{\varepsilon,\eta}_{\text{div}} + \nabla g^{\varepsilon,\eta}, \tag{B.3.12}$$

where

$$\operatorname{div}\mathbf{v}^{\varepsilon,\eta}_{\text{div}} = 0, \qquad \mathbf{v}^{\varepsilon,\eta}_{\text{div}} \text{ and } g^{\varepsilon,\eta} \text{ are } \Omega\text{-periodic}, \tag{B.3.13}$$

and

$$-\Delta g^{\varepsilon,\eta} = -\operatorname{div}\mathbf{v}^{\varepsilon,\eta} = \varepsilon\big|p^{\varepsilon,\eta}\big|^\alpha p^{\varepsilon,\eta}, \qquad \int_\Omega g^{\varepsilon,\eta}\,\mathrm{d}x = 0. \tag{B.3.14}$$

[15] There is a clear hint regarding the choice of α. Formally applying the divergence operator on $(\text{B.1.4})_2$ with $\mathbf{b} = \mathbf{0}$, we obtain

$$p = (-\Delta)^{-1}\operatorname{div}\operatorname{div}\big(\mathbf{v}\otimes\mathbf{v} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big). \tag{B.3.4}$$

Since the energy inequality implies that

$$\mathbf{v} \in L^\infty\big(0, T; L^2_{\text{per}}\big) \cap L^r\big(0, T; W^{1,r}_{\text{per}}\big), \tag{B.3.5}$$

the interpolation inequality $\|u\|_q \leqslant \|u\|_2^{2(3r-q)/(q(5r-6))}\|u\|_{3r/(3-r)}^{(3r/q)(q-2)/(5r-6)}$ (for $r < 3$) leads to

$$\mathbf{v} \in L^{5r/3}\big(0, T; L^{5r/3}_{\text{per}}\big). \tag{B.3.6}$$

Consequently,

$$\mathbf{v}\otimes\mathbf{v} \in L^{5r/6}\big(0, T; L^{5r/6}_{\text{per}}\big). \tag{B.3.7}$$

Due to (B.2.26), $\mathbf{S}(\mathbf{D}(\mathbf{v}))$ behaves as $|\nabla\mathbf{v}|^{r-1}$ and thus

$$\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) \in L^{r'}\big(0, T; L^{r'}_{\text{per}}\big). \tag{B.3.8}$$

It thus follows from (B.3.4), (B.3.7) and (B.3.8) that

$$p \in L^q\big(0, T; L^q_{\text{per}}\big), \quad \text{where } q = \min\left\{\frac{5r}{6}, \frac{r}{r-1}\right\}. \tag{B.3.9}$$

If $r \geqslant 11/5$, then $q = r/(r-1)$ while for $r < 11/5$, $q = 5r/6$. In both cases, we choose α in such way that $\alpha + 2 = q$.

Note that (B.3.10)–(B.3.11) are tantamount to

$$\mathbf{v}_{,t} + \operatorname{div}\big((\mathbf{v}_{\mathrm{div}} * \omega^{\eta}) \otimes \mathbf{v}\big) - \operatorname{div} \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) - \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \nabla\left(\frac{\operatorname{div}\mathbf{v}}{|\operatorname{div}\mathbf{v}|^{\alpha/(\alpha+1)}}\right) = \mathbf{b}, \tag{B.3.15}$$

with $p := -(\frac{1}{\varepsilon})^{1/(\alpha+1)} |\operatorname{div}\mathbf{v}|^{-\alpha/(\alpha+1)} \operatorname{div}\mathbf{v}$ defined after solving for $\mathbf{v} = \mathbf{v}^{\varepsilon,\eta}$ (B.3.15) together with (B.2.9) and (A.4.6). This kind of approximation is called a quasi-compressible approximation or the problem with penalized divergenceless constraint. Although three nonlinear operators appear in (B.3.15), the solvability of (A.3.10) is not difficult to establish in virtue of the fact that the first operator $\operatorname{div}((\mathbf{v}_{\mathrm{div}} * \omega^{\eta}) \otimes \mathbf{v})$ is compact, the second and third (for $\varepsilon > 0$ fixed) are monotone and the following estimates are available

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_2^2 + \int_{\Omega} \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) \cdot \mathbf{D}(\mathbf{v})\,\mathrm{d}x + \left\{ \begin{array}{l} \varepsilon \|p\|_{\alpha+2}^{\alpha+2} \\ (\frac{1}{\varepsilon})^{1/(\alpha+1)} \|\operatorname{div}\mathbf{v}\|_{(\alpha+2)/(\alpha+1)}^{(\alpha+2)/(\alpha+1)} \end{array} \right\} \\ &\quad = \langle \mathbf{b}, \mathbf{v}\rangle \leqslant \|\mathbf{b}\|_{-1,r'} \|\nabla \mathbf{v}\|_r. \end{aligned} \tag{B.3.16}$$

Note that

$$\alpha + 2 = \begin{cases} r', \\ \frac{5r}{6}, \end{cases} \qquad \alpha + 1 = \begin{cases} \frac{1}{r-1}, \\ \frac{5r-6}{6} \end{cases} \quad \text{and} \quad \frac{\alpha+2}{\alpha+1} = \begin{cases} r & \text{for } r \geqslant \frac{11}{5}, \\ \frac{5r}{5r-6} & \text{for } r < \frac{11}{5}. \end{cases} \tag{B.3.17}$$

Inequality (B.2.23) and Korn's inequality (B.2.27) then allow us to conclude from (B.3.16) that

$$\begin{aligned} &\sup_{t\in[0,T]} \big\|\mathbf{v}^{\varepsilon,\eta}(t)\big\|_2^2 + \int_0^T \big\|\nabla \mathbf{v}^{\varepsilon,\eta}(t)\big\|_r^r\,\mathrm{d}t + \varepsilon \int_0^T \big\|p^{\varepsilon,\eta}(t)\big\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}t \\ &\quad + \left(\frac{1}{\varepsilon}\right)^{1/(1+\alpha)} \int_0^T \big\|\operatorname{div}\mathbf{v}^{\varepsilon,\eta}(t)\big\|_{(\alpha+2)/(\alpha+1)}^{(\alpha+2)/(\alpha+1)}\,\mathrm{d}t \leqslant K, \end{aligned} \tag{B.3.18}$$

where K is an absolute constant depending on $\|\mathbf{b}\|_{L^{r'}(0,T;W^{-1,r'}_{\mathrm{per}})}$, $\|\mathbf{v}_0\|_2$ and the constant C_0^{-1} from (B.2.26).

It follows from the first two terms (implying that $\mathbf{v}^{\varepsilon,\eta}$ belongs to $L^{\infty}(0,T;L^2_{\mathrm{per}}) \cap L^r(0,T;W^{1,r}_{\mathrm{per}})$ uniformly w.r.t. ε and η) that for $r \in (1,3)$,

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}\|_{5r/3}^{5r/3}\,\mathrm{d}t \leqslant K. \tag{B.3.19}$$

In virtue of (B.3.18) and (B.2.26) we also have

$$\int_0^T \big\|\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big\|_{r'}^{r'}\,\mathrm{d}t \leqslant K. \tag{B.3.20}$$

On considering (B.3.15) and using the estimates (B.3.18)–(B.3.20) we see that the third and fifth term belong to $L^{r'}(0, T; W^{-1,r'}_{\text{per}}(\Omega))$ (even uniformly w.r.t. ε and η), the second term $\operatorname{div}((\mathbf{v}_{\text{div}} * \omega^\eta) \otimes \mathbf{v})$ belongs to $L^{r'}(0, T; W^{-1,r'}_{\text{per}}(\Omega))$ uniformly w.r.t. ε and η for $r \geqslant 11/5$, and for $r < 11/5$ it belongs to $L^{5r/6}(0, T; W^{-1,5r/6}_{\text{per}}(\Omega))$ (again uniformly w.r.t. ε and η). The term $(\varepsilon)^{-1/(\alpha+1)} \operatorname{div}(|\operatorname{div}\mathbf{v}|^{-\alpha/(\alpha+1)}(\operatorname{div}\mathbf{v})\mathbf{I})$ also belongs to $L^{r'}(0, T; W^{-1,r'}_{\text{per}}(\Omega))$ for $r \geqslant 11/5$, and to $L^{5r/6}(0, T; W^{-1,5r/6}_{\text{per}}(\Omega))$ otherwise, however *not* uniformly w.r.t. $\varepsilon > 0$.

As a consequence of this consideration, we have

$$\mathbf{v}^{\varepsilon,\eta}_{,t} \in \begin{cases} L^{r'}\big(0, T; W^{-1,r'}_{\text{per}}\big) & \text{for } r \geqslant \frac{11}{5}, \\ L^{5r/6}\big(0, T; W^{-1,5r/6}_{\text{per}}\big) & \text{for } r < \frac{11}{5}. \end{cases} \tag{B.3.21}$$

Obviously, if we eliminate the term $\varepsilon^{-1/(\alpha+1)} \operatorname{div}(|\operatorname{div}\mathbf{v}|^{-\alpha/(\alpha+1)}(\operatorname{div}\mathbf{v})\mathbf{I})$ using divergenceless test functions we obtain estimates that are uniform w.r.t. ε. On doing so, we conclude that

$$\int_0^T \big\|\mathbf{v}^{\varepsilon,\eta}_{,t}\big\|^{\alpha+2}_{W^{-1,\alpha+2}_{\text{per,div}}} \,\mathrm{d}t \leqslant K. \tag{B.3.22}$$

3.3. *Solvability of an approximate problem*

In this subsection ε and η are fixed, and thus we write $(\mathbf{v}, p)$ instead $(\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$. Based on (B.3.18) we set

$$X := \begin{cases} L^r\big(0, T; W^{1,r}_{\text{per}}\big) & \text{if } r \geqslant \frac{11}{5}, \\ \big\{\mathbf{u} \in L^r\big(0, T; W^{1,r}_{\text{per}}\big); \operatorname{div}\mathbf{u} \in L^{5r/(5r-6)}\big(0, T; L^{5r/(5r-6)}\big)\big\} & \text{if } r < \frac{11}{5}. \end{cases}$$

Let $\{\boldsymbol{\omega}^s\}_{s=1}^\infty$ be a basis for X. We construct a solution to (B.3.10)–(B.3.11), more precisely to (B.3.15) via Galerkin approximations $\{\mathbf{v}^N\}_{N=1}^\infty$ of the form

$$\mathbf{v}^N(t, x) = \sum_{s=1}^N c^N_s(t)\boldsymbol{\omega}^s(x),$$

where $\mathbf{c}^N := \{c^N_s(t)\}_{s=1}^\infty$ solve the system of ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{v}^N, \boldsymbol{\omega}^s\big) - \big(\big(\mathbf{v}^N_{\text{div}} * \omega^\eta\big) \otimes \mathbf{v}^N, \boldsymbol{\omega}^s\big) + \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^N\big)\big), \mathbf{D}\big(\boldsymbol{\omega}^s\big)\big) \\ &\quad + \frac{1}{\varepsilon^{1/(\alpha+1)}}\big(\big|\operatorname{div}\mathbf{v}^N\big|^{-\alpha/(\alpha+1)}\operatorname{div}\mathbf{v}^N, \operatorname{div}\boldsymbol{\omega}^s\big) = \big\langle \mathbf{b}, \boldsymbol{\omega}^s\big\rangle \quad \text{for } s = 1, 2, \ldots, N, \end{aligned} \tag{B.3.23}$$

completed by the initial conditions obtained by projecting $\mathbf{v}_0$ into the finite-dimensional space generated by the first N basis functions $\boldsymbol{\omega}^r$, $r = 1, \dots, N$.

Due to the linearity of the second component in all the expressions, we obtain (B.3.16) for $\mathbf{v}^N$ that leads to (B.3.18)–(B.3.20) for $\mathbf{v}^N$. Local-in-time existence of a solution to (B.3.23) follows from Carathéodory theory, global-in-time existence is then a consequence of (B.3.18), or its variant for $\mathbf{v}^N$. It also follows from (B.3.18), (B.3.20) and (B.3.21) that there is a subsequence $\{\mathbf{v}^n\}_{n=1}^{\infty} \subset \{\mathbf{v}^N\}_{N=1}^{\infty}$ and $\mathbf{v} \in X \cap L^{\infty}(0, T; L^2_{\text{per}})$, $\overline{\mathbf{S}} \in L^{r'}(0, T; L^{r'}(\Omega)^{3\times 3})$ and $\overline{P} \in L^{\alpha+2}(0, T; L^{\alpha+2})$ such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \begin{array}{l}\text{weakly in X} \\ \text{*-weakly in } L^{\infty}\big(0, T; L^2_{\text{per}}\big),\end{array} \tag{B.3.24}$$

$$\mathbf{v}^n_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } L^{\alpha+2}\big(0, T; W^{-1,\alpha+2}_{\text{per}}\big), \tag{B.3.25}$$

$$\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^n\big)\big) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}\big(0, T; L^{r'}(\Omega)^{3\times 3}\big), \tag{B.3.26}$$

$$P\big(\mathbf{v}^n\big) := \big|\operatorname{div} \mathbf{v}^n\big|^{-\alpha/(\alpha+1)} \operatorname{div} \mathbf{v}^n \rightharpoonup \overline{P} \quad \text{weakly in } L^{\alpha+2}\big(0, T; L^{\alpha+2}_{\text{per}}\big), \tag{B.3.27}$$

and in virtue of Aubin–Lions compactness lemma (cf. [82], Lemma 1.2.48, or [77], Section 1.5)

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } L^r\big(0, T; L^q_{\text{per}}\big) \text{ for all } q \in \left\langle 1, \frac{3r}{3-r}\right). \tag{B.3.28}$$

Simple arguments then lead to the conclusion that $\mathbf{v}$, $\overline{\mathbf{S}}$ and $\overline{P}$ fulfill (for almost all $t \in (0, T\rangle$)

$$\begin{aligned} 0 = \int_0^t \bigg[&\langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - \big(\big(\mathbf{v}_{\text{div}} * \omega^{\eta}\big) \otimes \mathbf{v}, \nabla \boldsymbol{\varphi}\big) \\ &+ \big(\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{\varphi})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(\overline{P}, \operatorname{div} \boldsymbol{\varphi}\big) - \langle \mathbf{b}, \boldsymbol{\varphi}\rangle \bigg] \,\mathrm{d}\tau \end{aligned} \tag{B.3.29}$$

for all $\boldsymbol{\varphi} \in X$. Particularly, for $\boldsymbol{\varphi} = \mathbf{v}$ we have[16]

$$\begin{aligned} 0 = {}& \frac{1}{2}\big(\big\|\mathbf{v}(t)\big\|_2^2 - \|\mathbf{v}_0\|_2^2\big) \\ &+ \int_0^t \bigg[\big(\overline{\mathbf{S}}, \mathbf{D}(\mathbf{v})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(\overline{P}, \operatorname{div} \mathbf{v}\big) - \langle \mathbf{b}, \mathbf{v}\rangle \bigg] \,\mathrm{d}\tau. \end{aligned} \tag{B.3.30}$$

[16] Proceeding similarly as in Section 3.10, one could conclude from (B.3.29) that $\mathbf{v}(0) = \mathbf{v}_0$.

Since for $\boldsymbol{\psi} \in X$

$$0 \leqslant \int_0^t \Bigg[\Big(\mathbf{S}\big(\mathbf{D}(\mathbf{v}^n)\big) - \mathbf{S}\big(\mathbf{D}(\boldsymbol{\psi})\big), \mathbf{D}\big(\mathbf{v}^n\big) - \mathbf{D}(\boldsymbol{\psi}) \big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(P\big(\mathbf{v}^n\big) - P(\boldsymbol{\psi}), \operatorname{div}\big(\mathbf{v}^n - \boldsymbol{\psi}\big) \Big) \Bigg] \, \mathrm{d}\tau,$$

we use (B.3.16) with $\mathbf{v}^n$ instead of $\mathbf{v}$ to replace the term

$$\int_0^t \left[\big(\mathbf{S}\big(\mathbf{D}(\mathbf{v}^n)\big), \mathbf{D}(\mathbf{v}^n)\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(P\big(\mathbf{v}^n\big), \operatorname{div} \mathbf{v}^n\big) \right] \mathrm{d}\tau$$

and take the limit as $n \to \infty$. Using (B.3.24)–(B.3.28) we conclude that

$$0 \leqslant \int_0^t \left[\big(\overline{\mathbf{S}} - \mathbf{S}\big(\mathbf{D}(\boldsymbol{\psi})\big), \mathbf{D}(\mathbf{v}) - \mathbf{D}(\boldsymbol{\psi})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(\overline{P} - P(\boldsymbol{\psi}), \operatorname{div}(\mathbf{v} - \boldsymbol{\psi})\big) \right] \mathrm{d}\tau \tag{B.3.31}$$

for all $\boldsymbol{\psi} \in X$.

A possible choice $\boldsymbol{\psi} = \mathbf{v} \pm \lambda \boldsymbol{\varphi}$, $\lambda > 0$, and continuity of the operators in (B.3.31) (for $\lambda \to 0_+$) then imply that

$$0 = \int_0^t \left[\big(\overline{\mathbf{S}} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\boldsymbol{\varphi})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(\overline{P} - P(\mathbf{v}), \operatorname{div} \boldsymbol{\varphi}\big) \right] \mathrm{d}\tau, \tag{B.3.32}$$

which means

$$\begin{aligned} &\int_0^t \left\{ \big(\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{\varphi})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(\overline{P}, \operatorname{div} \boldsymbol{\varphi}\big) \right\} \mathrm{d}\tau \\ &\quad = \int_0^t \left\{ \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\boldsymbol{\varphi})\big) + \left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} \big(P(\mathbf{v}), \operatorname{div} \boldsymbol{\varphi}\big) \right\} \mathrm{d}\tau. \end{aligned}$$

Consequently, (B.3.29) leads to the equation for $(\mathbf{v}, p) = (\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ of the form

$$\begin{aligned} &\int_0^t \langle \mathbf{v}_{,t}, \boldsymbol{\varphi} \rangle - \big((\mathbf{v}_{\mathrm{div}} * \omega^\eta) \otimes \mathbf{v}, \nabla \boldsymbol{\varphi} \big) + \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\boldsymbol{\varphi})\big) \, \mathrm{d}\tau \\ &\quad + \int_0^t \Bigg(\underbrace{\left(\frac{1}{\varepsilon}\right)^{1/(\alpha+1)} |\operatorname{div} \mathbf{v}|^{-\alpha/(1+\alpha)} \operatorname{div} \mathbf{v}}_{=:-p}, \operatorname{div} \boldsymbol{\varphi} \Bigg) \, \mathrm{d}\tau = \int_0^t \langle \mathbf{b}, \boldsymbol{\varphi} \rangle \, \mathrm{d}\tau, \end{aligned} \tag{B.3.33}$$

that is valid for all $\boldsymbol{\varphi} \in X$.

3.4. *Further uniform estimates w.r.t. ε and η*

We recall first that $\{\mathbf{v}^{\varepsilon,\eta}\}$ meet (B.3.18), (B.3.20) and (B.3.22), where K is independent of ε and η. Next, we focus on the uniform estimates of the pressure $p^{\varepsilon,\eta} := -\frac{1}{\varepsilon^{1/(\alpha+1)}}|\operatorname{div}\mathbf{v}^{\varepsilon,\eta}|^{-\alpha/(1+\alpha)}\operatorname{div}\mathbf{v}^{\varepsilon,\eta}$. We start by substituting $\boldsymbol{\varphi} = \nabla h^{\varepsilon,\eta}$ in (B.3.33), where $h^{\varepsilon,\eta}$ solves

$$\Delta h^{\varepsilon,\eta} = \left|p^{\varepsilon,\eta}\right|^{\alpha} p^{\varepsilon,\eta} \quad \text{in } \mathbb{R}^3, \quad h^{\varepsilon,\eta} \text{ being } \Omega\text{-periodic}, \int_\Omega h^{\varepsilon,\eta}(x)\,\mathrm{d}x = 0, \tag{B.3.34}$$

with the estimate

$$\left\|h^{\varepsilon,\eta}\right\|_{2,q} \leqslant C\left\|p^{\varepsilon,\eta}\right\|_{(\alpha+1)q}^{\alpha+1}, \quad q \in (1,\infty), \tag{B.3.35}$$

where C is independent of ε and η, but may depend on q and L. As a consequence we obtain

$$\begin{aligned}
&\int_0^t \left\|p^{\varepsilon,\eta}\right\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}\tau \\
&\quad\leqslant \int_0^t \left\langle \mathbf{v}^{\varepsilon,\eta}_{,t}, \nabla h^{\varepsilon,\eta}\right\rangle \mathrm{d}\tau + \int_0^t\int_\Omega \left|\mathbf{v}^{\varepsilon,\eta}\right|\left|\mathbf{v}^{\varepsilon,\eta}_{\mathrm{div}} * \omega^\eta\right|\left|\nabla^{(2)} h^{\varepsilon,\eta}\right|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad+ \int_0^t\int_\Omega \left|\mathbf{S}\left(\mathbf{D}\left(\mathbf{v}^{\varepsilon,\eta}\right)\right)\right|\left|\mathbf{D}\left(\nabla h^{\varepsilon,\eta}\right)\right|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad:= I_1 + I_2 + I_3.
\end{aligned} \tag{B.3.36}$$

Terms I_2 and I_3 and (B.3.35) suggest that we ought to set q such that the $q(\alpha+1)$-norm for p on the right-hand side of (B.3.35) equals to the $(\alpha+2)$-norm of p on the left-hand side of (B.3.36). This gives $q := (\alpha+2)/(\alpha+1)$ and it is easy to check that using (B.3.19) and (B.3.20), I_2 and I_3 can be bounded.[17]

Focusing on I_1, we first appeal to the Helmholtz decomposition (B.3.12) of $\mathbf{v}^{\varepsilon,\eta}$. Then it follows from (B.3.34) and (B.3.14) and the unique solvability of the Laplace equation for

[17]To be more explicit, considering, for example, the term I_3 we have

$$\begin{aligned}
|I_3| &\leqslant \left(\int_0^t \left\|\mathbf{S}\left(\mathbf{D}\left(\mathbf{v}^{\varepsilon,\eta}\right)\right)\right\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}\tau\right)^{1/(\alpha+2)}\left(\int_0^t \left\|\nabla^2 h^{\varepsilon,\eta}\right\|_{(\alpha+2)/(\alpha+1)}^{(\alpha+2)/(\alpha+1)}\,\mathrm{d}\tau\right)^{(\alpha+1)/(\alpha+2)} \\
&\leqslant KC\left(\int_0^t \left\|p^{\varepsilon,\eta}\right\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}\tau\right)^{(\alpha+1)/(\alpha+2)},
\end{aligned}$$

where we used the fact that $\alpha+2 \geqslant r'$ for arbitrary $r > 1$.

the class of functions being considered that $g^{\varepsilon,\eta}/\varepsilon = -h^{\varepsilon,\eta}$. Consequently,

$$\begin{aligned}\int_0^t \langle \mathbf{v}^{\varepsilon,\eta}_{,t}, \nabla h^{\varepsilon,\eta}\rangle \,\mathrm{d}\tau \\ &= \int_0^t \langle \nabla g^{\varepsilon,\eta}_{,t}, \nabla h^{\varepsilon,\eta}\rangle \,\mathrm{d}\tau = -\varepsilon \int_0^t \Big\langle \nabla \frac{g^{\varepsilon,\eta}_{,t}}{\varepsilon}, \nabla\big(-h^{\varepsilon,\eta}\big)\Big\rangle \,\mathrm{d}\tau \\ &= -\varepsilon \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \nabla \frac{g^{\varepsilon,\eta}_{,t}}{\varepsilon}\right\|_2^2 \mathrm{d}\tau = -\frac{1}{2\varepsilon}\big(\|g^{\varepsilon,\eta}(t)\|_2^2 - \|g^{\varepsilon,\eta}(0)\|_2^2\big) \leqslant 0\end{aligned} \tag{B.3.37}$$

as $\Delta g^{\varepsilon,\eta}(0) = \operatorname{div} \mathbf{v}^{\varepsilon,\eta}(0) = 0 \Rightarrow g^{\varepsilon,\eta}(0) = 0$. (The reader may wish to verify this argument concerning the treatment of the term I_1 first for smooth approximations $(\mathbf{v}^{\varepsilon,\eta}_m, p^{\varepsilon,\eta}_m)$ and then taking the limit as $m \to \infty$, whereas $\mathbf{v}^{\varepsilon,\eta}_m$ follows from the density of smooth functions in $L^r(0,T; W^{1,r}_{\mathrm{per}})$ and $p^{\varepsilon,\eta}_m := -(\varepsilon)^{1/(\alpha+1)}|\operatorname{div}\mathbf{v}^{\varepsilon,\eta}_m|^{-\alpha/(\alpha+1)}\operatorname{div}\mathbf{v}^{\varepsilon,\eta}_m$.)

Thus, it follows from (B.3.36)–(B.3.37) that[18]

$$\int_0^T \|p^{\varepsilon,\eta}\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}\tau \leqslant K. \tag{B.3.38}$$

Consequently, we can strengthen (B.3.22) to conclude from (B.3.33) that

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}_{,t}\|_{W^{-1,\alpha+2}_{\mathrm{per}}}^{\alpha+2}\,\mathrm{d}\tau \leqslant K. \tag{B.3.39}$$

Estimates (B.3.38) and (B.3.39) are uniform w.r.t. η. If we however relax this requirement and use the fact that $\mathbf{v} * \omega^\eta$ is a smooth function for $\eta > 0$ fixed, we obtain, proceeding as above,

$$\int_0^T \|p^{\varepsilon,\eta}\|_{r'}^{r'}\,\mathrm{d}\tau \leqslant C\big(\eta^{-1}\big) \tag{B.3.40}$$

and

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}_{,t}\|_{W^{-1,r'}_{\mathrm{per}}}^{r'}\,\mathrm{d}\tau \leqslant C\big(\eta^{-1}\big). \tag{B.3.41}$$

[18]Note that this step can be repeated without any change for Navier's boundary conditions, it is however open in general for the no-slip boundary condition due to the fact that ∇h is not an admissible function.

3.5. *Limit* $\varepsilon \to 0$

For fixed $\eta > 0$, we establish in this section the existence of a (suitable) weak solution to the problem

$$\begin{cases} \operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}\big((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}\big) - \operatorname{div}\big(\mathbf{S}(\mathbf{D}(\mathbf{v}))\big) = -\nabla p + \mathbf{b}, \\ v_i,\, p \text{ are } \Omega\text{-periodic} \quad \text{with } \int_\Omega v_i \,\mathrm{d}x = \int_\Omega p \,\mathrm{d}x = 0 \text{ for } i = 1,2,3, \\ \mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad \text{in } \Omega, \end{cases} \tag{$\mathcal{P}^\eta$}$$

if the parameter r appearing in (B.2.8) fulfills

$$r > \frac{8}{5}. \tag{B.3.42}$$

Using the estimates (B.3.18), (B.3.20), (B.3.40) and (B.3.41) uniform w.r.t. $\varepsilon > 0$, and letting $\varepsilon \to 0$ we can find a sequence $\{\mathbf{v}^n, p^n\}$ chosen from $\{\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta}\}$, and a limit element $\{\mathbf{v}, p\} := \{\mathbf{v}^\eta, p^\eta\}$ such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \qquad \begin{array}{l} \text{weakly in } L^r\big(0,T; W^{1,r}_{\mathrm{per}}\big) \\ \text{*-weakly in } L^\infty\big(0,T; L^2_{\mathrm{per}}\big) \end{array} \tag{B.3.43}$$

$$\mathbf{v}^n_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } L^{r'}\big(0,T; W^{-1,r'}_{\mathrm{per}}\big) \tag{B.3.44}$$

$$\mathbf{v}^n \to \mathbf{v} \qquad \text{strongly in } \begin{cases} L^r\big(0,T; L^q_{\mathrm{per}}\big) & \text{for all } q \in \big\langle 1, \frac{3r}{3-r}\big), \\ L^s\big(0,T; L^s_{\mathrm{per}}\big) & \text{for all } s \in \big\langle 1, \frac{5r}{3}\big), \end{cases} \tag{B.3.45}$$

$$\mathbf{S}\big(\mathbf{D}(\mathbf{v}^n)\big) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}\big(0,T; L^{r'}_{\mathrm{per}}\big) \tag{B.3.46}$$

and

$$p^n \rightharpoonup p \quad \text{weakly in } L^{r'}\big(0,T; L^{r'}_{\mathrm{per}}\big). \tag{B.3.47}$$

It also follows from the fourth term in (B.3.18) (as $\varepsilon \to 0$) that

$$\operatorname{div} \mathbf{v} = 0 \quad \text{a.e. in } (0,T) \times \Omega. \tag{B.3.48}$$

On employing (B.3.33) with $\{\mathbf{v}^n, p^n\}$ instead of $\{\mathbf{v}, p\} = \{\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta}\}$, we can take the limit as $n \to \infty$ and obtain with help of (B.3.43)–(B.3.47) that

$$\begin{aligned} &\int_0^t \big\{ \langle \mathbf{v}_{,t}, \boldsymbol{\varphi} \rangle - \big((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}, \nabla \boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi})\big) - (p, \operatorname{div} \boldsymbol{\varphi}) \big\} \,\mathrm{d}\tau \\ &\quad = \int_0^t \langle \mathbf{b}, \boldsymbol{\varphi} \rangle \,\mathrm{d}\tau \quad \text{for all } \boldsymbol{\varphi} \in L^r\big(0,T, W^{1,r}_{\mathrm{per}}\big) \end{aligned} \tag{B.3.49}$$

provided that we show that

$$\overline{\mathbf{S}} = \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) \quad \text{a.e. in } (0,T)\times\Omega. \tag{B.3.50}$$

Towards this purpose, we consider (B.3.33) for $(\mathbf{v}^n, p^n)$ and set $\boldsymbol{\varphi} = \mathbf{v}^n - \mathbf{v}$ therein. Then

$$\begin{aligned}
&\int_0^T \big\langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\big\rangle \,\mathrm{d}t + \int_0^T \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^n\big)\big) - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}\big(\mathbf{v}^n - \mathbf{v}\big)\big)\,\mathrm{d}t \\
&\quad = -\int_0^T \big\langle \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\big\rangle \,\mathrm{d}t + \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}\big(\mathbf{v}^n - \mathbf{v}\big)\big)\,\mathrm{d}t + \big\langle \mathbf{b}, \mathbf{v}^n - \mathbf{v}\big\rangle \,\mathrm{d}t \\
&\qquad - \int_0^T \big(\big(\mathbf{v}^n_{\mathrm{div}} * \omega^\eta\big)\otimes \mathbf{v}^n, \nabla\big(\mathbf{v}^n - \mathbf{v}\big)\big)\,\mathrm{d}t.
\end{aligned} \tag{B.3.51}$$

Let $n\to\infty$. The first term on the right-hand side of (B.3.51) vanishes due to the weak convergence documented in (B.3.43), the last integral that equals to $\int_0^T (\operatorname{div}(\mathbf{v}^n_{\mathrm{div}} * \omega^\eta)\mathbf{v}^n, \mathbf{v}^n - \mathbf{v})\,\mathrm{d}t + \int_0^T ((\mathbf{v}^n_{\mathrm{div}} * \omega^\eta)\otimes(\mathbf{v}^n - \mathbf{v}), \nabla\mathbf{v}^n)\,\mathrm{d}t$ also vanishes due to (B.3.45) and $|\nabla\mathbf{v}^n||\mathbf{v}^n|$ is uniformly integrable if $r > 8/5$.

Consequently, using (B.3.51) and (B.2.20) (resp. (B.2.21)), we have[19]

$$\lim_{n\to\infty}\int_0^T \big\|\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v})\big\|_r^r \,\mathrm{d}t = 0. \tag{B.3.52}$$

Thus $\mathbf{D}(\mathbf{v}^n)\to\mathbf{D}(\mathbf{v})$ a.e. in $(0,T)\times\Omega$ (at least for the subsequence) and Vitali's lemma (see [82], Lemma 2.1, or [28]) and (B.3.20) give $\mathbf{S}(\mathbf{D}(\mathbf{v}^n))\to\mathbf{S}(\mathbf{D}(\mathbf{v}))$ a.e. in $(0,T)\times\Omega$ that implies (B.3.50).

Taking $\boldsymbol{\varphi} = \mathbf{v}\phi$, $\phi\in\mathcal{D}(-\infty,\infty;\mathcal{C}^\infty_{\mathrm{per}})$ in (B.3.49) we conclude that the *local energy equality* is fulfilled. Thus, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big(|\mathbf{v}|^2\phi\big)(t,x)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\cdot\mathbf{D}(\mathbf{v})\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad = \frac{1}{2}\int_\Omega \big|\mathbf{v}_0(x)\big|^2\phi(0,x)\,\mathrm{d}x + \frac{1}{2}\int_0^t\!\!\int_\Omega |\mathbf{v}|^2\phi_{,t}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \langle \mathbf{b}, \mathbf{v}\phi\rangle\,\mathrm{d}\tau \\
&\qquad + \int_0^t\!\!\int_\Omega \left(\frac{|\mathbf{v}|^2}{2}\big(\mathbf{v}*\omega^\eta\big) + p\mathbf{v} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\mathbf{v}\right)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{B.3.53}$$

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[19] We have also used the fact that

$$\int_0^T \big\langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\big\rangle\,\mathrm{d}t = \frac{1}{2}\big\|\mathbf{v}^n(T) - \mathbf{v}(T)\big\|_2^2 - \frac{1}{2}\big\|\mathbf{v}^n(0) - \mathbf{v}(0)\big\|_2^2 = \frac{1}{2}\big\|\mathbf{v}^n(T) - \mathbf{v}(T)\big\|_2^2.$$

This requires one to check that $\mathbf{v}^n(0) = \mathbf{v}(0) = \mathbf{v}_0$. We skip it here however and show it later for the more difficult case.

Also setting $\boldsymbol{\varphi} = \mathbf{v}$ in (B.3.49) we have the global energy equality

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\mathbf{v})\big)\,\mathrm{d}\tau = \frac{1}{2}\|\mathbf{v}_0\|_2^2 + \int_0^t \langle \mathbf{b}, \mathbf{v}\rangle\,\mathrm{d}\tau \tag{B.3.54}$$

and in view of the lower-semicontinuity of the norms w.r.t. weak convergence it follows from (B.3.18)–(B.3.20), (B.3.38) and (B.3.39) that $(\mathbf{v}, p) = (\mathbf{v}^\eta, p^\eta)$ fulfills the following estimates that are uniform w.r.t. $\eta > 0$:

$$\sup_{t\in[0,T]} \|\mathbf{v}^\eta(t)\|_2^2 + \int_0^T \|\nabla \mathbf{v}^\eta\|_r^r\,\mathrm{d}t + \int_0^T \|\mathbf{v}^\eta\|_{5r/3}^{5r/3}\,\mathrm{d}t \leqslant K, \tag{B.3.55}$$

$$\int_0^T \big\|\mathbf{S}\big(\mathbf{D}(\mathbf{v}^\eta)\big)\big\|_{r'}^{r'}\,\mathrm{d}t \leqslant K, \tag{B.3.56}$$

$$\int_0^T \|p^\eta\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}t \leqslant K \quad \text{with } \alpha + 2 = \begin{cases} r' & \text{if } r \geqslant \frac{11}{5}, \\ \frac{5r}{6} & \text{if } r < \frac{11}{5}, \end{cases} \tag{B.3.57}$$

$$\int_0^T \|\mathbf{v}^\eta_{,t}\|_{W^{-1,\alpha+2}_{\mathrm{per}}}^{\alpha+2}\,\mathrm{d}t \leqslant K. \tag{B.3.58}$$

3.6. *Limit* $\eta \to 0$*, the case* $r \geqslant 11/5$

If $r \geqslant 11/5$, the uniform estimates that are available coincide with those needed to take the limit in Section 3.5. Thus, we proceed as above. The quadratic convective term requires

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } L^2\big(0, T; L^2(\Omega)\big). \tag{B.3.59}$$

This follows from the Aubin–Lions lemma provided that

$$r > \frac{6}{5},$$

which is of course trivially satisfied here (and in the next subsection as well). The other arguments coincide with those used in Section 3.5. For $r \geqslant 11/5$, we have thus the existence of a weak solution fulfilling the energy equality, the local energy equality, etc. The proof of Theorem 3.1 for the case $r \geqslant 11/5$ is complete.

3.7. *Limit* $\eta \to 0$*, the case* $8/5 < r < 11/5$

We start by observing that if $\mathbf{u} \in L^{5r/3}(0, T; L^{5r/3}_{\mathrm{per}})$ and $\nabla\mathbf{u} \in L^r(0, T; L^r_{\mathrm{per}})$ then

$$[\nabla \mathbf{u}]\big(\mathbf{u} * \omega^\eta\big) \in L^1\big(0, T; L^1_{\mathrm{per}}\big) \tag{B.3.60}$$

uniformly w.r.t. $\eta > 0$ provided that

$$r \geqslant \frac{8}{5}. \tag{B.3.61}$$

Thus, introducing, for $r > 8/5$ and $\delta \in (0, \frac{5}{8}(r - \frac{8}{5}))$, the space of divergenceless functions

$$X_\delta := \left\{\boldsymbol{\varphi} \in L^r\left(0, T; W^{1,r}_{\mathrm{per,div}}\right) \cap L^{(1+\delta)/\delta}\left(0, T; L^{(1+\delta)/\delta}_{\mathrm{per}}\right)\right\}, \tag{B.3.62}$$

and using the fact that

$$-\left(\left(\mathbf{v} * \omega^\eta\right) \otimes \mathbf{v}, \nabla\boldsymbol{\varphi}\right) = \left([\nabla\mathbf{v}]\left(\mathbf{v} * \omega^\eta\right), \boldsymbol{\varphi}\right), \tag{B.3.63}$$

it follows from (B.3.49) that

$$\left\|\mathbf{v}^\eta_{,t}\right\|_{X^*_\delta} \leqslant K \quad \text{uniformly w.r.t. } \eta > 0. \tag{B.3.64}$$

Letting η tend to zero, and using (B.3.55)–(B.3.58), (B.3.64) and the Aubin–Lions compactness lemma, we find a subsequence $\{(\mathbf{v}^k, p^k)\}_{k\in\mathbb{N}}$ and "its weak limit" $\{(\mathbf{v}, p)\}$ such that $(r < 11/5)$

$$\mathbf{v}^k \rightharpoonup \mathbf{v} \qquad \begin{array}{l} \text{weakly in } L^r\left(0, T; W^{1,r}_{\mathrm{per}}\right), \\ \text{*-weakly in } L^\infty\left(0, T; L^2_{\mathrm{per}}\right), \end{array} \tag{B.3.65}$$

$$\mathbf{v}^k_{,t} \rightharpoonup \mathbf{v}_{,t} \qquad \text{weakly in } L^{5r/6}\left(0, T; W^{-1,5r/6}_{\mathrm{per}}\right) \text{ and in } X^*_\delta, \tag{B.3.66}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } L^r\left(0, T; L^q_{\mathrm{per}}\right) \text{ for all } q \in \left\langle 1, \frac{3r}{3-r}\right), \tag{B.3.67}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } L^s\left(0, T; L^2_{\mathrm{per}}\right) \text{ for all } s > 1, \text{ if } r > \frac{6}{5}, \tag{B.3.68}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{a.e. in } (0, T) \times \Omega, \tag{B.3.69}$$

$$p^k \rightharpoonup p \qquad \text{weakly in } L^{5r/6}\left(0, T; L^{5r/6}_{\mathrm{per}}\right), \tag{B.3.70}$$

and there is an $\overline{\mathbf{S}} \in L^{r'}(0, T; L^{r'}_{\mathrm{per}})$ such that

$$\mathbf{S}\left(\mathbf{D}\left(\mathbf{v}^k\right)\right) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}\left(0, T; L^{r'}_{\mathrm{per}}\right). \tag{B.3.71}$$

Also, it follows from (B.3.68) that

$$\mathbf{v}^k(t) \to \mathbf{v}(t) \quad \text{strongly in } L^2_{\mathrm{per}} \text{ for all } t \in [0, T] \setminus N, \tag{B.3.72}$$

where N has zero one-dimensional Lebesgue measure.

In order to identify $\overline{\mathbf{S}}$ with $\mathbf{S}(\mathbf{D}(\mathbf{v}))$ we recall the analysis in previous sections that this would follow from

$$\lim_{n\to\infty}\int_0^T\int_\Omega \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^n\big)\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big)\cdot\big(\mathbf{D}\big(\mathbf{v}^n\big)-\mathbf{D}(\mathbf{v})\big)\,\mathrm{d}x\,\mathrm{d}t=0, \tag{B.3.73}$$

on using the fact that this integral operator is uniformly monotone (note that it would suffice to know that this operator is strictly monotone).

Here, we will establish a condition weaker than (B.3.73), namely

$$\begin{aligned}&\text{for every } \varepsilon^*>0 \text{ and for some } \theta\in\big(\tfrac{1}{r},1\big)\\ &\text{there is a subsequence } \big\{\mathbf{v}^n\big\}_{n=1}^\infty \text{ of } \big\{\mathbf{v}^k\big\}_{k=1}^\infty \text{ such that}\\ &\lim_{n\to\infty}\int_0^T\int_\Omega \big[\big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^n\big)\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big)\cdot\big(\mathbf{D}\big(\mathbf{v}^n\big)-\mathbf{D}(\mathbf{v})\big)\big]^\theta\,\mathrm{d}x\,\mathrm{d}t\leqslant\varepsilon^*.\end{aligned} \tag{$*$}$$

Once $(*)$ is proved, we can take $\varepsilon_m^*\to 0$ and for each $m\in\mathbb{N}$ select gradually (not relabeled) subsequences so that the Cantor diagonal sequence (again not relabeled) fulfills

$$\mathbf{D}\big(\mathbf{v}^n\big)\to\mathbf{D}(\mathbf{v}) \quad \text{a.e. in } (0,T)\times\mathbb{R}^3. \tag{B.3.74}$$

Vitali's theorem and (B.3.74) then imply that $\overline{\mathbf{S}}=\mathbf{S}(\mathbf{D}(\mathbf{v}))$ a.e. in $(0,T)\times\mathbb{R}^3$. The convergence given in (B.3.65)–(B.3.72) and (B.3.74) clearly suffice to take the limit from the weak formulation of the problem $(\mathcal{P}^\eta)$ to the weak form of the problem $(\mathcal{P})$.

It remains to verify $(*)$. For this purpose we set

$$g^k:=\big(\big|\nabla\mathbf{v}^k\big|^r+|\nabla\mathbf{v}|^r+\big(\big|\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^k\big)\big)\big|+\big|\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big|\big)\big(\big|\mathbf{D}\big(\mathbf{v}^k\big)\big|+\big|\mathbf{D}(\mathbf{v})\big|\big)\big). \tag{B.3.75}$$

Clearly $g^k\geqslant 0$ and

$$0\leqslant\int_0^T\int_\Omega g^k\,\mathrm{d}x\,\mathrm{d}t\leqslant K, \qquad K>1. \tag{B.3.76}$$

We prove the following property (K is the constant referred to in (B.3.76))

$$\begin{aligned}&\text{for every } \varepsilon^{**}>0 \text{ there is } L\leqslant\tfrac{\varepsilon^{**}}{K},\ \big\{\mathbf{v}^n\big\}_{n=1}^\infty\subset\big\{\mathbf{v}^k\big\}_{k=1}^\infty \text{ and sets}\\ &E^n:=\big\{(x,t)\in(0,T)\times\Omega;\, L^2\leqslant\big|\mathbf{v}^n(t,x)-\mathbf{v}(t,x)\big|<L\big\} \text{ such that}\\ &\int_{E^n} g^n\,\mathrm{d}x\,\mathrm{d}t\leqslant\varepsilon^{**}.\end{aligned} \tag{$**$}$$

To see that this is indeed so, we fix $\varepsilon^{**} \in (0,1)$, set $L_1 = \varepsilon^*/K$ and take $N \in \mathbb{N}$ such that $N\varepsilon^{**} > K$ (K that appears in (B.3.76)). Defining iteratively $L_i = L_{i-1}^2$ for $i = 2, 3, \dots, N$, we set

$$E^{k,i} = \left\{(t,x) \in (0,T) \times \Omega;\ L_i^2 \leqslant \left|\mathbf{v}^k(t,x) - \mathbf{v}(t,x)\right| < L_i\right\}, \quad i = 1, 2, \dots, N.$$

For $k \in \mathbb{N}$ fixed, $E^{k,i}$ are mutually disjoint. Consequently,

$$\sum_{i=1}^{N} \int_{E^{k,i}} g^k \,\mathrm{d}x\,\mathrm{d}t \leqslant K.$$

As $N\varepsilon^{**} > K$, for each $k \in \mathbb{N}$ there is $i_0(k) \in \{1, \dots, N\}$ such that

$$\int_{E^{k,i_0(k)}} g^k \,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon^{**}.$$

However, $i_0(k)$ are taken from a finite set of indices. Then, there has to be a sequence $\{\mathbf{v}^n\} \subset \{\mathbf{v}^k\}$ such that $i_0(n) = i_0^*$ for each n ($i_0^* \in \{1, 2, \dots, N\}$ fixed). The property $(**)$ is then proved by setting $L = L_{i_0^*}$ and $E^n = E^{n,i_0^*}$.

Returning to our aim, the verification of $(*)$, we consider $(\mathbf{v}^n, p^n)$ satisfying (B.3.49), (B.3.53), (B.3.54) and having all convergence properties stated in (B.3.65)–(B.3.72) and $(**)$, and we set $\boldsymbol{\varphi}$ in (B.3.49) to be of the form

$$\boldsymbol{\varphi}^n := \mathbf{h}^n - \nabla z := \left(\mathbf{v}^n - \mathbf{v}\right)\left(1 - \min\left(\frac{|\mathbf{v}^n - \mathbf{v}|}{L}, 1\right)\right) - \nabla z^n, \tag{B.3.77}$$

where L comes from $(**)$, and z^n solves the problem defined through

$$-\Delta z^n = f^n, \quad z^n \text{ being } \Omega\text{-periodic}, \int_\Omega z^n \,\mathrm{d}x = 0, \tag{B.3.78}$$

where

$$f^n := \operatorname{div}\left(\left(\mathbf{v}^n - \mathbf{v}\right)\left(1 - \min\left(\frac{|\mathbf{v}^n - \mathbf{v}|}{L}, 1\right)\right)\right) = \operatorname{div} \mathbf{h}^n.$$

We summarize the properties of $\mathbf{h}^n$, z^n and $\boldsymbol{\varphi}^n$. Introducing Q^n through $Q^n := \{(t,x) \in (0,T) \times \Omega;\ |\mathbf{v}^n(t,x) - \mathbf{v}(t,x)| < L\}$, we first note that

$$\mathbf{h}^n = \mathbf{0} \quad \text{on } (0,T) \times \Omega \setminus Q^n \tag{B.3.79}$$

and

$$\left|\mathbf{h}^n(t,x)\right| \leqslant L \quad \text{for all } (t,x) \in (0,T) \times \Omega. \tag{B.3.80}$$

Consequently, owing to (B.3.69) and Lebesgue's theorem we have, for all $s \in \langle 1, \infty)$,

$$\int_0^T \|\mathbf{h}^n\|_s^s \,\mathrm{d}t \to 0 \quad \text{as } n \to \infty, \tag{B.3.81}$$

and due to L^s-theory for the Laplace operator, it follows from (B.3.78) that

$$\int_0^T \|\nabla z^n\|_s^s \to 0 \quad \text{as } n \to \infty. \tag{B.3.82}$$

From (B.3.81) and (B.3.82) it follows ($\boldsymbol{\varphi}^n$ defined in (B.3.77)) that

$$\int_0^T \|\boldsymbol{\varphi}^n\|_s^s \,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{B.3.83}$$

Next (χ_Z denotes the characteristic function of a set Z),

$$f^n = \operatorname{div} \mathbf{h}^n = \left(\mathbf{v}^n - \mathbf{v}\right) \cdot \frac{(\mathbf{v}^n - \mathbf{v})_j}{L} \frac{\nabla(\mathbf{v}^n - \mathbf{v})_j}{|\mathbf{v}^n - \mathbf{v}|} \chi_{Q^n}.$$

Splitting Q^n into E^n (introduced in $(**)$) and its complement, and using the fact that $|f^n|^r \leqslant |\nabla(\mathbf{v}^n - \mathbf{v})|^r \leqslant C(|\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r)$ on E^n and $|f^n|^r \leqslant L(|\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r) \leqslant \frac{\varepsilon^*}{K}(|\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r)$ on $Q^n \setminus E^n$ we conclude from $(**)$ that

$$\int_0^T \|f^n\|_{r,Q^n}^r \,\mathrm{d}t \leqslant 2\varepsilon^*, \tag{B.3.84}$$

and using the L^r-regularity for the Laplace operator and (B.3.78)

$$\int_0^T \|\nabla^{(2)} z^n\|_{r,Q^n}^r \,\mathrm{d}t \leqslant 2C_{\mathrm{reg}}\varepsilon^*. \tag{B.3.85}$$

We also note that

$$\boldsymbol{\varphi}^n \rightharpoonup 0 \quad \text{weakly in } L^r\left(0, T; W^{1,r}_{\mathrm{per}}\right) \text{ and also in } X_\delta, \tag{B.3.86}$$

where X_δ is defined in (B.3.62).

Inserting $\boldsymbol{\varphi}^n$ of the form (B.3.77) into (B.3.49) we obtain (note that the term with pressure vanishes as $\operatorname{div} \boldsymbol{\varphi}^n = 0$) that

$$\begin{aligned}
&\int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \,\mathrm{d}t + \int_0^T \left(\mathbf{S}\left(\mathbf{D}\left(\mathbf{v}^n\right)\right) - \mathbf{S}\left(\mathbf{D}(\mathbf{v})\right), \mathbf{D}\left(\boldsymbol{\varphi}^n\right)\right) \mathrm{d}t \\
&\quad = -\int_0^T \left(\left(\mathbf{v}^n * \omega^{1/n}\right)\left[\nabla \mathbf{v}^n\right], \boldsymbol{\varphi}^n\right) \mathrm{d}t + \int_0^T \langle \mathbf{b}, \boldsymbol{\varphi}^n \rangle \,\mathrm{d}t \\
&\qquad - \int_0^T \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \,\mathrm{d}t - \int_0 \left(\mathbf{S}\left(\mathbf{D}(\mathbf{v})\right), \mathbf{D}\left(\boldsymbol{\varphi}^n\right)\right) \mathrm{d}t.
\end{aligned} \tag{B.3.87}$$

It is not difficult to see that all the terms on the right-hand side of (B.3.87) tend to 0: the first one due to (B.3.60) and (B.3.83), the third one due to (B.3.86) and the fact that $\mathbf{v}_{,t} \in X^*_\delta$, and the second and fourth terms also tend to zero as a consequence of (B.3.86).

Let $H : \langle 0, \infty) \to \mathbb{R}$ satisfy $H(0) = 0$ and $H'(s) = (1 - \min(\sqrt{s}/L, 1))$. Then the first term on the left-hand side is nonnegative as

$$\begin{aligned}\int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \,\mathrm{d}t &= \int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{h}^n \rangle \,\mathrm{d}t \\ &= \frac{1}{2} \int_\Omega H\big(|\mathbf{v}^n - \mathbf{v}|^2 (T)\big) \,\mathrm{d}x \geqslant 0.\end{aligned}$$

We thus conclude from (B.3.87) that

$$\begin{aligned}&\lim_{n\to\infty} \int_{Q^n} \big(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\big)\big(\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v})\big) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \lim_{n\to\infty} \int_{Q^n} \big(|\mathbf{S}(\mathbf{D}(\mathbf{v}^n))| + |\mathbf{S}(\mathbf{D}(\mathbf{v}))|\big)\big(|\nabla(\mathbf{v}^n - \mathbf{v})| + |\nabla^{(2)} z^n|\big) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{B.3.88}$$

Arguing in a manner analogous to that for obtaining (B.3.84) and (B.3.85), we find that

$$\lim_{n\to\infty} \int_{Q^n} \big(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\big)\big(\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v})\big) \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^*. \tag{B.3.89}$$

Since

$$\begin{aligned}&\int_0^T \int_\Omega \big[\big(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\big)\big(\mathbf{D}(\mathbf{v}^n - \mathbf{v})\big)\big]^\theta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q^n} [\cdots]^\theta \,\mathrm{d}x\,\mathrm{d}t + \int_{(0,T)\times\Omega\setminus Q^n} [\cdots]^\theta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \overset{\text{Hölder}}{\leqslant} \left(\int_{Q^n} [\cdots] \,\mathrm{d}x\,\mathrm{d}t\right)^\theta |Q^n|^{1-\theta} \\ &\qquad + \left(\int_{(0,T)\times\Omega\setminus Q^n} [\cdots] \,\mathrm{d}x\,\mathrm{d}t\right)^\theta \big|\{(t,x); |\mathbf{v}^n - \mathbf{v}| > L\}\big|^{1-\theta} \\ &\quad \leqslant C^*\varepsilon^*,\end{aligned}$$

where we apply (B.3.89) to bound the first term and appeal to the convergence in measure to treat the second term, we finally conclude that $(*)$ holds.

3.8. *Continuity w.r.t. time in the weak topology of* L^2_{per}

With all convergences established in the previous sections, particularly those given by (B.3.65)–(B.3.70) and (B.3.74), it is straightforward to conclude that $(\mathbf{v}, p)$ fulfills the weak identity (B.2.15). Taking $\boldsymbol{\varphi}$ of the form

$$\boldsymbol{\varphi}(\tau, x) = \chi_{\langle t_0, t\rangle}(\tau)\tilde{\boldsymbol{\varphi}}(x), \quad \text{where } \tilde{\boldsymbol{\varphi}} \in W^{1,s}_{\mathrm{per}} \text{ and } t_0, t \in \langle 0, T\rangle,$$

we obtain

$$\begin{aligned}&\big(\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}\big) - \big(\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}\big)\\ &\quad= \int_{t_0}^{t} \big(\mathbf{v}(\tau)\otimes\mathbf{v}(\tau), \nabla\tilde{\boldsymbol{\varphi}}\big) - \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\tilde{\boldsymbol{\varphi}})\big) + \big\langle \mathbf{b}(\tau), \tilde{\boldsymbol{\varphi}}\big\rangle + \big(p(\tau), \operatorname{div}\tilde{\boldsymbol{\varphi}}\big)\,\mathrm{d}\tau.\end{aligned}$$

This implies (for $r > 6/5$) that

$$\begin{aligned}&\big|\big(\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}\big) - \big(\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}\big)\big|\\ &\quad\leqslant c\int_{t_0}^{t} \big(\big\|\mathbf{v}(\tau)\big\|^2_{5r/3} + \big\|\nabla\mathbf{v}(\tau)\big\|^{r-1}_r + \big\|\mathbf{b}(\tau)\big\|_{-1,r'} + \big\|p(\tau)\big\|_{\alpha+2}\big)\,\mathrm{d}\tau\,\|\tilde{\boldsymbol{\varphi}}\|_{1,s}.\end{aligned} \tag{B.3.90}$$

Also, using Hölder's inequality over time, we have

$$\begin{aligned}&\big|\big(\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}\big) - \big(\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}\big)\big|\\ &\quad\leqslant c\Bigg(|t-t_0|^{(5r-6)/(5r)}\bigg(\int_{t_0}^{t}\big\|\mathbf{v}(\tau)\big\|^{5r/3}_{5r/3}\,\mathrm{d}\tau\bigg)^{6/(5r)}\\ &\qquad + |t-t_0|^{1/r}\bigg(\int_{t_0}^{t}\big\|\nabla\mathbf{v}(\tau)\big\|^{r}_{r}\,\mathrm{d}\tau\bigg)^{1/r'}\\ &\qquad + |t-t_0|^{1/r}\bigg(\int_{t_0}^{t}\big\|\mathbf{b}(\tau)\big\|^{r'}_{-1,r'}\,\mathrm{d}\tau\bigg)^{1/r'}\\ &\qquad + |t-t_0|^{(\alpha+1)/(\alpha+2)}\bigg(\int_{t_0}^{t}\big\|p(\tau)\big\|^{\alpha+2}_{\alpha+2}\,\mathrm{d}\tau\bigg)^{1/(\alpha+2)}\Bigg)\|\tilde{\boldsymbol{\varphi}}\|_{1,s}.\end{aligned} \tag{B.3.91}$$

Since all the integrals are finite, (B.3.91) leads to the conclusion that $(\mathbf{v}(\cdot), \tilde{\boldsymbol{\varphi}})$ is continuous at t_0 for all $\tilde{\boldsymbol{\varphi}} \in W^{1,s}_{\mathrm{per}}(\Omega)$. In other words,

$$\mathbf{v} \in \mathcal{C}\big(0, T; \big(W^{1,s}_{\mathrm{per}}\big)^*_{\mathrm{weak}}\big) \tag{B.3.92}$$

or

$$\lim_{t\to t_0}\big(\mathbf{v}(t) - \mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}\big) = 0 \quad \text{for all } \tilde{\boldsymbol{\varphi}} \in W^{1,s}_{\mathrm{per}} \text{ and for all } t_0 \in \langle 0, T\rangle. \tag{B.3.93}$$

Since $\mathbf{v} \in L^\infty(0, T; L^2_{\mathrm{per}})$ and $W^{1,s}_{\mathrm{per}}$ is dense in L^2_{per}, we observe that $\mathbf{v} \in \mathcal{C}(\langle 0, T\rangle; L^2_{\mathrm{weak}})$, which is $(\mathrm{B}.2.12)_1$.

3.9. *(Local) Energy equality and inequality*

If $r \geqslant 11/5$, (B.2.15) permits us to take $\boldsymbol{\varphi} = \mathbf{v}$ or $\boldsymbol{\varphi} = \mathbf{v}\phi$ which implies both the energy equality and its local version. If $11/5 > r > 8/5$, we take $\limsup_{n\to\infty}$ of (B.3.54) where $\mathbf{v}$ stands for $\mathbf{v}^n$, and $t \in \langle 0, T\rangle \setminus N$ with N introduced in (B.3.72). Since

- $\limsup\limits_{n\to\infty}\{a_n + b_n\} \geqslant \limsup\limits_{n\to\infty}\{a_n\} + \liminf\limits_{n\to\infty}\{b_n\}, \quad a_n \geqslant 0, b_n \geqslant 0,$
- $\displaystyle\int_0^t\int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) \cdot \mathbf{D}(\mathbf{v})\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \liminf_{n\to\infty} \int_0^t\int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{v}^n)\big) \cdot \mathbf{D}(\mathbf{v}^n)\,\mathrm{d}x\,\mathrm{d}\tau,$
- $\mathbf{v}_0^n = \mathbf{v}^n(0) \quad$ for all $n \in \mathbb{N}$,
- (B.3.72) and (B.3.65) hold,

we see that the energy inequality (B.2.17) follows directly.

Similarly we can argue by taking the limit $n \to \infty$ in (B.3.53). Here, in addition, we need to take the limit of the terms in

$$\int_0^t\int_\Omega \left(\frac{|\mathbf{v}^n|^2}{2}\big(\mathbf{v}^n * \omega^{1/n}\big) + p^n\mathbf{v}^n\right) \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau.$$

This, however, follows from (B.3.68), (B.3.69) and $(\mathrm{B}.3.55)_3$ provided that

$$r > \frac{9}{5}.$$

3.10. *Initial conditions*

The property (B.2.14) is an easy consequence of the energy inequality (B.2.17) and the following operations

$$\begin{aligned} \big\|\mathbf{v}(t) - \mathbf{v}_0\big\|_2^2 &= \big\|\mathbf{v}(t)\big\|_2^2 + \|\mathbf{v}_0\|_2^2 - 2\big(\mathbf{v}(t), \mathbf{v}_0\big) \\ &= \big\|\mathbf{v}(t)\big\|_2^2 - \|\mathbf{v}_0\|_2^2 - 2\big(\mathbf{v}(t) - \mathbf{v}_0, \mathbf{v}_0\big) \\ &\overset{(\mathrm{B}.2.17)}{\leqslant} -2\int_0^t \big[\big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{D}(\mathbf{v})\big) - \langle \mathbf{b}, \mathbf{v}\rangle\big]\,\mathrm{d}\tau - 2\big(\mathbf{v}(t) - \mathbf{v}_0, \mathbf{v}_0\big). \end{aligned} \tag{B.3.94}$$

Letting $t \to 0_+$ in (B.3.94) we conclude (B.2.14) from $(\mathrm{B}.2.12)_1$ and the fact that $(\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v})) - \langle \mathbf{b}, \mathbf{v}\rangle \in L^1(0, T)$.

4. On the smoothness of flows

4.1. *A survey of regularity results*

The alternate title for this section could be *higher differentiability* of weak solution $(\mathbf{v}, p)$ of the problem $(\mathcal{P})$.

For simplicity, we set

$$\mathbf{b} = \mathbf{0}.$$

Since we deal with the spatially periodic problem we do not confront technical difficulties due to localization.

For $j = 1, 2, 3$, let $\mathbf{e}^j$ denote the basis vector in $\mathbb{R}^3$ ($\mathbf{e}^j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$, δ_{ij} being the Kronecker delta). Let $\delta_0 > 0$ be fixed. Introducing for $h \in (0, \delta_0)$ the notation

$$\Delta_j^h z(t,x) = z^{[+h\mathbf{e}^j]}(t,x) - z(t,x) := z\big(t, x + h\mathbf{e}^j\big) - z(t,x),$$

we observe that (B.2.15) implies that

$$\begin{aligned}\big\langle\big[\Delta_j^h\mathbf{v}\big]_{,t}, \boldsymbol{\varphi}\big\rangle &+ \int_\Omega \big[\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)^{[+h\mathbf{e}^j]} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big]\cdot\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}x \\ &= \big(\big(\Delta_j^h\mathbf{v}\big)\otimes\mathbf{v}^{[+h\mathbf{e}^j]}, \nabla\boldsymbol{\varphi}\big) + \big(\mathbf{v}\otimes\Delta_j^h\mathbf{v}, \nabla\boldsymbol{\varphi}\big) + \big(\Delta_j^h p, \operatorname{div}\boldsymbol{\varphi}\big) \end{aligned}\tag{B.4.1}$$

for all $\boldsymbol{\varphi} \in L^s(0,T; W^{1,s}_{\mathrm{per}})$ with $s = r$ if $r \geqslant 11/5$ and $s = 5r/(5r-6)$ if $6/5 \leqslant r < 11/5$ almost everywhere in $(0,T)$. It is a direct consequence of the requirements on $\boldsymbol{\varphi}$ in (B.4.1) and (B.2.15) that we can set $\boldsymbol{\varphi} = \Delta_j^h\mathbf{v}$ in (B.4.1) only if $r \geqslant 11/5$. In order to relax such an a priori bound on r, we can use instead of (B.4.1) the weak formulation of the problem $(\mathcal{P}^\eta)$. Then for $(\mathbf{v}, p) = (\mathbf{v}^\eta, p^\eta)$ we have

$$\begin{aligned}\big\langle\big[\Delta_j^h\mathbf{v}\big]_{,t}, \boldsymbol{\varphi}\big\rangle &+ \int_\Omega \big[\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)^{[+h\mathbf{e}^j]} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)\big]\cdot\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}x \\ &= \big(\Delta_j^h p, \operatorname{div}\boldsymbol{\varphi}\big) + \big(\big(\big(\Delta_j^h\mathbf{v}\big)*\omega^\eta\big)\otimes\mathbf{v}^{[+h\mathbf{e}^j]}, \nabla\boldsymbol{\varphi}\big) + \big(\big(\mathbf{v}*\omega^\eta\big)\otimes\Delta_j^h\mathbf{v}, \nabla\boldsymbol{\varphi}\big) \end{aligned}\tag{B.4.2}$$

valid for a.a. $t \in (0,T)$ and for all $\boldsymbol{\varphi} \in L^r(0,T; W^{1,r}_{\mathrm{per}})$.

In (B.4.2), we are allowed to take $\boldsymbol{\varphi} = \Delta_j^h\mathbf{v}$, having the same restriction on r needed for the solvability of (B.4.2), i.e., $r > 8/5$. We aim to obtain higher differentiability estimates uniformly w.r.t. $\eta > 0$. Inserting $\boldsymbol{\varphi} = \Delta_j^h\mathbf{v}$ into (B.4.2), noting that $\operatorname{div}\Delta_j^k\mathbf{v} = 0$ implies that the term involving the pressure as well as the last term appearing in (B.4.2) vanish, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Delta_j^h\mathbf{v}\big\|_2^2 &+ \big(\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big)^{[+h\mathbf{e}^j]} - \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \big[\mathbf{D}(\mathbf{v})\big]^{[+h\mathbf{e}^j]} - \mathbf{D}(\mathbf{v})\big) \\ &= -\big(\big(\Delta_j^h\mathbf{v}*\omega^\eta\big)\otimes\Delta_j^h\mathbf{v}, \nabla\mathbf{v}^{[+h\mathbf{e}^j]}\big) \leqslant \|\nabla\mathbf{v}\|_r\big\|\Delta_j^h\mathbf{v}\big\|^2_{2r/(r-1)}. \end{aligned}\tag{B.4.3}$$

Using (B.2.21) for $r \geqslant 2$ (or (B.2.22) for $r < 2$), it follows from (B.4.3) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta_j^h \mathbf{v}\right\|_2^2 + \left\|\Delta_j^h \mathbf{D}(\mathbf{v})\right\|_2^2 + \left\|\Delta_j^h \mathbf{D}(\mathbf{v})\right\|_r^r \leqslant \|\nabla \mathbf{v}\|_r \left\|\Delta_j^h \mathbf{v}\right\|_{2r/(r-1)}^2. \tag{B.4.4}$$

If one deduces from (B.4.4) higher differentiability estimates (even fractional ones suffice), the compact embedding theorem (the Aubin–Lions lemma) then leads to almost everywhere convergence for the velocity gradients. We can then take to the limit, as $\eta \to 0$, from (B.3.49) to (B.2.15). The higher differentiability estimates thus represent another method to establish the long-time and large-data existence of a weak solution to the problem $(\mathcal{P})$.

Building on the original contribution by Málek, Nečas and Růžička [83] and Bellout, Bloom and Nečas [9], see also [82], where however smoother approximations of the problem $(\mathcal{P})$ are considered,[20] it seems very likely that the procedure outlined above will bear fruit and one can thus find (fractional) higher differentiability estimates for $r > 9/5$. It is worth noting that this range for r's coincides with that required for the existence of a suitable weak solution.

To be more precise, the following results are in place (see [9,82,83,87]).

THEOREM 4.1. (i) *If* $r \geqslant 11/5$ *then there is a weak solution* $(\mathbf{v}, p)$ *to the problem* $(\mathcal{P})$ *fulfilling*

$$\left.\begin{array}{l} \sup_{t\in\langle 0,T\rangle} \left\|\nabla \mathbf{v}(t)\right\|_2^2 \\ \int_0^T \left(\kappa \left\|\nabla^2 \mathbf{v}\right\|_2^2 + \left\|\mathbf{D}(\mathbf{v})\right\|_{\mathcal{N}^{2/r,r}(\Omega)}^r + \|\nabla \mathbf{v}\|_{3r}^r\right) \mathrm{d}t \end{array}\right\} \leqslant C\left(\|\nabla \mathbf{v}_0\|_2\right), \tag{B.4.5}$$

$$\int_0^T \|\mathbf{v}_{,t}\|_2^2 \,\mathrm{d}t + \sup_{t\in\langle 0,T\rangle} \left\|\nabla \mathbf{v}(t)\right\|_r^r \leqslant C\left(\|\nabla \mathbf{v}_0\|_r\right), \tag{B.4.6}$$

$$\left.\begin{array}{l} \sup_{t\in\langle 0,T\rangle} \|\mathbf{v}_{,t}\|_2^2 \\ \kappa \int_0^T \|\nabla \mathbf{v}_{,t}\|_2^2 \,\mathrm{d}t + \int_0^T\int_\Omega \left|\mathbf{D}(\mathbf{v})\right|^{r-2} \left|\mathbf{D}(\mathbf{v}_{,t})\right|^2 \mathrm{d}x\,\mathrm{d}t \end{array}\right\} \leqslant C\left(\|\mathbf{v}_0\|_{2,q}\right), \tag{B.4.7}$$

where $\kappa = 0$ *or* 1 *according to* (B.2.8), $q > 3$ *and*

$$\|z\|_{\mathcal{N}^{\alpha,r}} := \left(\sup_{0<h\leqslant\delta_0} \int_\Omega \frac{|z(x+h) - z(x)|^r}{h^{\alpha r}} \,\mathrm{d}x\right)^{1/r}.$$

(ii) *If* $r \in \langle 2, \frac{11}{5})$, *then there is a weak solution* $(\mathbf{v}, p)$ *to the problem* $(\mathcal{P})$ *such that* $\mathbf{v}$ *fulfills*

$$\kappa \int_0^T \left\|\nabla^2 \mathbf{v}\right\|_2^{2(3r-5)/(r+1)} \mathrm{d}t + \int_0^T \|\nabla \mathbf{v}\|_{W^{s,r}}^{r^2(3r-5)/(3(r^2-3r+4))} \mathrm{d}t < \infty, \quad s \in \left(0, \frac{2}{r}\right). \tag{B.4.8}$$

[20] In [83], the estimates are derived directly for the Galerkin approximations using a smooth basis of functions. In [9], a multipolar fluid model is used as a smooth approximation. Both approximations thus allow us to differentiate the equations of the approximate problems.

(iii) *If* $r \in (\frac{9}{5}, 2)$, *then there is* $(\mathbf{v}, p)$ *to the problem* $(\mathcal{P})$ *such that* $\mathbf{v}$ *satisfies*

$$\int_0^T \|\nabla^2 \mathbf{v}\|_r^{r(5r-9)/(-r^2+8r-9)}\,\mathrm{d}t \leqslant \infty. \tag{B.4.9}$$

In particular, for the spatially-periodic problem described by the Navier–Stokes equations in three dimensions, it follows from Theorem 4.1 that (set $r = 2$ in (B.4.8)) there is a weak solution $(\mathbf{v}, p)$ such that

$$\int_0^T \|\nabla^2 \mathbf{v}\|_2^{2/3}\,\mathrm{d}t < \infty, \tag{B.4.10}$$

the result was established by Foiaş, Guillopé and Temam [38].

Málek, Nečas and Růžička [84] considered the no-slip boundary condition for the case $r \geqslant 2$ with $\kappa = 1$ and showed that

- if $r \geqslant 9/4$ (and $r < 3$) then there is a weak solution that fulfills

$$\begin{aligned}
&\mathbf{v} \in L^{2/(2-r)}\big(0, T; W^{2,6/(r+1)}(\Omega)\big)^3 \cap L^2\big(0, T; W^{2,2}_{\mathrm{loc}}\big),\\
&\mathbf{v}_{,t} \in L^2\big(0, T; L^2(\Omega)^3\big),\\
&\int_0^T\int_{\Omega_0} \big(1 + |\mathbf{D}(\mathbf{v})|\big)^{r-2} |\mathbf{D}(\nabla\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant K \quad \text{for all } \Omega_0 \subset\subset \Omega;
\end{aligned} \tag{B.4.11}$$

- if $r \in \langle 2, \frac{9}{4})$ then

$$\int_0^T \|\nabla^2 \mathbf{v}\|_{6/(r+1)}^{(2/3)(2r-3)/(r-1)}\,\mathrm{d}t \leqslant K < \infty. \tag{B.4.12}$$

Note that (B.4.12) implies (B.4.10) even for the Dirichlet (no-slip) boundary value problem.

Studies relevant to other boundary conditions can be found in [104] and [8].

Instead of proving (B.4.5)–(B.4.10) rigorously, we rather provide a cascade of formal inequalities that form however the essence of the arguments. Details and many extensions can be found[21] in [82], [84] and [25]. This cascade consists of three levels of inequalities and starts with the energy inequality at level zero.

Level 1. Differentiate $(\text{B.1.4})_2$ w.r.t. x_s and form the scalar product of the result with $\partial\mathbf{v}/\partial x_s$.

Level 2. Form the scalar product of $(\text{B.1.4})_2$ with $\mathbf{v}_{,t}$.

Level 3. Differentiate $(\text{B.1.4})_2$ w.r.t. time t and use $\mathbf{v}_{,t}$ to form the scalar product.

For $r \geqslant 11/5$ the procedure leads to (B.4.5)–(B.4.7). For the Navier–Stokes equations, (B.4.5) *is not* available and there are plenty of results in literature that address the question

[21] Dealing with approximations different than those used in Section 3.

as to what conditions imply (B.4.5). The well-known Prodi–Serrin conditions[22] state that (B.4.5) holds provided that

$$\mathbf{v} \in L^q\big(0, T; L^s\big) \quad \text{with } \frac{2}{q} + \frac{3}{s} \leqslant 1, s \geqslant 3. \tag{B.4.13}$$

For $s > 3$, the result was established by Serrin [132]. The most interesting limiting case $L^\infty(0, T; L^3)$ has been analyzed recently by Escauriaza, Serëgin and Šverák [31,32,129]. Other regularity criteria are expressed in terms of the velocity gradient (see [7], for example), the vorticity, (see [19]), the pressure [10,56,95,130], or just one component of the velocity [97,98] or the velocity gradient (see [19,31]). The result in [63] and [138] extends (B.4.13) to the class $L^2(0, T; BMO)$. The regularity criteria expressed in terms of eigenvalues and eigenfunctions of the symmetric part of the velocity gradient were established in [98,99].

While fractional higher differentiability result, as that mentioned in (B.4.10), gives compactness of velocity gradient, say in all $L^q(0, T; L^q_{\mathrm{per}})$, $q < 2$, they do not give any improvement with regard to the regularity of the velocity or its gradient. In terms of our “level” inequalities (Doering and Gibbon [26] talk about the ladder where each split bar corresponds to a level above) for the Navier–Stokes equations in three dimensions, it is not known how to get to the first level from level zero. However, once level 1 is established (in fact (B.4.13) or other criteria suffice), $L^\infty(0, T; L^2_{\mathrm{per}})$ integrability of any spatial or time derivatives of any order is available, provided that data ($\mathbf{v}_0$ and $\mathbf{b}$) are sufficiently smooth.

For Ladyzhenskaya’s equations, or for the problem $(\mathcal{P})$ with $\kappa = 1$, Theorem 4.1 states that if $r \geqslant 11/5$, the first three levels (B.4.5)–(B.4.9) (of the ladder) are accessible. It is however open as to how to proceed to higher levels. More precisely, using (B.4.7) and using (B.1.4) we rewrite the problem $(\mathcal{P})$ as

$$\operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S}\big(\mathbf{D}(\mathbf{v})\big) + \nabla p = -\mathbf{v}_{,t} \in L^\infty\big(0, T; L^2_{\mathrm{per}}\big), \tag{B.4.14}$$

and we observe that we can apply the higher differentiability technique to almost all time instants. Doing so we establish the following theorem.

THEOREM 4.2. *If $r \geqslant 11/5$ then there is a solution $(\mathbf{v}, p)$ to the problem $(\mathcal{P})$ (with $\kappa = 1$) such that*

$$\sup_t \big\| \nabla^2 \mathbf{v}(t) \big\|_2^2 + \sup_t \big\| \nabla \mathbf{v}(t) \big\|^r_{\mathcal{N}^{2/r,r}(\Omega)} + \sup_t \big\| \nabla \mathbf{v}(t) \big\|^r_{3r} \leqslant K < \infty. \tag{B.4.15}$$

In particular, $\mathbf{v}$ *is bounded in* $(0, T) \times \mathbb{R}^3$.

Thus, the task (III) long-time and large-data regularity in a sense given in Section 1.2 (Part B) is fulfilled. The question whether $\nabla \mathbf{v}$ is bounded or Hölder continuous has not been answered as yet.

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[22] Prodi asks for conditions implying uniqueness of a weak solution, see [107]. It follows that the criterion coincides with that for regularity.

In the following subsections we formally establish (B.4.5)–(B.4.7), and also (B.4.15). We also discuss related results concerning local-in-time existence of solutions with integrable second derivatives.

4.2. *A cascade of inequalities for Ladyzhenskaya's equations*

In this subsection we consider Ladyzhenskaya's equations (B.1.6). It means we deal with the system (B.1.4) where $\mathbf{S}(\mathbf{D}(\mathbf{v})) = \nu_0 + \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v})$. Note also that (B.2.8) holds with $\kappa = 1$. In the sequel we sometimes use the specific structure of $\mathbf{S}$, sometimes we refer to (B.2.8).

Derivation of (B.4.5). We formally differentiate (B.1.4) with respect to the spatial variable x_s and take the scalar product of the result with $\frac{\partial \mathbf{v}}{\partial x_s}$. After summing over $s = 1, 2, 3$ and integrating by parts we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + \int_\Omega \frac{\partial \mathbf{S}(\mathbf{D}(\mathbf{v}))}{\partial \mathbf{D}} \cdot \mathbf{D}(\nabla\mathbf{v}) \otimes \mathbf{D}(\nabla\mathbf{v})\,\mathrm{d}x\\ &\quad = -\int_\Omega \frac{\partial v_k}{\partial x_s}\frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_s}.\end{aligned} \tag{B.4.16}$$

Using (B.2.8) we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + C_1\left\|\nabla^2\mathbf{v}\right\|_2^2 + C_1 J_r(\mathbf{v}) \leqslant \|\nabla\mathbf{v}\|_3^3, \tag{B.4.17}$$

where

$$J_r(\mathbf{v}) = \int_\Omega \left|\mathbf{D}(\mathbf{v})\right|^{r-2}\left|\mathbf{D}(\nabla\mathbf{v})\right|^2\mathrm{d}x.$$

Since

$$J_r(\mathbf{v}) \geqslant c^*\|\nabla\mathbf{v}\|_{3r}^r, \tag{B.4.18}$$

see [82], p. 227, and

$$J_r(\mathbf{v}) \geqslant c^{**}\left\|\mathbf{D}(\mathbf{v})\right\|_{\mathcal{N}^{2/r,r}}^r, \tag{B.4.19}$$

see [87] for the proof, then (B.4.17) and the energy inequality (B.2.17) imply (B.4.5) if $r \geqslant 3$. If $r < 3$, then we incorporate the interpolation inequalities

$$\begin{aligned}\|z\|_3 &\leqslant \|z\|_r^{(r-1)/2}\|z\|_{3r}^{(3-r)/2},\\ \|z\|_3 &\leqslant \|z\|_2^{2(r-1)/(3r-2)}\|z\|_{3r}^{r/(3r-2)},\end{aligned} \tag{B.4.20}$$

and use the splitting $\|\nabla\mathbf{v}\|_3^3 = \|\nabla\mathbf{v}\|_3^{3\alpha}\|\nabla\mathbf{v}\|_3^{3(1-\alpha)}$ for $\alpha \in \langle 0, 1\rangle$. We then obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + C_1\big\|\nabla^2\mathbf{v}\big\|_2^2 + \frac{C_1}{2}J_r(\mathbf{v}) + C_1^*\|\nabla\mathbf{v}\|_{3r}^r$$
$$\leqslant \|\nabla\mathbf{v}\|_r^{3\alpha(r-1)/2}\|\nabla\mathbf{v}\|_2^{6(1-\alpha)(r-1)/(3r-2)}\|\nabla\mathbf{v}\|_{3r}^{(3\alpha/2)(3-r)+3r(1-\alpha)/(3r-2)}. \tag{B.4.21}$$

Setting $Q_1 := \frac{3\alpha}{2}\frac{(3-r)}{r} + \frac{3(1-\alpha)}{3r-2}$, $Q_2 := \frac{3\alpha}{2}\frac{r-1}{r}$ and $Q_3 := 3(1-\alpha)\frac{r-1}{3r-2}$, we apply Young's inequality with $\delta = 1/Q_1$. Requiring that $Q_2\delta' = 1$, i.e., $Q_2 + Q_1 = 1$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + 2C_1\big\|\nabla^2\mathbf{v}\big\|_2^2 + C_1J_r(\mathbf{v}) + C_1^*\|\nabla\mathbf{v}\|_r^r \leqslant c\|\nabla\mathbf{v}\|_r^r\|\nabla\mathbf{v}\|_2^{2\lambda}, \tag{B.4.22}$$

where

$$\lambda := 2\frac{3-r}{3r-5}. \tag{B.4.23}$$

Since

$$\lambda \leqslant 1 \quad \Longleftrightarrow \quad r \geqslant \frac{11}{5},$$

we obtain (B.4.5) by applying Gronwall's lemma.

Derivation of (B.4.6). The scalar multiplication of (B.1.4) with $\mathbf{v}_{,t}$ and integration over Ω leads to

$$\|\mathbf{v}_{,t}\|_2^2 - \big(\operatorname{div}\mathbf{S}\big(\mathbf{D}(\mathbf{v})\big), \mathbf{v}_{,t}\big) + (\mathbf{v}_{,t}\otimes\mathbf{v}, \nabla\mathbf{v}) = 0. \tag{B.4.24}$$

Using the specific form of $\mathbf{S}$ and integrating by parts we obtain

$$\|\mathbf{v}_{,t}\|_2^2 + \frac{\nu_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + \frac{\nu_1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\mathbf{D}(\mathbf{v})\big\|_r^r = (\mathbf{v}_{,t}\otimes\mathbf{v}, \nabla\mathbf{v})$$
$$\leqslant \frac{1}{2}\|\mathbf{v}_{,t}\|_2^2 + \big\||\mathbf{v}|\,|\nabla\mathbf{v}|\big\|_2^2. \tag{B.4.25}$$

Since the estimate for the last term can be established with the help of $W^{1,3r} \hookrightarrow L^\infty$

$$\int_\Omega |\mathbf{v}|^2|\nabla\mathbf{v}|^2\,\mathrm{d}x \leqslant \|\mathbf{v}\|_\infty^2\|\nabla\mathbf{v}\|_2^2 \leqslant C\Big(\sup_t\big\|\nabla\mathbf{v}(t)\big\|_2^2\Big)\|\nabla\mathbf{v}\|_{3r}^2, \tag{B.4.26}$$

we see that (B.4.6) follows after integrating (B.4.25) over time and applying (B.4.5).

It is also possible to conclude from (B.4.24) that for $q > d$,

$$\big\|\mathbf{v}_{,t}(0)\big\|_2^2 \leqslant C\big(\|\mathbf{v}_0\|_{2,q}\big). \tag{B.4.27}$$

Derivation of (B.4.7). A formal differentiation of (B.1.4) with respect to time and forming the scalar product of the result with $\mathbf{v}_{,t}$ lead to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_{,t}\|_2^2+\int_\Omega \frac{\partial \mathbf{S}(\mathbf{D}(\mathbf{v}))}{\partial \mathbf{D}}\cdot \mathbf{D}(\mathbf{v}_{,t})\otimes \mathbf{D}(\mathbf{v}_{,t})\,\mathrm{d}x=(\mathbf{v}_{,t}\otimes \mathbf{v},\nabla \mathbf{v}_{,t}) \tag{B.4.28}$$

since $(p_{,t},\operatorname{div}\mathbf{v}_{,t})=0$ and $(\mathbf{v}\otimes\mathbf{v}_{,t},\nabla\mathbf{v}_{,t})=(\mathbf{v},\nabla\frac{|\mathbf{v}_{,t}|^2}{2})=-(\operatorname{div}\mathbf{v},\frac{|\mathbf{v}_{,t}|^2}{2})=0$. Using (B.2.8), and applying Hölder's inequality to the right-hand side of (B.4.28) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_{,t}\|_2^2+\kappa\|\nabla\mathbf{v}_{,t}\|_2^2+\int_\Omega\big|\mathbf{D}(\mathbf{v})\big|^{r-2}\big|\mathbf{D}(\mathbf{v}_{,t})\big|^2\,\mathrm{d}x\\
&\quad\leqslant \frac{\kappa}{2}\|\nabla\mathbf{v}_{,t}\|_2^2+\|\mathbf{v}\|_\infty^2\|\mathbf{v}_{,t}\|_2^2\\
&\quad\leqslant \frac{\kappa}{2}\|\nabla\mathbf{v}_{,t}\|_2^2+C\|\nabla\mathbf{v}\|_{3r}^2\|\mathbf{v}_{,t}\|_2^2.
\end{aligned} \tag{B.4.29}$$

The use of Gronwall's lemma and (B.4.27) complete the formal proof of (B.4.7).

4.3. *Boundedness of the velocity*

Derivation of (B.4.15). Since $r\geqslant 11/5$, (B.4.5)–(B.4.7) hold. We proceed in a manner similar to that for obtaining (B.4.16), the term with the time derivative $\mathbf{v}_{,t}$ is however treated differently,

$$(\nabla\mathbf{v}_{,t},\nabla\mathbf{v})=-(\mathbf{v}_{,t},\Delta\mathbf{v})\leqslant\|\mathbf{v}_{,t}\|_2\big\|\nabla^2\mathbf{v}\big\|_2.$$

Using Hölder's inequality and (B.4.18) we have, instead of (B.4.17),

$$C_1\big\|\nabla^2\mathbf{v}\big\|_2^2+C_1J_r(\mathbf{v})+\|\nabla\mathbf{v}\|_{3r}^r\leqslant\|\mathbf{v}_{,t}\|_2^2+\|\nabla\mathbf{v}\|_3^3. \tag{B.4.30}$$

The interpolation inequality $\|z\|_3\leqslant\|z\|_r^{(r-1)/2}\|z\|_{3r}^{(3-r)/2}$ then gives

$$\begin{aligned}
&C_1\big\|\nabla^2\mathbf{v}(t)\big\|_2^2+C_1J_r\big(\mathbf{v}(t)\big)+C_1\|\nabla\mathbf{v}\|_{3r}^r\\
&\quad\leqslant\big\|\mathbf{v}_{,t}(t)\big\|_2^2+\big(\big\|\nabla\mathbf{v}(t)\big\|_r^r\big)^{3(r-1)/(2r)}\big(\big\|\nabla\mathbf{v}(t)\big\|_{3r}^r\big)^{(3-r)/(2r)}.
\end{aligned} \tag{B.4.31}$$

As $\sup_t\|\mathbf{v}_{,t}(t)\|_2^2<\infty$ due to (B.4.7) and $\sup_t\|\nabla\mathbf{v}(t)\|_r^r<K\leqslant\infty$ in virtue of (B.4.6), and $(3-r)/2r<1$, we obtain (B.4.15). Since $W^{2,2}(\Omega)\hookrightarrow\mathcal{C}^{0,1/6}(\Omega)$, we conclude from (B.4.15) that

$$\mathbf{v}\in L^\infty\big(0,T;\mathcal{C}^{0,1/6}(\Omega)\big). \tag{B.4.32}$$

In particular, $\mathbf{v}$ is bounded in $(0,T)\times\Omega$.

4.4. *Fractional higher differentiability*

Let $5/3 < r < 11/5$ ($\Leftrightarrow \lambda > 1$). Since

$$J_r(\mathbf{v}) \geqslant C_{11} \begin{cases} \kappa \left\| \nabla^2 \mathbf{v} \right\|_2^2 + \left\| \mathbf{D}(\mathbf{v}) \right\|^r_{\mathcal{N}^{2/r,r}(\Omega)} & \text{if } r \geqslant 2, \\ C \frac{\|\nabla^2 \mathbf{v}\|_r^2}{(1+\|\nabla \mathbf{v}\|_r)^{2-r}} & \text{if } r < 2, \end{cases}$$

it follows from (B.4.22), see [82] for details, that

$$\kappa \int_0^T \frac{\|\nabla^2 \mathbf{v}\|_2^2}{(1+\|\nabla \mathbf{v}\|_2^2)^\lambda} + \frac{\|\mathbf{D}(\mathbf{v})\|^r_{\mathcal{N}^{2/r,r}(\Omega)}}{(1+\|\nabla \mathbf{v}\|_2^2)^\lambda} \, \mathrm{d}t \leqslant \infty \quad \text{if } r \geqslant 2 \tag{B.4.33}$$

and

$$\int_0^T \frac{\|\nabla^2 \mathbf{v}\|_r^2}{(1+\|\nabla \mathbf{v}\|_r)^{2-r}} \frac{1}{(1+\|\nabla \mathbf{v}\|_2^2)^\lambda} \, \mathrm{d}t \leqslant \infty \quad \text{if } r < 2. \tag{B.4.34}$$

Hölder's inequality and the energy inequality then lead to (B.4.8) and (B.4.10). Using such estimates, we can then apply the interpolation inequalities to obtain fractional higher differentiability with the exponent greater than one. For example, for the Navier–Stokes equations we know from (B.4.8) and (B.2.12) that

$$\int_0^T \|\mathbf{v}\|_{1,2}^2 \, \mathrm{d}t < \infty \quad \text{and} \quad \int_0^T \|\mathbf{v}\|_{2,2}^{2/3} \, \mathrm{d}t < \infty.$$

This then implies that

$$\int_0^T \|\mathbf{v}\|_{1+s,2}^{2/(2s+1)} \, \mathrm{d}t < \infty \quad \text{and} \quad \frac{2}{2s+1} \geqslant 1 \quad \Longleftrightarrow \quad s \leqslant \frac{1}{2}, \quad s \in \langle 0, 1\rangle.$$

4.5. *Short-time or small-data existence of a "smooth" solution*

Inequalities of the type (B.4.22) that can be rewritten in the simplified form

$$y'(t) \leqslant g(t) y(t)^\lambda, \quad \text{where } y(t) \geqslant 0 \text{ and } g \in L^1(0,T), \tag{B.4.35}$$

serve, if $\lambda > 1$, as the key in proving either short-time and large-data or long-time and small-data existence of a "smooth" solution.

Note that (B.4.22) takes the form of (B.4.35) with $y(t) = \|\nabla \mathbf{v}\|_2^2$, $g(t) = \|\nabla \mathbf{v}\|_r^r$ and $\lambda = 2\frac{3-r}{3r-5}$, and the energy inequality (B.2.17) implies that for all $T > 0$,

$$\int_0^T \|\nabla \mathbf{v}\|_r^r \leqslant c \|\mathbf{v}_0\|_2^2. \tag{B.4.36}$$

If $\lambda > 1$, (B.4.35) implies that

$$y(t) \leqslant \frac{y(0)}{(1 - I(t)(\lambda-1)[y(0)]^{\lambda-1})^{1/(\lambda-1)}} \quad \text{with } I(t) := \int_0^t g(\tau)\,\mathrm{d}\tau, \tag{B.4.37}$$

and we observe that

$$\sup_t y(t) \leqslant K < \infty,$$

provided that

$$1 - I(t)(\lambda-1)\big[y(0)\big]^{\lambda-1} \geqslant \frac{1}{2}. \tag{B.4.38}$$

In the case of (B.4.22), condition (B.4.38) reads

$$2(\lambda-1)\left(\int_0^t \|\nabla\mathbf{v}\|_r^r\,\mathrm{d}\tau\right)\|\nabla\mathbf{v}_0\|_2^{2(\lambda-1)} \leqslant 1. \tag{B.4.39}$$

It follows from (B.4.36) that (B.4.39) holds for all $t > 0$ provided that

$$2(\lambda-1)c\|\mathbf{v}_0\|_2^2\|\nabla\mathbf{v}_0\|_2^{2(\lambda-1)} \leqslant 1. \tag{B.4.40}$$

Thus, if $\mathbf{v}_0 \in W^{1,2}_{\mathrm{per}}$ meets (B.4.40), there is a solution $\mathbf{v}$ such that for all $T > 0$,

$$\sup_{t\in(0,T)} \big\|\nabla\mathbf{v}(t)\big\|_2^2 \leqslant 2\|\nabla\mathbf{v}_0\|_2^2.$$

Since $\int_0^t \|\nabla\mathbf{v}\|_r^r\,\mathrm{d}\tau \to 0$ as $t \to 0_+$, it also follows from (B.4.37) and (B.4.39) that for any $\mathbf{v}_0 \in W^{1,2}_{\mathrm{per}}$ there is $t^* > 0$ such that a weak solution $\mathbf{v}$ fulfills

$$\sup_{t\in(0,t^*)} \big\|\nabla\mathbf{v}(t)\big\|_2^2 \leqslant 2\|\nabla\mathbf{v}_0\|_2^2. \tag{B.4.41}$$

In order to have an explicit bound on the length t^*, one can proceed slightly differently starting again from inequality (B.4.17). If we apply only the second interpolation inequality from (B.4.20) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_2^2 + C_1\big\|\nabla^2\mathbf{v}\big\|_2^2 + \frac{C_1}{2}J_r(\mathbf{v}) + \|\nabla\mathbf{v}\|_{3r}^r \\ &\quad \leqslant c\|\nabla\mathbf{v}\|_2^{6(r-1)/(3r-2)}\|\nabla\mathbf{v}\|_{3r}^{3r/(3r-2)}. \end{aligned} \tag{B.4.42}$$

Since $\frac{3}{3r-2} < 1$ if and only if $r > 5/3$, Young's inequality leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathbf{v}\|_2^2 + 2C_1\left\|\nabla^2\mathbf{v}\right\|_2^2 + C_1 J_r(\mathbf{v}) + \|\nabla \mathbf{v}\|_{3r}^r \leqslant c\|\nabla \mathbf{v}\|_2^{6(r-1)/(3r-5)}. \quad \text{(B.4.43)}$$

This is an inequality of the type

$$y' \leqslant c y^{\mu} \quad \text{with } \mu = \frac{3(r-1)}{3r-5} > 1.$$

Proceeding as above we observe that (B.4.41) holds provided

$$0 < t^* \leqslant \frac{1}{2(\mu-1)\|\nabla \mathbf{v}_0\|_2^{2(\mu-1)}}.$$

To summarize, the following results follow from (B.4.22) (and a discussion of the level inequalities) for the Navier–Stokes equations and Ladyzhenskaya's equations:

- three-dimensional flows of the Navier–Stokes fluids starting with smooth initial flow $\mathbf{v}_0$ are smooth on a certain time interval $(0, t^*)$. If the smooth initial condition $\mathbf{v}_0$ also fulfills (B.4.40), long-time and small-data existence of the smooth solution takes place. See [23,61,67,77,78,103,144] or [142];
- three-dimensional flows of a power-law fluid or a Ladyzhenskaya's fluid with $r \in (\frac{5}{3}, \frac{11}{5})$ fulfill[23] (B.4.5)–(B.4.7) on a certain $(0, t^*)$ for any smooth initial data. In particular, $\mathbf{v}$ is bounded on $(0, t^*)$. If $\mathbf{v}_0$ also fulfills (B.4.40), long-time (and small-data) existence of flows $\mathbf{v}$ meeting (B.4.5)–(B.4.7) and (B.4.15) are possible. Again $\mathbf{v}$ remains bounded provided $\mathbf{v}_0 \in W^{2,q}$, $q > 3$.

Long-time existence of $\mathcal{C}(0, T; W^{2,q})$-solutions for a small data $\mathbf{v}_0 \in W^{2,q}$, $q > 3$, is also established in [1]. An improvement in the short-time and large-data existence for the range $r > 5/3$ up to $r > 7/5$ is presented in [25].

5. Uniqueness and large-data behavior

The aim, to show internal mathematical consistency for Ladyzhenskaya's equations if $r \geqslant 11/5$, will be completed by establishing two results on the continuous dependence of flows on data, implying uniqueness. As a consequence, the asymptotic structure of all possible flows as $t \to \infty$ can be studied. We present results on the existence of an *exponential attractor*. This is a *compact* set in the function space of initial conditions, *invariant* with respect to a solution semigroup, having *finite fractal dimension* and attracting all trajectories *exponentially*.

[23] Strictly speaking inequalities (B.4.5)–(B.4.7) hold only for $r \geqslant 2$. If $r < 2$, different norms appear in (B.4.5)–(B.4.7) (see [82,89]).

5.1. *Uniquely determined flows for Ladyzhenskaya's equations*

THEOREM 5.1. *Let* $(\mathbf{v}^1, p^1)$ *and* $(\mathbf{v}^2, p^2)$ *be two weak solutions to the problem* $(\mathcal{P})$ *corresponding to the data* $(\mathbf{v}_0^1, \mathbf{b}^1)$ *and* $(\mathbf{v}_0^2, \mathbf{b}^2)$, *respectively. If*

$$r \geqslant \frac{5}{2}, \tag{B.5.1}$$

and

$$\mathbf{v}_0^i \in L^2_{\mathrm{per}} \quad \textit{and} \quad \mathbf{b}^i \in \big(L^{r'}\big(0, T; W^{-1,r'}_{\mathrm{per}}\big)\big), \quad i = 1, 2, \tag{B.5.2}$$

then

$$\sup_t \big\| \mathbf{v}^1(t) - \mathbf{v}^2(t) \big\|_2^2 \leqslant h\big(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2, \mathbf{v}_0^2, \mathbf{b}^2\big), \tag{B.5.3}$$

where

$$\begin{aligned} &h\big(\boldsymbol{\omega}_0, \mathbf{g}, \mathbf{v}_0^2, \mathbf{b}^2\big) \\ &\quad := c_1\left(\|\boldsymbol{\omega}_0\|_2^2 + \int_0^T \|\mathbf{g}\|^{r'}_{(W^{1,r}_{\mathrm{per}})^*}\right) \exp c_2\left(\big\|\mathbf{v}_0^2\big\|_2^2 + \int_0^T \big\|\mathbf{b}^2\big\|^{r'}_{(W^{1,r}_{\mathrm{per}})^*}\right). \end{aligned}$$

Also,

$$\int_0^T \big\|\nabla\big(\mathbf{v}^1 - \mathbf{v}^2\big)\big\|_2^2 + \big\|\nabla\big(\mathbf{v}^1 - \mathbf{v}^2\big)\big\|_r^r \,\mathrm{d}t \leqslant ch\big(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2\big) \tag{B.5.4}$$

and

$$\int_0^T \big\|p^1 - p^2\big\|_{r'}^{r'} \,\mathrm{d}t \leqslant ch\big(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2\big). \tag{B.5.5}$$

In particular, the problem $(\mathcal{P})$ *is uniquely solvable in the class of weak solutions.*

PROOF. Taking the difference of (B.2.15) for $(\mathbf{v}^1, p^1)$ from (B.2.15) for $(\mathbf{v}^2, p^2)$ we find, for $r \geqslant 11/5$, an identity for $\boldsymbol{\omega} = \mathbf{v}^2 - \mathbf{v}^1$ and $q = p^2 - p^1$, namely

$$\begin{aligned} &\langle \boldsymbol{\omega}_{,t}, \boldsymbol{\varphi}\rangle + \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^2\big)\big) - \mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^1\big)\big), \mathbf{D}(\boldsymbol{\varphi})\big) \\ &\quad = (q, \operatorname{div}\boldsymbol{\varphi}) + \big\langle \mathbf{b}^2 - \mathbf{b}^2, \boldsymbol{\varphi}\big\rangle + \big(\boldsymbol{\omega} \otimes \mathbf{v}^2, \nabla\boldsymbol{\varphi}\big) + \big(\mathbf{v}^1 \otimes \boldsymbol{\omega}, \nabla\boldsymbol{\varphi}\big) \end{aligned} \tag{B.5.6}$$

valid for all $\boldsymbol{\varphi} \in W^{1,r}_{\mathrm{per}}$ and a.a. $t \in \langle 0, T\rangle$. Taking $\boldsymbol{\varphi} = \boldsymbol{\omega}$ and observing $(q, \operatorname{div}\boldsymbol{\omega}) = 0$ and

$(\mathbf{v}^1 \otimes \boldsymbol{\omega}, \nabla\boldsymbol{\omega}) = 0$, (B.5.6) implies

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_2^2 + \big(\mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^2\big)\big) - \mathbf{S}\big(\mathbf{D}\big(\mathbf{v}^1\big)\big), \mathbf{D}\big(\mathbf{v}^2 - \mathbf{v}^1\big)\big)\\ &\quad = \big\langle \mathbf{b}^1 - \mathbf{b}^2, \boldsymbol{\omega}\big\rangle - \big(\boldsymbol{\omega}\otimes\boldsymbol{\omega}, \nabla\mathbf{v}^2\big).\end{aligned} \tag{B.5.7}$$

Monotone properties of $\mathbf{S}$, i.e., (B.2.21) and (B.2.22), Korn's inequality and duality estimates that allow us to treat the term with $\mathbf{b}^1 - \mathbf{b}^2$, then yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_2^2 + \nu_0\|\nabla\boldsymbol{\omega}\|_2^2 + \frac{\nu_1}{2}\|\nabla\boldsymbol{\omega}\|_r^r\\ &\quad \leqslant c\big\|\mathbf{b}^2 - \mathbf{b}^1\big\|_{(W^{1,r}_{\mathrm{per}})^*}^{r'} + \int_\Omega |\boldsymbol{\omega}|^2\big|\nabla\mathbf{v}^2\big|\,\mathrm{d}x.\end{aligned} \tag{B.5.8}$$

Also, on using

$$\begin{aligned}\int_\Omega |\boldsymbol{\omega}|^2\big|\nabla\mathbf{v}^2\big|\,\mathrm{d}x &\leqslant \big\|\nabla\mathbf{v}^2\big\|_r\|\boldsymbol{\omega}\|_{2r/(r-1)}^2\\ &\leqslant \big\|\nabla\mathbf{v}^2\big\|_r\|\boldsymbol{\omega}\|_2^{(2r-3)/r}\|\boldsymbol{\omega}\|_6^{3/r}\\ &\leqslant c\big\|\nabla\mathbf{v}^2\big\|_r\|\boldsymbol{\omega}\|_2^{(2r-3)/r}\|\nabla\boldsymbol{\omega}\|_2^{3/r}\\ &\leqslant \frac{\nu_0}{2}\|\nabla\boldsymbol{\omega}\|_2^2 + c\big\|\nabla\mathbf{v}^2\big\|_r^{2r/(2r-3)}\|\boldsymbol{\omega}\|_2^2,\end{aligned}$$

it follows from (B.5.8) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_2^2 + \big[\nu_0\|\nabla\boldsymbol{\omega}\|_2^2 + \nu_1\|\nabla\boldsymbol{\omega}\|_r^r\big]\\ &\quad \leqslant c\Big(\big\|\mathbf{b}^1 - \mathbf{b}^2\big\|_{(W^{1,r}_{\mathrm{per}})^*}^{r'} + \big\|\nabla\mathbf{v}^2\big\|_r^{2r/(2r-3)}\|\boldsymbol{\omega}\|_2^2\Big).\end{aligned} \tag{B.5.9}$$

Neglecting the terms within the square brackets, the Gronwall lemma then completes the proof of (B.5.3) provided that $\frac{2r}{2r-3} \leqslant r$, which is exactly the condition (B.5.1). The energy inequality (B.2.17) and (B.2.23) to estimate $\int_0^T \|\nabla\mathbf{v}^2\|_r^{(2r)/(2r-3)}\,\mathrm{d}t$ is also used.

Integrating (B.5.9) over time between 0 and T, using (B.5.3) to control $\sup_t \|\boldsymbol{\omega}\|_2^2$, leads then to (B.5.4).

To conclude (B.5.5), we set $\boldsymbol{\varphi} := \nabla h$ in (B.5.6), where h solves

$$\begin{aligned}&\Delta h = |q|^{(2-r)/(r-1)}q - \frac{1}{|\Omega|}\int_\Omega |q|^{(2-r)/(r-1)}q,\\ &\quad h \text{ is } \Omega\text{-periodic}, \int_\Omega h\,\mathrm{d}x = 0.\end{aligned} \tag{B.5.10}$$

Then

$$\|\boldsymbol{\varphi}\|_{1,r} \leqslant \|h\|_{2,r} \leqslant c\left\||q|^{1/(r-1)}\right\|_r \leqslant c\|q\|_{r'}^{1/(r-1)}. \tag{B.5.11}$$

Since $\langle \boldsymbol{\omega}_{,t}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\omega}_{,t}, \nabla h\rangle = 0$, (B.5.6) with $\boldsymbol{\varphi} = \nabla h$ then leads to

$$\begin{aligned}
\|q\|_{r'}^{r'} &\leqslant c \int_\Omega |\nabla\boldsymbol{\omega}||\nabla\boldsymbol{\varphi}| + \left(\left|\mathbf{D}\left(\mathbf{v}^1\right)\right| + \left|\mathbf{D}\left(\mathbf{v}^2\right)\right|\right)^{r-2}\left|\mathbf{D}(\boldsymbol{\omega})\right|\left|\mathbf{D}(\boldsymbol{\varphi})\right| \mathrm{d}x \\
&\quad + \left\|\mathbf{b}^1 - \mathbf{b}^2\right\|_{W^{-1,r'}_{\mathrm{per}}} \|\nabla\boldsymbol{\varphi}\|_r + \left\|\mathbf{v}^1 + \mathbf{v}^2\right\|_{3r/(2(2r-3))} \|\boldsymbol{\omega}\|_{3r/(3-r)} \|\nabla\boldsymbol{\varphi}\|_r \\
&\leqslant \|\nabla\boldsymbol{\omega}\|_2 \|\nabla\boldsymbol{\varphi}\|_2 + \|\nabla\boldsymbol{\omega}\|_r \left\|\nabla\mathbf{v}^1 + \nabla\mathbf{v}^2\right\|_r^{r-2} \|\nabla\boldsymbol{\varphi}\|_r \\
&\quad + \left\|\mathbf{b}^1 - \mathbf{b}^2\right\|_{W^{-1,r'}_{\mathrm{per}}} \|\nabla\boldsymbol{\varphi}\|_r + \left\|\mathbf{v}^1 + \mathbf{v}^2\right\|_{3r/(2(2r-3))} \|\nabla\boldsymbol{\omega}\|_r \|\nabla\boldsymbol{\varphi}\|_r.
\end{aligned} \tag{B.5.12}$$

Using (B.5.11), Young's inequality, the fact that $\mathbf{v}^1 + \mathbf{v}^2 \in L^{5r/3}(0, T; L^{5r/3}_{\mathrm{per}}) \cap L^r(0, T; W^{1,r}_{\mathrm{per}})$, and finally (B.5.4), we obtain (B.5.5). □

THEOREM 5.2. *Let* $(\mathbf{v}^1, p^1)$ *and* $(\mathbf{v}^2, p^2)$ *be two weak solutions to the problem* $(\mathcal{P})$ *corresponding to the data* $(\mathbf{v}_0^1, \mathbf{b}^1)$ *and* $(\mathbf{v}_0^2, \mathbf{b}^2)$, *respectively. If*

$$r \geqslant \frac{11}{5}, \tag{B.5.13}$$

$(\mathbf{v}_0^1, \mathbf{b}^1)$ *fulfills* (B.5.2) *and*

$$\mathbf{v}_0^2 \in W^{1,2}_{\mathrm{per}}(\Omega) \quad \textit{and} \quad \mathbf{b}^2 \in L^2\left(0, T; L^2_{\mathrm{per}}\right), \tag{B.5.14}$$

then for all $t \in (0, T\rangle$ *inequalities* (B.5.3)–(B.5.5) *hold with*

$$h\left(\boldsymbol{\omega}_0, \mathbf{g}, \mathbf{v}_0^2, \mathbf{b}^2\right) := c_1\left(\|\boldsymbol{\omega}_0\|_2^2 + \int_0^T \|\mathbf{g}\|^{r'}_{(W^{1,r}_{\mathrm{per}})^*}\right) \exp c_2\left(\left\|\mathbf{v}_0^2\right\|_{1,2}^2 + \int_0^T \left\|\mathbf{b}^2\right\|_2^2\right).$$

Consequently, a weak solution fulfilling in addition (B.4.5) *is unique in the class of weak solutions. In other words, if the data fulfill* (B.5.14), *the problem* $(\mathcal{P})$ *is uniquely solvable.*

PROOF. Since $(\mathbf{v}_0^2, \mathbf{b}^2)$ fulfills (B.5.14), Theorem 4.1 implies the existence of a weak solution $\mathbf{v}^2$ fulfilling

$$\int_0^T \left\|\nabla\mathbf{v}^2\right\|_{3r}^r \mathrm{d}t \leqslant c\left(\left\|\nabla\mathbf{v}_0^2\right\|_2^2 + \int_0^T \left\|\mathbf{b}^2\right\|_2^2 \mathrm{d}t\right).$$

Proceeding step by step as in the proof of Theorem 5.1, we estimate the right-hand side of (B.5.8) as follows

$$\begin{aligned}\int_\Omega |\boldsymbol{\omega}|^2 \left|\nabla \mathbf{v}^2\right| \mathrm{d}x &\leqslant \left\|\nabla \mathbf{v}^2\right\|_{3r} \|\boldsymbol{\omega}\|^2_{6r/(3r-1)} \\ &\leqslant \left\|\nabla \mathbf{v}^2\right\|_{3r} \|\boldsymbol{\omega}\|_2^{2(2r-1)/(2r)} \|\nabla \boldsymbol{\omega}\|_2^{2/r} \\ &\leqslant \frac{\nu_0}{2} \|\nabla \boldsymbol{\omega}\|_2^2 + c\left\|\nabla \mathbf{v}^2\right\|_{3r}^{2r/(2r-1)} \|\boldsymbol{\omega}\|_2^2.\end{aligned}$$

As $\frac{2r}{2r-1} \leqslant r$ for $r \geqslant 11/5$, the remaining part of the proof coincides with that of Theorem 5.1. □

Uniqueness of a weak solution of the problem $(\mathcal{P})$ for $r \geqslant 5/2$ is stated in [67], see also [77], uniqueness for $r \geqslant 11/5$ can be found in [79] and [82].

In this context it is worth mentioning a counter-example to the uniqueness of these dimensional flows that take place in special domains that vary with time, to the Navier–Stokes equations that is due to O.A. Ladyzhenskaya (see [66]).

5.2. *Long-time behavior – the method of trajectories*

Not only are the Navier–Stokes equations the first system of nonlinear partial differential equations for which the methods of functional analysis were applied and developed,[24] the Navier–Stokes equations, at least in two dimensions, serve also as the first system of equations of mathematical physics to which the theory of dynamical systems was applied and further extended.[25] The restriction to two-dimensional flows is due to the lack of uniqueness and regularity results in three spatial dimensions.

Owing to the uniqueness of the flows $(\mathbf{v}, p)$ of the Navier–Stokes fluid in two spatial dimensions, the mapping

$$S_t : L^2_{\mathrm{per}} \to L^2_{\mathrm{per}} \quad \text{such that} \quad S_t \mathbf{v}_0 = \mathbf{v}(t)$$

possesses the semigroup property, i.e.,

$$S_0 = Id \quad \text{and} \quad S_{t+s} = S_t S_s \quad \text{for all } t, s \geqslant 0. \tag{B.5.15}$$

We recall definitions of several basic notions. For later use, let $(X, \|\cdot\|_X)$ be a normed space and $S_t : X \to X$ have the properties (B.5.15). A bounded set $B \subset X$ is said to be *uniformly absorbing* if for all $B_0 \subset X$ bounded there is $t_0 = t(B_0)$ such that $S_t B_0 \subset B$ for all $t \geqslant t_0$. A set $\widetilde{B} \subset X$ is *positively invariant* w.r.t. S_t if $S_t \widetilde{B} \subset \widetilde{B}$ for all $t \geqslant 0$. If there is a

[24] We refer the reader to [23,53,67,74,75,77,141], etc.

[25] As general reference, we provide [6,22,30,40,47,51,68,70,71,143].

bounded set $B^* \subset Y \hookrightarrow\hookrightarrow X$ that is uniformly absorbing all bounded sets in X and that is also positively invariant, then

$$\mathcal{A} := \bigcap_{s>0} \overline{\bigcup_{t \geqslant s} S_t B^*}$$

is called *global attractor* as it shares the following properties: (i) $\mathcal{A}$ is compact in X, (ii) $S_t\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$, i.e., $\mathcal{A}$ is invariant w.r.t. S_t, and (iii) $\mathcal{A}$ attracts all bounded sets of X, which means that[26] for all $B \subset X$ bounded,

$$\operatorname{dist}_X(S_t B, \mathcal{A}) \to 0 \quad \text{as } t \to \infty.$$

Compactness of the global attractor is related to the question of the finite dimension of long-time dynamics. For a compact set $C \subset X$, the fractal dimension $d_f^X(C)$ is defined as

$$d_f^X(C) := \limsup_{\varepsilon \to 0+} \frac{\log N_\varepsilon^X(C)}{\log(1/\varepsilon)},$$

where $N_\varepsilon^X(C)$ is the minimal number of ε-balls needed to cover C. According to Foiaş and Olson [39] if $d_f^X(C) < m/2, m \in \mathbb{N}$, then C can be placed into the graph of a Hölder continuous mapping from $\mathbb{R}^m$ onto C. This mapping is a projector if X is a Hilbert space. Thus the finiteness of the fractal dimension $d_f^X(\mathcal{A})$ and its estimates from above (and even more importantly from below) give a characterization of the long-time dynamics. The following elementary criterion holds (see [85], Lemma 1.3):

$$\begin{aligned} &\text{Let } (Y, \|\cdot\|_Y) \hookrightarrow\hookrightarrow (X, \|\cdot\|_X) \text{ and } C \subset X \text{ be bounded.}\\ &\text{If there is } L : X \to Y, \text{ being Lipschitz continuous on } C,\\ &\text{and } LC \subset C, \text{ then } d_f^X(C) < \infty. \end{aligned} \tag{$***$}$$

To ensure an exponential rate of attraction, Eden, Foiaş, Nikolaenko and Temam [30] enlarge the global attractor and introduce the notion that they call an *exponential attractor*. This is a subset of B^* having the following properties: (i) $\mathcal{E}$ is compact in X, (ii) $\mathcal{E}$ is positively invariant w.r.t. S_t, (iii) $d_f^X(\mathcal{E}) < \infty$, and (iv) there are constants $\alpha_1, \alpha_2 > 0$ such that $\operatorname{dist}_X(S_t B^*, \mathcal{E}) \leqslant \alpha_1 e^{-\alpha_2 t}$ for all $t \geqslant 0$.

For two-dimensional flows of the Navier–Stokes fluid, the existence of a global (minimal B) attractor $\mathcal{A} \subset L^2_{\text{per}}$ was established by Ladyzhenskaya [68]. Estimates on its fractal dimension were first studied by Foiaş and Temam [41], see also [70] for a similar criterion. The best estimates up-to-date, based on the method of Lyapunov exponents, are due to Constantin and Foiaş (see [23], for example). A proof of the existence of an exponential attractor is presented in [30].

[26] The Hausdorff distance $\operatorname{dist}_X(A, B)$ of two sets $A, B \subset X$ is defined as

$$\operatorname{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$

It is natural to ask if the long-time dynamics of three-dimensional flows of a Ladyzhenskaya's fluid share the same properties as the two-dimensional NSEs. The first result in this direction was due to Ladyzhenskaya [69,72] who proved the existence of a global attractor for $r \geqslant 5/2$, leaving however open the question of its dimension. Neither Ladyzhenskaya's criterion requiring that orthonormal projectors commute with the nonlinear operator $\operatorname{div}(\mathbf{S}(\mathbf{D}(\mathbf{v})))$ nor the method of Lyapunov exponents requiring the regularity results (not available) for the linearized problem, can be employed to show the finiteness of the fractal dimension of the attractors. Neither can the criterion $(***)$ be used for $X = L^2_{\text{per}}$ and $Y = W^{1,2}_{\text{per}}$. It is however elementary to verify $(***)$ for $X = L^2(0, \ell; L^2_{\text{per}})$ and $Y := \{\mathbf{u} \in L^2(0, \ell; W^{1,2}_{\text{per}}), \mathbf{u}_{,t} \in L^1(0, \ell; (W^{3,2}_{\text{per}})^*)\}$ where $\ell > 0$ is fixed.

This is the first motivation for working with the set of ℓ-trajectories rather than with single values $\mathbf{v}(t) \in L^2_{\text{per}}$. Further motivation comes from the uniqueness result established in Theorem 5.2 for $r \in \langle \frac{11}{5}, \frac{5}{2})$. We are not sure if just one trajectory starts from any $\mathbf{v}_0 \in L^2_{\text{per}}$ (Theorem 5.2 says it is true if $\mathbf{v}_0$ is smoother, namely $\mathbf{v}_0 \in W^{1,2}_{\text{per}}$). However, once we fix any ℓ-trajectory starting at $\mathbf{v}_0 \in L^2_{\text{per}}$, we know that it has a uniquely defined continuation, as almost all values of the ℓ-trajectory belong to $W^{1,2}_{\text{per}}$. Thus the operators

$$L_t : L^2\big(0, \ell; L^2_{\text{per}}\big) \to L^2\big(0, \ell; L^2_{\text{per}}\big), \tag{B.5.16}$$

that append to any ℓ-trajectory χ its uniquely defined shift at time t, have the semigroup property (B.5.15).

Following Málek and Pražák [85], using the semigroup (B.5.16) it is not only possible to find $\mathcal{A}_\ell \subset L^2(0, \ell; L^2_{\text{per}})$, the global attractor with respect to the semigroup L_t and with help of $(***)$ to show that its fractal dimension is finite, but introducing $\mathcal{A} \subset L^2_{\text{per}}$ as the set of all end-points of ℓ-trajectories belonging to $\mathcal{A}_\ell$, it easily follows from Lipschitz (or at least Hölder) continuity of the mapping $e : \chi \in \mathcal{A}_\ell \to \chi(\ell) \in \mathcal{A}$ that $\mathcal{A}$ is an attractor with respect to the original dynamics, with finite fractal dimension. The same approach also gives the existence of an exponential attractor.

THEOREM 5.3. *Let* $\mathbf{b} \in L^2_{\text{per}}$ *be time independent. Consider the problem* $(\mathcal{P})$ *with* $r \geqslant 11/5$ *and* $\kappa = 1$ *in* (B.2.8), *and with* $\mathbf{v}_0 \in L^2_{\text{per}}$. *Then this dynamical system possesses*

- *a global attractor* $\mathcal{A} \subset L^2_{\text{per}}$ *with finite fractal dimension,*
- *an exponential attractor* $\mathcal{E}$.

In both cases, explicit upper bounds on $d_f^X(\mathcal{A})$ *and* $d_f^X(\mathcal{E})$ *with* $X = L^2_{\text{per}}$ *are available.*

We refer to Málek and Pražák [85] for an explanation of the method of trajectories, that was introduced in [79], and for the proof of Theorem 5.3. Explicit upper bounds for $d_f^{L^2_{\text{per}}}(\mathcal{A})$ are given in [86]. See also [14] for a comparison of the estimates for two-dimensional flows obtained by the method of Lyapunov exponents on the one hand and the method $(***)$ on the other hand.

As an extreme case of the method of trajectories one can consider Sell's study of ∞-trajectories of three-dimensional Navier–Stokes equations, see [128], which is suitable for treating ill-posed problem.

6. On the structure of possible singularities for flows of the Navier–Stokes fluid

It is hardly possible to cover all the aspects related to the mathematical analysis of the Navier–Stokes equations. For other important aspects, different viewpoints and further references we refer the reader to the monographs by Constantin and Foiaş [23], Temam [144], von Wahl [149], Lions [78], Sohr [136], Ladyzhenskaya [67], Lemarié-Rieusset [74] and Cannone [18], as well as to the survey (or key) articles by Leray [75], Serrin [132], Heywood [52], Galdi [46], Wiegner [150], Kozono [62], among others.

Consider a (suitable) weak solution of the Navier–Stokes equations with $\mathbf{b} = \mathbf{0}$ and with an initial condition $\mathbf{v}_0 \in W^{k,2}(\Omega)$ for all $k \in \mathbb{N}$. Then, following the discussion in Section 5, see also [75] or [133], there is certainly a $T^* > 0$ such that $\mathbf{v}$ is a smooth flow on $[0, T^*]$. Even more, such a $\mathbf{v}$ is uniquely determined in the class of weak solutions. Since $\mathbf{v} \in L^2(0, \infty; W^{1,2}_{\mathrm{per}})$ there is T^{**} such that $\mathbf{v}_0 := \mathbf{v}(T^{**})$ fulfills (B.4.40) implying that $\mathbf{v}$ is smooth on $[T^{**}, \infty)$. Thus, possible singularities lie somewhere between T^* and T^{**}. Set

$$\sigma = \Big\{ t \in \langle 0, \infty), \limsup_{\tau \to t} \big\| \nabla \mathbf{v}(\tau) \big\|_2 = +\infty \Big\}.$$

Since $\mathbf{v} \in L^2(0, \infty; W^{1,2}_{\mathrm{per}})$, the Lebesgue measure of σ is zero.

The program to study the structure of possible singularities was initiated by Leray [75], who showed that even the $\frac{1}{2}$ Hausdorff dimension of σ is zero, $\langle 0, T\rangle \setminus \sigma$ can be written as $\bigcup_{j=1}^{\infty} (a_j, b_j)$ and if $t^* \in \sigma$ then $\|\mathbf{v}(t)\|_{1,2} \geqslant C/\sqrt{t^* - t}$ as $t \to t^*_-$. Leray proposed the construction of a weak solution exhibiting the singularity at t^* of the form[27]

$$\begin{aligned} &\mathbf{v}(t, x) = \lambda(t)\mathbf{U}\big(\lambda(t)x\big), \qquad p(t, x) = \lambda^2(t) P\big(\lambda(t)x\big) \\ &\quad \text{with } \lambda(t) = \sqrt{2a\big(t^* - t\big)}, \end{aligned} \tag{B.6.1}$$

where $a > 0$, and showed that if there is a nontrivial solution $(\mathbf{U}, P)$ of the system

$$\begin{aligned} &\operatorname{div} \mathbf{U} = 0, \\ &-2a\mathbf{U} + \operatorname{div}(\mathbf{y} \otimes \mathbf{U}) + \operatorname{div}(\mathbf{U} \otimes \mathbf{U}) - \nu_0 \Delta \mathbf{U} + \nabla P = 0 \end{aligned} \tag{B.6.2}$$

($\mathbf{y}$ is a generic point of $\mathbb{R}^3$), and if $\mathbf{U} \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^2)$, then $(\mathbf{v}, p)$ of the form (B.6.1) is a weak solution of the Navier–Stokes equations, being singular at $t = t^*$.

Based on an observation that $|\mathbf{U}|^2/2 + P + a\mathbf{y} \cdot \mathbf{U}$ satisfies the maximum principle, Nečas, Růžička and Šverák [96] show that in the class of weak solutions satisfying $\mathbf{U} \in L^3(\mathbb{R}^3)$, system (B.6.2) admits only the trivial solution, $\mathbf{U} \equiv \mathbf{0}$. Tsai [148] proves the same under more general assumptions namely if $\mathbf{U} \in L^q(\mathbb{R}^3)$ for $q > 3$ or if $\mathbf{v}$ fulfills the energy inequality considered on any ball $B \subset \mathbb{R}^3$. Clearly, the implication $\mathbf{U} \in W^{1,2}(\mathbb{R}^3) \Rightarrow \mathbf{U} \equiv \mathbf{0}$ follows from the result established in [96]. An elementary proof of this implication is given in [80], where also the so-called pseudo-self-similar solutions are introduced. Their nonexistence is established in [91].

[27] The form of $(\mathbf{v}, p)$ can be also motivated by the self-similar scaling (B.1.23).

Note that $\mathbf{v}$ of the form as in (B.6.1) is not only in $L^\infty(0,T;L^2)$ but also in $L^\infty(0,T;L^3)$ provided $\mathbf{U} \in L^3$. Note also that the self-similar transformation (B.1.23) is meaningful in any conical domain. This suggests the possibility of constructing singular solutions of the form (B.6.1) in cones. Escauriaza, Serëgin and Šverák [31–33] show, using an approach different from that used in [96], that such a solution does not exist, at least in the half-space.

Consider all points (t,x) such that $\mathbf{v}$ is bounded (or Hölder continuous) in a certain parabolic neighborhood of (t,x). Let S be the complement of such a set in $\langle 0,+\infty) \times \mathbb{R}^3$. Scheffer [123–125] started to study the Hausdorff dimension of the set of singularities S. Caffarelli, Kohn and Nirenberg [17], introducing the notion of a suitable weak solution and proving its existence, finalized these studies by showing that the one-dimensional parabolic Hausdorff measure of S is zero. Simplification of the proof and certain improvements of the technique called *partial regularity* can be found in [73,76,131] or [20].

To give a better description of the result by Caffarelli, Kohn and Nirenberg, we recall the definition of (parabolic) Hausdorff measures and related statements.

For a countable collection $\mathcal{Q} = \bigcup_{i\in\mathbb{N}} B_{\rho_i}(\mathbf{y}_i)$ in $\mathbb{R}^s$, set $S(\alpha) = \sum_{i=1}^\infty \rho_i^\alpha$. Then the α-dimensional Hausdorff measure $H^\alpha(F)$ of a Borel set $F \subset \mathbb{R}^s$ is defined as

$$H^\alpha(F) = \lim_{\delta\to 0+} \inf_{\mathcal{Q}} \left\{ S(\alpha);\, F \subset \bigcup_{i=1}^\infty B_{r_i}(x_i), \sup_{i\in\mathbb{N}} r_i < \delta \right\}.$$

Similarly, for a countable collection $\mathcal{Q}^{\mathrm{par}} = \bigcup_{i\in\mathbb{N}} Q_{r_i}(t_i,x_i)$ of parabolic balls $Q_{r_i}(t_i, x_i) = \{(\tau,y);\, \tau \in (t_i - r_i^2, t_i), |y - x_i| < r_i\}$, set $S^{\mathrm{par}}(\alpha) = \sum_{i=1}^\infty r_i^\alpha$. Then the α-dimensional parabolic Hausdorff measure $P^\alpha(E)$ of a Borel set $E \subset \mathbb{R}\times\mathbb{R}^3$ is defined as

$$P^\alpha(E) = \lim_{\delta\to 0+} \inf_{\mathcal{Q}^{\mathrm{par}}} \left\{ S^{\mathrm{par}}(\alpha);\, E \subset \bigcup_{i=1}^\infty Q_{r_i}(t_i,x_i), \sup_{i\in\mathbb{N}} r_i < \delta \right\}.$$

Clearly,

$$P^\alpha(E) = 0 \quad\Longleftrightarrow\quad \forall\varepsilon > 0\ \exists \mathcal{Q}^{\mathrm{par}} = \bigcup_{i=1}^\infty Q_{r_i}(t_i,x_i) \quad \text{such that} \quad \sum_{i\in\mathbb{N}} r_i^\alpha < \varepsilon. \tag{B.6.3}$$

If $0 < P^\alpha(E) < \infty$, then $P^{\alpha'} = 0$ for all $\alpha' > \alpha$ and $P^{\alpha''}(E) = +\infty$ for all $\alpha'' < \alpha$. If $\alpha \in \mathbb{N}$ and $P^\alpha(E) < \infty$, then E is homeomorphic to a subset in $\mathbb{R}^\alpha$.

The following characterization of smooth points is due to Caffarelli, Kohn and Nirenberg [17].

THEOREM 6.1. *Let* $(\mathbf{v}, p)$ *be a suitable weak solution to the Navier–Stokes equations. There is a universal constant* $\varepsilon^* > 0$ *such that if*

$$\limsup_{R\to 0} \frac{1}{R} \int_{Q_R(t_0,x_0)} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t < \varepsilon^*, \tag{B.6.4}$$

then for any $k \in \mathbb{N} \cup \{0\}$, *the functions* $(t, x) \mapsto \nabla^k \mathbf{v}(t, x)$ *are Hölder continuous in* $Q_{R/2}(t_0, x_0)$ *and*

$$\sup_{(\tau,\mathbf{y})\in Q_{R/2}(t_0,x_0)} \left|\nabla^k \mathbf{v}\right| \leqslant C_k R^{-(k+1)}, \tag{B.6.5}$$

C_k *being a universal constant*.

Thus, if (t^*, x^*) is a singular point, there is $Q_{R^*}(t^*, x^*)$ such that

$$\int_{Q_{R^*}(t^*,x^*)} |\nabla \mathbf{v}|^2 \,\mathrm{d}\tau \,\mathrm{d}x \geqslant \varepsilon^* R^*. \tag{B.6.6}$$

Clearly, $\bigcup_{(t^*,x^*)\in S} Q_{R^*}(t^*, x^*)$ is a collection of (parabolic) balls that cover S. Since the four-dimensional Lebesgue measure of S is zero, the four-dimensional Lebesgue measure of this covering collection can be made arbitrarily small. Vitali's covering lemma then provides the existence of a countable subcollection of mutually disjoint balls such that

$$S \subset \bigcup_{i=1}^{\infty} Q_{5R_i}\left(t^i, x^i\right), \qquad \left(t^i, x^i\right) \in S,$$

and the four-dimensional Lebesgue measure of $\bigcup_{i=1}^{\infty} Q_{R_i}(t^i, x^i)$ is so small that

$$\int_{\bigcup_{i=1}^{\infty}\{Q_{R_i}(t^i,x^i),\, R_i<\delta\}} |\nabla \mathbf{v}|^2 \,\mathrm{d}x \,\mathrm{d}t < \frac{\varepsilon^* \varepsilon}{5},$$

$\varepsilon > 0$ arbitrary. Then

$$\begin{aligned}\sum_{i=1}^{\infty} 5R_i &\leqslant \frac{5}{\varepsilon^*} \sum_{i=1}^{\infty} \int_{Q_{R_i}(t^i,x^i)} |\nabla \mathbf{v}|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &= \frac{5}{\varepsilon^*} \int_{\bigcup_{i=1}^{\infty}\{Q_{R_i}(t^i,x^i),\, R_i<\delta\}} |\nabla \mathbf{v}|^2 \,\mathrm{d}x \,\mathrm{d}t < \varepsilon.\end{aligned}$$

According to (B.6.3), $P^1(S) = 0$ and S cannot be a curve in $\mathbb{R}^+ \times \mathbb{R}^3$. Consequently,

- weak solutions of the Navier–Stokes equations in two dimensions are smooth,
- axially symmetric flows cannot have the singularity outside the set $r = 0$,
- the result concerning the zero $\frac{1}{2}$-dimensional Hausdorff measure of singular times σ follows due to the inequality $H^{1/2}(\sigma) \leqslant cP^1(S)$.

Scheffer [126] constructs an irregular (nonphysical) $\mathbf{b}$ satisfying $\mathbf{b} \cdot \mathbf{v} \leqslant 0$ so that for any $\delta > 0$ the Hausdorff dimension of singular points is above $1 - \delta$ showing the optimality of the Caffarelli, Kohn and Nirenberg result.

We refer to the above mentioned literature for further details.

7. Other incompressible fluid models

As we have been invited to address both physical and analytical aspects concerning fluids with pressure dependent viscosities for the Volume 4 of *Handbook of Mathematical Fluid Dynamics* (edited by S. Friedlander and D. Serre), without getting into any details we comment briefly on the results available for such fluids here.

7.1. *Fluids with pressure-dependent viscosity*

To our knowledge, there is no long-time and large-data existence result to the system of partial differential equations of the form as (B.1.3). Furthermore, no results concerning long-time existence for small data or short-time existence for large-data seems to be in place. Renardy [121] obtained local existence and uniqueness results in higher Sobolev spaces by assuming that the viscosity satisfies

$$\lim_{p\to+\infty}\frac{\nu(p)}{p}=0, \tag{B.7.1}$$

that clearly contradicts experimental results that are available[28] while also requiring an additional condition on eigenvalues of $\mathbf{D}(\mathbf{v})$ in terms of $\partial\nu/\partial p$.

Gazzola does not assume (B.7.1). He however establishes only short time existence of a smooth solution for small data under very restrictive conditions, both on the almost conservative specific body force $\mathbf{b}$ and the initial data.

7.2. *Fluids with pressure and shear dependent viscosities*

In the case of fluids that have a more complicated structure, namely (B.1.2), where the viscosity is not only a function of p, but depends also on the shear rate, it has been observed by Málek, Nečas and Rajagopal [81] that for certain specific forms of viscosities, long-time and large-data existence can be established. More precisely, assuming that for a $\mathcal{C}^1$-function $\mathbf{S}$ of the form $\mathbf{S}(p,\mathbf{D}(\mathbf{v}))=\nu(p,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$ there are two positive constants C_1, C_2 such that for all $\mathbf{0}\neq\mathbf{A},\mathbf{B}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$ and for all $q\in\mathbb{R}$,

$$C_1\big(1+|\mathbf{A}|^2\big)^{(r-2)/2}|\mathbf{B}|^2\leqslant\frac{\partial\mathbf{S}(q,\mathbf{A})}{\partial\mathbf{A}}\cdot(\mathbf{B}\otimes\mathbf{B})\leqslant C_2\big(1+|\mathbf{A}|^2\big)^{(r-2)/2}|\mathbf{B}|^2 \tag{B.7.3}$$

and

$$\left|\frac{\partial\mathbf{S}(q,\mathbf{A})}{\partial q}\right|\leqslant\gamma_0\big(1+|\mathbf{A}|^2\big)^{(r-2)/4}\quad\text{with }\gamma_0=\min\left(\frac{1}{2},\frac{C_1}{4C_2}\right), \tag{B.7.4}$$

[28] In fact, in most popular engineering models the relationship between ν and p is exponential, i.e.,

$$\nu(p)=\exp(\alpha_0 p),\quad \alpha_0>0. \tag{B.7.2}$$

Málek, Nečas and Rajagopal [81] established the following result.

THEOREM 7.1. *Let* $\mathbf{S}$ *satisfy* (B.7.3) *and* (B.7.4) *with* $r \in (\frac{9}{5}, 2)$. *Let* $\mathbf{v}_0 \in W^{1,2}_{\mathrm{per}}$ *and* $g \in L^2(0,T)$. *Then there is a* (*suitable*) *weak solution* $(\mathbf{v}, p)$ *to* (B.1.2) *subject to spatially periodic conditions* (B.2.9) *and the requirement* $\int_\Omega p(t,x)\,\mathrm{d}x = g(t)$ *for* $t \in (0,T)$ *such that*

$$\mathbf{v} \in \mathcal{C}\big(0,T; L^2_{\mathrm{weak}}\big) \cap L^r\big(0,T; W^{1,r}_{\mathrm{per}}\big) \cap L^{5r/3}\big(0,T; L^{5r/3}\big), \tag{B.7.5}$$

$$p \in L^{5r/6}\big(0,T; L^{5r/6}\big). \tag{B.7.6}$$

Moreover, if $r \in \langle \frac{5}{3}, 2)$ *there is a solution* $(\mathbf{v}, p)$ *such that*

$$\mathbf{v} \in L^\infty\big(0,T^*; W^{1,2}_{\mathrm{per,div}}\big) \cap L^r\big(0,T^*; W^{2,r}_{\mathrm{div}}\big), \tag{B.7.7}$$

$$p \in L^2\big(0,T^*; W^{1,2}\big). \tag{B.7.8}$$

Here, $T^* > 0$ *is arbitrary if* $\mathbf{v}_0$ *is sufficiently small or* T^* *is small enough if* $\mathbf{v}_0$ *is arbitrary.*

It is worth remarking that if one considers viscosity of the form

$$\nu\big(p, \left|\mathbf{D}(\mathbf{v})\right|^2\big) = \begin{cases} \big(1 + A + |\mathbf{D}|^2\big)^{(r-2)/2} & \text{if } p < 0, \\ \big(A + \exp(-\alpha q p) + |\mathbf{D}|^2\big)^{(r-2)/2} & \text{if } p \geqslant 0, \end{cases}$$

instead of (B.7.2), assumptions (B.7.3) and (B.7.4) are fulfilled provided that

$$2\alpha q(2-r) \leqslant (r-1)A^{(2-r)/2}.$$

This can be achieved by taking one of the parameters α or q to be sufficiently small, or A sufficiently large or r close enough to 2.

Other examples and the proof of Theorem 7.1 can be found in [81]. Two-dimensional flows are studied in [54] and [14]. In the latter, long-time behavior, based on uniqueness results, is also studied via the method of trajectories. A step towards the treatment of other boundary conditions is carried out in [42].

7.3. *Inhomogeneous incompressible fluids*

Here, we give references to results relevant to analysis of the partial differential equations (B.1.1). The first result deals with $\mathbf{T}$ of the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu(\rho)\mathbf{D}(\mathbf{v}).$$

Long-time and large-data existence of a weak solution established by the Novosibirsk school prior 1990 is presented by Antontsev, Kazhikov and Monakhov [5]. A detailed exposition is given in the first chapter of the monograph by Lions [78].

Fluids with μ depending on $|\mathbf{D}(\mathbf{v})|^2$ were analyzed in [37], where Fernadéz-Cara, Guillén and Ortega prove the existence of a weak solution to (B.1.1) with

$$\mathbf{T} = -\mathbf{I} + \left(\mu_0 + \mu_1 \left|\mathbf{D}(\mathbf{v})\right|^{r-2}\right)\mathbf{D}(\mathbf{v})$$

for $r \geqslant 12/5$. This result, treating homogeneous Dirichlet, i.e., (no-slip) boundary conditions, was recently improved upon by Guillén-Gonzáles [50], in the case of the spatially periodic problem for $r \geqslant 2$.

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CHAPTER 6

Evolution of Rate-Independent Systems

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1. Introduction

Rate-independent systems occur as limit problems in many physical and mechanical problems if the interesting time scales are much longer than the intrinsic time scales in the system. Rate-independent systems are sometimes also called quasistatic systems, however, the term "quasistatic" is often used in a more general sense, namely if the inertial terms in a system are neglected but viscous effects might still be present.

This chapter considers only systems which satisfy the following exact definition of rate independence. The definition is formulated in terms of input functions $\ell : [0,\infty) \to \mathcal{X}$ and output functions $y : [0,\infty) \to \mathcal{Y}$. The usage of input and output functions is necessary, since rate-independent systems have no own dynamics, they rather respond to changes in the input.

DEFINITION 1.1. A system $\mathcal{H}$ is called a *rate-independent system* with input data $y_0 \in \mathcal{Y}$ and $\ell \in \mathrm{C}^0([t_1,t_2],\mathcal{X})$ if the set $\mathcal{O}([t_1,t_2],y_0,\ell) \subset \mathrm{C}^0([t_1,t_2],\mathcal{Y}) \cap \{y(t_1)=y_0\}$ of possible outputs satisfies for all strictly monotone time reparametrizations $\alpha : [t_1,t_2] \to [t_1^*,t_2^*]$ with $\alpha(t_1)=t_1^*$ and $\alpha(t_2)=t_2^*$ the relation

$$y \in \mathcal{O}\big([t_1,t_2],y_0,\ell\big) \quad \Longleftrightarrow \quad y\circ\alpha \in \mathcal{O}\big(\big[t_1^*,t_2^*\big],y_0,\ell\circ\alpha\big).$$

We call the system a *multivalued evolutionary system* if the following additional conditions hold.

Concatenation:

$$\begin{aligned}&\hat{y} \in \mathcal{O}\big([t_1,t_2],y_1,\ell\big), \qquad \tilde{y} \in \mathcal{O}\big([t_2,t_3],y_2,\ell\big), \quad \hat{y}(t_2)=\tilde{y}(t_2)=y_2\\ &\quad\Longrightarrow \quad y \in \mathcal{O}\big([t_1,t_3],y_1,\ell\big), \quad \text{where } y(t)=\begin{cases}\hat{y}(t) & \text{for } t\in[t_1,t_2],\\ \tilde{y}(t) & \text{for } t\in[t_2,t_3].\end{cases}\end{aligned}$$

Restriction:

$$\begin{aligned}&t_1<t_2<t_3<t_4 \quad \text{and} \quad y \in \mathcal{O}\big([t_1,t_4],y_1,\ell\big)\\ &\quad\Longrightarrow \quad y|_{[t_2,t_3]} \in \mathcal{O}\big([t_2,t_3],y(t_2),\ell\big).\end{aligned}$$

Note that the definition is such that the system may have several solutions for a given initial value $y(t_1)$ and a given input function ℓ. Since rate-independent systems occur as limit problems, it is to be expected that the solutions are not unique without strong further assumptions.

Rate-independent systems occur on the level of ordinary differential equations as well as for partial differential equations. The simplest systems of this type arise in the limit $\varepsilon=0$ in the following systems

$$\begin{aligned}&\varepsilon M\big(t,y(t)\big)\ddot{y}(t)+D\big(t,y(t)\big)\dot{y}(t)=G\big(t,y(t)\big)\dot{\ell}(t) \quad \text{or}\\ &\varepsilon\dot{y}(t)=-\mathrm{D}\mathcal{U}\big(y(t)\big)+\ell(t),\end{aligned}$$

which appears in the slow-time limit of rigid-body dynamics. However, such smooth systems are in some sense trivial, since $y(t)$ can be obtained as a function of $\ell(t)$ without any dynamical effects. Interesting problems occur only if nonsmoothness comes into play, like in dry friction, where the frictional force R is a multivalued, nonsmooth function of the velocity v, namely $R = \mathrm{Sign}(v)$, where Sign is the multivalued signum function. Thus, replacing $\varepsilon \dot{y}(t)$ above by $R(\dot{y}) = \mathrm{Sign}(\dot{y})$, the corresponding system takes the form

$$0 \in \mathrm{Sign}\big(\dot{y}(t)\big) + \mathrm{D}\mathcal{U}\big(y(t)\big) - \ell(t), \quad y(0) = y_1.$$

Since $\mathrm{Sign}(\gamma v) = \mathrm{Sign}(v)$ for all $\gamma > 0$ and v, it is easy to see that the problem is rate independent.

Applications in partial differential equations arise naturally in the theory of elastoplasticity or if an elastic body like a rubber is drawn slowly over a rough surface such that dry friction acts but inertia does not matter. In fact, the driving problems in the theory of rate-independent hysteresis have been the theory of elastoplasticity on the one hand, and hysteresis effects in magnetism on the other hand. While in the former theory the aspect of partial differential equations was always a focus of attention, in magnetism a proper theory for the field equations was attacked only recently. Instead of this, highly complex scalar-valued hysteresis operators like the Preisach and the Prandtl–Ishlinskii operators were developed. In the latter case the ordering properties of $\mathbb{R}^1$ are essential whereas in the former theory convexity methods in Hilbert spaces are the main tool and thus vector-valued and tensor-valued generalizations of the Preisach and Prandtl–Ishlinskii operator can be treated, see [KraP89,Vis94,BrS96,Kre96]. We will survey the scalar-valued theory only little, since our focus is on methods for problems in continuum mechanics, where complex hysteretic behavior occurs through spatial variations of the internal variables.

This chapter brings together different aspects of rate-independent models or hysteresis operators in the context of continuum mechanics. In fact, there are several areas in these fields, which have evolved quite independently and have developed their own languages and notation. Here we try to compare these different approaches by translating them into one language and thus hope to provide a useful overview of the different methods in the field. We will not try to survey the whole theory of hysteresis operators and rate-independent models which started on the mechanical side more than 100 years ago but had major mathematical achievements only in the mid-1970s [Mor74,Mor76,Joh76]. The theory was formulated on the level of research monographs only 15 years later starting with [KraP89]. Afterwards, several books [Mon93,Vis94,BrS96,Kre96,Kre99] appeared, which cover a variety of different aspects. We also refer to these works for the historical background. Note that most of these books treat also a lot of models which are not rate-independent in the sense we have defined previously, but they usually involve a rate-independent operator which is embedded into a larger system which is rate independent. Nevertheless, the guiding theme of these works are the common difficulties one has in treating hysteretic behavior which is intrinsically nonsmooth. In [Vis94,Alb98] the emphasis on applications in continuum mechanics is quite similar to ours, but we restrict ourselves to pure rate independence.

The unified approach in this chapter will be a new energetic approach developed within the last five years in [MieT99,MieTL02,MieT04]. It combines in a natural way several different approaches. Classically, rate-independent systems are either written as an evolutionary (quasi)variational inequality

$$\big\langle \mathrm{D}\mathcal{E}\big(t, y(t)\big), v - \dot{y}(t)\big\rangle + \Psi\big(y(t), v\big) - \Psi\big(y(t), \dot{y}(t)\big) \geqslant 0 \quad \text{for all } v \in Y,$$

where Y is a Banach space with dual pairing $\langle\cdot,\cdot\rangle$, $\mathcal{E} : [0, T] \times Y \to \mathbb{R}$ is an energy-storage potential with Gateaux derivative $\mathrm{D}\mathcal{E}(t, y) \in Y^*$ and $\Psi : Y \times Y \to [0, \infty)$ is the dissipation (pseudo)potential. Rate independence is implemented through the assumption that $\Psi(y, \cdot)$ is homogeneous of degree 1, i.e.,

$$\Psi(y, \gamma v) = \gamma \Psi(y, v) \quad \text{for } \gamma \geqslant 0, \text{ and } \Psi(y, v) = \sup\big\{\langle\sigma, v\rangle \;\big|\; \sigma \in C_*(y)\big\},$$

where $C_*(y) \subset Y^*$ is often called the elastic domain. The equivalent formulation using subdifferentials is

$$0 \in \partial_v \Psi(y, \dot{y}) + \mathrm{D}\mathcal{E}(t, y) \subset Y^*,$$

which is a slight generalization of the doubly nonlinear form studied in [ColV90]. We continue to use $\partial_v \Psi(y, \dot{y})$ to indicate the subdifferential of $v \mapsto \Psi(y, v)$ at the point $v = \dot{y}$, i.e., only with respect to the second variable.

Using the Legendre transform $\mathcal{L}$, such that $\Psi(y, \cdot) = \mathcal{L} I_{C_*(y)}$, one arrives at the following differential inclusion, also called generalized sweeping process,

$$\dot{y}(t) \in \partial I_{C_*(y(t))}\big(-\mathrm{D}\mathcal{E}\big(t, y(t)\big)\big).$$

If the potential $\mathcal{E}(t, \cdot)$ is convex and if Ψ does not depend on $y \in Y$, then the above equations are equivalent to the following energetic formulation:

Find $y : [0, T] \to Y$ with $y(0) = y_0$ such that for all $t \in [0, T]$
the stability (S) and the energy balance (E) hold:
(S) $\mathcal{E}(t, y(t)) \leqslant \mathcal{E}(t, \hat{y}) + \Psi(\hat{y} - y(t))$ for all $\hat{y} \in \mathcal{Y}$;
(E) $\mathcal{E}(t, y(t)) + \int_0^t \Psi(\dot{y}(s))\,\mathrm{d}s = \mathcal{E}(0, y_0) + \int_0^t \partial_s \mathcal{E}(s, y(s))\,\mathrm{d}s$.

For general potentials $\mathcal{E}$, the energetic formulation may be considered as a weak form of the variational inequality, since it is derivative free for the solution y as well as for the functionals $\mathcal{E}$ and Ψ. However, the smoothness of the loadings has to be a little higher since the power of the external forces, given via $t \mapsto \partial_t \mathcal{E}(t, y)$, must be well defined.

In Section 2 we study these systems in the standard case with a quadratic energy $\mathcal{E}(t, y) = \frac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle$ on a Hilbert space Y. We compare several equivalent formulations, address their basic properties and explain the typical approaches to prove existence and uniqueness of solutions. Thus, the hysteresis operator $\mathcal{H}$ with $y = \mathcal{H}(y_0, \ell)$ can be defined and in Section 2.4 we discuss the mapping properties of $\mathcal{H}$ in different Banach spaces.

However, the main emphasis of this work will not be the continuity properties of the solution operator. We focus mainly on the question of solvability in general nonconvex problems where uniqueness does not hold and where even existence of solutions is questionable. Thus, we will mostly make simple assumptions on the temporal behavior of the loading function which often can be generalized. Instead we want to be most general in terms of the behavior of the energy $\mathcal{E}(t, y)$ on the state variable y, such that we are able to deal with generally nonconvex problems like finite-strain elastoplasticity. Also the dissipation law has to be understood in a more general setting. In particular, the dissipation potential is replaced by a more general dissipation distance $\mathcal{D}: \mathcal{Y} \times \mathcal{Y} \to [0, \infty]$ which generalizes Ψ via $\mathcal{D}(y_0, y_1) = \Psi(y_1 - y_0)$. The main emphasis will be on the topological and analytical properties of the functionals

$$\mathcal{E}: [0, T] \times \mathcal{Y} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\} \quad \text{and} \quad \mathcal{D}: \mathcal{Y} \times \mathcal{Y} \to [0, \infty].$$

In Section 3 we set up the abstract formulation and show how first a priori estimates can be used to estimate possible solutions. Moreover, we introduce a time-incremental minimization problems (IP) which will be the basis of most of our existence proofs, namely

(IP) For a given y_0 and a partition $0 = t_0 < t_1 < \cdots < t_N = T$ find $y_1, y_2, \ldots, y_N \in \mathcal{Y}$ such that $y_k \in \operatorname{Arg\,min}\{\mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \mid y \in \mathcal{Y}\}$.

Under natural conditions it is possible to derive a priori estimates for the solutions $(y_k)_{k=1,\ldots,N}$ of (IP) in the form

$$\mathcal{E}(t_k, y_k) \leqslant E_* \quad \text{and} \quad \sum_{k=1}^{N} \mathcal{D}(y_{k-1}, y_k) \leqslant E^*,$$

i.e., they are independent of the partitions. Moreover, uniform convexity of $\mathcal{E}(t, \cdot) + \mathcal{D}(y_0, \cdot)$ will provide a Lipschitz bound.

Thus, we see that the problem is governed by three different topologies (or function spaces). For instance in a Banach space setting, the energy-storage functional $\mathcal{E}$ will be coercive with respect to a Banach space Y_1. Moreover, it might be uniformly convex with respect to a norm of a larger Banach space Y_2. Finally, the dissipation distance $\mathcal{D}$ might be bounded from below by a norm of a Banach space Y_3. Then, for solutions of (S) and (E) or for piecewise constant interpolants of solutions to (IP) we can expect to obtain the following, typical a priori estimates

$$\begin{aligned} &\|y(t)\|_{Y_1} \leqslant C_1, \qquad \|y(t) - y(s)\|_{Y_2} \leqslant C_2|t - s|, \\ &\operatorname{Var}_{Y_3}\big(y; [0, T]\big) \leqslant c_3 \operatorname{Diss}_{\mathcal{D}}\big(z; [0, T]\big) \leqslant C_3, \end{aligned}$$

for $t, s \in [0, T]$. Here, the total dissipation is defined as

$$\begin{aligned} &\operatorname{Diss}_{\mathcal{D}}\big(y; [r, s]\big) \\ &\quad = \sup\left\{ \sum_{j=1}^{n} \mathcal{D}\big(y(t_{j-1}), y(t_j)\big) \,\middle|\, n \in \mathbb{N}, r \leqslant t_0 < t_1 < \cdots < t_n \leqslant s \right\} \end{aligned}$$

and $\mathrm{Var}_{Y_3}(y;[r,s])$ is obtained similarly by replacing $\mathcal{D}(y(t_{j-1}),y(t_j))$ via $\|y(t_j)-y(t_{j-1})\|_{Y_3}$. Thus, there will be two quite different approaches to show existence of solutions. The first one (see Section 4) is based on convexity, uses the norm in Y_2 and yields solutions in $\mathrm{C}^{\mathrm{Lip}}([0,T],Y_2)$. The second approach (see Section 5) works without convexity, relies on the dissipation estimate and provides solutions in $\mathrm{BV}([0,T],Y_3)$.

In Section 4 we study the convex cases in more detail. Under suitable additional smoothness assumptions it is then possible to prove existence and uniqueness of solutions. In this part no compactness arguments are needed to establish convergence; in fact, the error between the incremental solutions and the true solution can be estimated in terms of the fineness of the time discretization. This part is based on work in [MieT04,BrKS04,MieR05]. Section 4.4 shows that in the best case the solutions y have derivatives of bounded variations, i.e., $\dot{y}\in\mathrm{BV}([0,T],Y)$. Adapting the proofs in [HanR95,AlbeC00] we provide a convergence of the incremental solutions which is linear in the fineness of the partition.

In Section 5 we study general nonconvex and nonsmooth systems, where uniqueness is not to be expected. The basic existence result relies on compactness assumptions and is based on work in [MieT99,MieTL02,MaiM05,DalFT05,FM05]. In terms of the above-mentioned Banach spaces Y_1 and Y_3 the compactness assumption roughly means, that Y_1 is compactly embedded in Y_3.

It can be seen easily that the solutions of (S) and (E) may have jumps if $\mathcal{E}$ is nonconvex. Thus, in Section 5.4 we explain how rate-independent limits of viscous problems have been obtained in [EfM04]. They turn jumps into suitable continuous paths in state space. In Section 5.5 we study situations where the state space $\mathcal{Y}$ may depend on time. Finally, Section 5.6 addresses the question of relaxations of rate-independent problems, since many applications in continuum mechanics lead to systems in which the incremental problem (IP) does not have solutions due to formation of microstructure.

Section 6 is devoted to dissipation laws which are nonassociated, i.e., they cannot be derived from a principle of maximal dissipation. In particular, the energetic formulation is no longer available, since the set of frictional forces $\mathcal{R}(t,y,\dot{y})$ is no longer given by the subdifferential $\partial_v\Psi(y,\dot{y})$. In this area there is known much less, but it is of great importance in queuing theory, in plasticity models in soil mechanics and in the area of Coulomb friction of sliding elastic bodies or structures. The last application was a major stimulant for the theory of nonassociated flow rules over the last 15 years, cf. [MarO87,And91,AndK97, MarPS02].

The final Section 7 presents a selection of applications in continuum mechanics which are meant to illustrate the abstract theory developed in the previous sections. In Section 7.1 we recall the classical theory of linearized elastoplasticity, which was the main driving forces in the early mathematical developments, and in Section 7.2 we mention some result in finite-strain elastoplasticity. We also discuss some models for shape-memory alloys (cf. Section 7.3) and for ferromagnetic materials (cf. Section 7.4). Finally, we show that certain damage problems can be also put into the energetic framework, namely a delamination problem (cf. Section 7.5) as well as a problem of rate-independent crack growth in brittle materials (cf. Section 7.6). The latter application is especially interesting, as it provides a true need for the abstract formulation of the energetic problem in Section 5. In the crack problem the state space $\mathcal{Y}$ is far from being a subset of a Banach space, since $y = (u, \Gamma) \in \mathcal{Y}$ consists of subsets Γ of the body $\overline{\Omega} \subset \mathbb{R}^d$ and a deformation $u \in \mathrm{W}^{1,p}(\Omega \setminus \Gamma; \mathbb{R}^d)$.

2. The simple case with a quadratic energy

In this section we survey the classical results on evolutionary variational inequalities which can be formulated in several equivalent ways. We collect these formulations for later reference, since each of the formulations has advantages when generalizations have to be done. In the following subsection we shortly address the existence theory via monotone operators and via time-incremental methods. Finally, we will review some results on the continuity of the solution operator in different function spaces.

2.1. *Equivalent formulations*

We start with a Hilbert space Y with dual Y^* and dual pairing $\langle \cdot, \cdot \rangle : Y^* \times Y \to \mathbb{R}$ and a positive definite operator $A \in \mathrm{Lin}(Y, Y^*)$, i.e., $A = A^*$ and there exists a constant $\alpha > 0$ such that $\langle Ay, y \rangle \geqslant \alpha \|y\|^2$ for all $y \in Y$. Often, Y^* and Y are identified, A is taken to be the identity $\mathbf{1}$ and instead of the dual pairing the scalar product is used. However, as is common practice in mechanics, we prefer to distinguish the space and its dual.

For a function $\ell \in \mathrm{C}^1([0, T], Y^*)$ we define the energy functional

$$\mathcal{E}(t, y) = \frac{1}{2} \langle Ay, y \rangle - \big\langle \ell(t), y \big\rangle.$$

Here ℓ serves as input datum and is called external loading in mechanics. We will use $\Sigma = -\mathrm{D}_y \mathcal{E}(t, y) = \ell(t) - Ay$ to denote the force generated by the potential.

Moreover, let a dissipation functional $\Psi : Y \to [0, \infty]$ be given which is convex, lower semicontinuous and positively homogeneous of degree 1, i.e.,

$$\Psi(\gamma v) = \gamma \Psi(v) \quad \text{for all } \gamma \geqslant 0 \text{ and } v \in Y.$$

(Throughout we assume that "convex" already means that the function is also "proper", i.e., not identically to $+\infty$.) Its subdifferential is given via $\partial \Psi(v) = \{\sigma \in Y^* \mid \forall w \in Y\colon\ \Psi(w) \geqslant \Psi(v) + \langle \sigma, w - v \rangle\}$ and we set $C_* = \partial \Psi(0) \subset Y^*$, which is convex and closed. Duality theory shows that Ψ is the Legendre transform of the characteristic function $I_{C_*} : Y^* \to [0, \infty]$, i.e., $\Psi(v) = \sup\{\langle \sigma, v \rangle \mid \sigma \in C_*\}$.

The *subdifferential formulation* (SF) of the rate-independent hysteresis problem associated with $\mathcal{E}$ and Ψ reads

$$\text{(SF)} \qquad 0 \in \partial \Psi\big(\dot{y}(t)\big) + \mathrm{D}_y \mathcal{E}\big(t, y(t)\big) = \partial \Psi\big(\dot{y}(t)\big) + Ay - \ell(t) \subset Y^*. \tag{2.1}$$

In mechanics, (SF) is a force balance which may be written as $\Sigma \in \partial \Psi(\dot{y})$.

Using the definition of the subdifferential $\partial\Psi(\dot y)$ leads to the *variational inequality*

$$\text{(VI)} \qquad \forall v \in Y\colon \quad \big\langle Ay - \ell(t), v - \dot y\big\rangle + \Psi(v) - \Psi(\dot y) \geqslant 0. \tag{2.2}$$

This formulation is called the *primal form*, since $y \in Y$ is the primal variable while $\Sigma \in Y^*$ is the dual variable.

Using the Legendre transform $\Psi = \mathcal{L}(I_{C_*})$ we can rewrite (2.1) (which reads in short form $\Sigma \in \partial\Psi(\dot y)$) as the *differential inclusion*

$$\text{(DI)} \qquad \dot y(t) \in \partial I_{C_*}(\Sigma) = \partial I_{C_*}\big(-\mathrm{D}\mathcal{E}\big(t, y(t)\big)\big) = \mathrm{N}_{C_*}\big(\Sigma(t)\big) \subset Y, \tag{2.3}$$

where we used the standard result that for closed convex sets C_* the subdifferential $\partial I_{C_*}(\sigma)$ equals the outward normal cone $\mathrm{N}_{C_*}(\sigma) = \{v \in Y \mid \forall\hat\sigma \in C_*\colon \langle\hat\sigma - \sigma, v\rangle \leqslant 0\}$.

Introducing the variable $u = -Ay \in Y^*$ and the moving sets $C_*(t) = -\ell(t) + C_* \subset Y^*$, we arrive at the *sweeping-process* formulation

$$\text{(SW)} \qquad -\dot u(t) \in A\mathrm{N}_{C_*(t)}\big(u(t)\big), \tag{2.4}$$

which is used in [Mon93] with $A = \mathbf{1}$ since $Y = Y^*$ is assumed.

Using the definition of the subdifferential ∂I_{C_*} in (2.3) we see that (VI) is equivalent to the *dual variational inequality*

$$\text{(DVI)} \qquad \Sigma = \ell - Ay \in C_* \quad \text{and} \quad \langle\Sigma - \hat\sigma, \dot y\rangle \geqslant 0 \quad \text{for all } \hat\sigma \in C_*. \tag{2.5}$$

Integration over $[0, T]$ leads to a weakened form which allows y to lie in $\mathrm{BV}([0, T], X)$ by employing a suitable Stieltjes integral,

$$\begin{aligned} &\Sigma = \ell - Ay \in C_* \quad \text{and} \\ &\int_0^T \big\langle\Sigma(t) - \tilde\sigma(t), \mathrm{d}y(t)\big\rangle \geqslant 0 \quad \text{for all } \tilde\sigma \in \mathrm{C}^0\big([0, T], C_*\big). \end{aligned} \tag{2.6}$$

In (DVI) we may also eliminate completely the primal variable y by using $\dot y = A^{-1}(\dot\ell - \dot\Sigma)$,

$$\Sigma \in C_* \quad \text{and} \quad \forall\hat\sigma \in C_*, \quad \big\langle\Sigma - \hat\sigma, A^{-1}\big(\dot\Sigma - \dot\ell\big)\big\rangle \geqslant 0. \tag{2.7}$$

Finally, we derive the energetic formulation which is the basis for the more recent approach to general nonconvex problems. Whereas the equivalence of the above problems is well known, see, e.g., [DuL76,HanR99], the equivalence to the energetic formulation is less known. Thus, we explain it in more detail.

It can be easily seen that (VI) is equivalent to the following two local conditions:

$$\begin{aligned} &(\mathrm{S})_{\mathrm{loc}} \quad \forall\hat v \in Y\colon \quad \big\langle\mathrm{D}\mathcal{E}\big(t, y(t)\big), \hat v\big\rangle + \Psi(\hat v) \geqslant 0; \\ &(\mathrm{E})_{\mathrm{loc}} \quad \big\langle\mathrm{D}\mathcal{E}\big(t, y(t)\big), \dot y(t)\big\rangle + \Psi\big(\dot y(t)\big) \leqslant 0. \end{aligned} \tag{2.8}$$

For $(S)_{loc}$ simply let $v = \alpha\hat{v}$ with $\alpha \to \infty$ in (VI), and for $(E)_{loc}$ let $v = 0$. However, since $\mathcal{E}(t,\cdot)$ and Ψ are convex, we conclude that $y(t)$ is a global minimizer of $\hat{y} \mapsto \mathcal{E}(t,\hat{y}) + \Psi(\hat{y} - y(t))$ by letting $\hat{y} = y(t) + \hat{v}$. Moreover, $(S)_{loc}$ and $(E)_{loc}$ together imply $\langle D\mathcal{E}(t, y(t)), \dot{y}(t)\rangle + \Psi(\dot{y}(t)) = 0$, which leads to

$$\frac{d}{dt}\mathcal{E}\big(t, y(t)\big) = \partial_t \mathcal{E}\big(t, y(t)\big) - \Psi\big(\dot{y}(t)\big) = -\big\langle \dot{\ell}(t), y(t)\big\rangle - \Psi\big(\dot{y}(t)\big).$$

This leads to the *energetic formulation* which is based on the *global stability condition* (S) and the *global energy balance* (E), which is obtained by integration over $t \in [0, T]$:

$$\begin{aligned}
&\text{(S)} \quad \forall \hat{y} \in Y\colon \quad \mathcal{E}\big(t, y(t)\big) \leqslant \mathcal{E}(t,\hat{y}) + \Psi\big(\hat{y} - y(t)\big);\\
&\text{(E)} \quad \mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_\Psi\big(y; [0,t]\big) = \mathcal{E}\big(0, y(0)\big) + \int_0^t \partial_s \mathcal{E}\big(s, y(s)\big)\, ds,
\end{aligned} \tag{2.9}$$

where $\mathrm{Diss}_\Psi(y; [r,s]) = \int_r^s \Psi(\dot{y}(t))\, dt$ and $\int_0^t \partial_s\mathcal{E}(s, y(s))\, ds = -\int_0^t \langle \dot{\ell}(s), y(s)\rangle\, ds$.

The stability condition can be formulated in terms of the *sets of stable states*

$$\begin{aligned}
&\mathcal{S}(t) = \big\{ y \in Y \mid \forall \hat{y} \in Y\colon\, \mathcal{E}(t,y) \leqslant \mathcal{E}(t,\hat{y}) + \Psi(\hat{y} - y)\big\} \subset Y,\\
&\mathcal{S}_{[0,T]} = \bigcup_{t\in[0,T]} \big(t, \mathcal{S}(t)\big) \subset [0,T] \times Y.
\end{aligned}$$

Now, (S) just means $y(t) \in \mathcal{S}(t)$. The major simplification in the theory of quadratic energies arises from the fact that $\mathcal{S}(t)$ can be given explicitly in the form

$$\mathcal{S}(t) = \big\{ y \in Y \mid \ell(t) - Ay \in C_*\big\} = A^{-1}\big(\ell(t) - C_*\big).$$

Hence, $\mathcal{S}(t)$ is a closed convex set and thus it is weakly closed.

A typical situation in continuum mechanical problems is that the state variables $y \in Y$ consist of two components, namely an elastic (or nondissipative) component $u \in U$ and an internal (or dissipative) component $z \in Z$. The splitting is such that $\Psi : Y \to [0,\infty]$ depends only on $\dot{z}$ but not on $\dot{y}$. In particular, we have

$$y = (u,z) \in U \times Z = Y \quad \text{and} \quad \Psi(\dot{y}) = \Psi\big((\dot{u},\dot{z})\big) = \widetilde{\Psi}(\dot{z}), \tag{2.10}$$

where $\widetilde{\Psi}$ satisfies $\widetilde{\Psi}(\dot{z}) > 0$ for $\dot{z} \neq 0$. The linear operator A takes the form $\begin{pmatrix} A_{UU} & A_{ZU} \\ A_{UZ} & A_{ZZ}\end{pmatrix} \in \mathrm{Lin}(U \times Z, U^* \times Z^*)$, $\ell(t) = (\ell_U(t), \ell_Z(t)) \in U^* \times Z^*$ and for the subdifferential $\partial\Psi$ we have

$$\partial\Psi\big((\dot{u},\dot{z})\big) = \{0\} \times \partial\widetilde{\Psi}(\dot{z}) \in U^* \times Z^*.$$

With these definitions the subdifferential formulation (SF), see (2.1), takes the form

$$\begin{aligned}
&0 = A_{UU}u + A_{UZ}z - \ell_U(t) \in U^* \quad \text{and}\\
&0 \in \partial\widetilde{\Psi}(\dot{z}) + A_{ZU}u + A_{ZZ}z - \ell_Z(t) \subset Z^*.
\end{aligned} \tag{2.11}$$

The first equation is then called the elastic equilibrium equation, and for given z and ℓ_U it can be solved uniquely for $u \in U$, since A_{UU} is again positive definite. The second relation is the flow rule for the internal variable z.

2.2. *Basic a priori estimates and uniqueness*

We first provide a few a priori estimates for the solutions of the above formulations. In particular, we will obtain uniqueness of solutions as well as continuous dependence on the data. For this, we make the following assumption on the dissipation functional:

$$\exists c_1 > 0, \forall v \in Y: \quad \Psi(v) \geqslant c_1 \|v\|_X,$$

where $\|\cdot\|_X$ denotes a seminorm. Note that in the case $X = Y$ this implies that $C_* = \partial\Psi(0)$ satisfies $\{\sigma \in Y^* \mid \|\sigma\|_* \leqslant c_1\} \subset C_*$. Similarly, if Ψ is bounded from above by $c_2\|\cdot\|_Y$, then C_* is contained inside a ball of radius c_2.

The assumptions on A and on ℓ imply

$$\mathcal{E}(t, y) \geqslant \frac{\alpha}{2}\|y\|^2 - \|\ell(t)\|_*\|y\| \geqslant \Lambda\|y\| - \frac{1}{2\alpha}\big(\Lambda + \|\ell(t)\|_*\big)^2 \tag{2.12}$$

for any $\Lambda \geqslant 0$. Thus, with $\Lambda_0 = \|\ell\|_{L^\infty}$ and $\Lambda_1 = \|\dot{\ell}\|_{L^\infty}$, we obtain

$$\big|\partial_t\mathcal{E}(t, y)\big| \leqslant \big|\langle\dot{\ell}(t), y\rangle\big| \leqslant \Lambda_1\|y\| \leqslant \frac{\Lambda_1}{\Lambda}\left(\mathcal{E}(t, y) + \frac{(\Lambda + \Lambda_0)^2}{2\alpha}\right), \tag{2.13}$$

where $\Lambda > 0$ is still arbitrary. Further on we choose $\Lambda = \Lambda_0$.

Since any solution $y : [0, T] \to Y$ satisfies the energy balance (E) we find, using $\mathrm{Diss}_\Psi \geqslant 0$, the estimate $\mathcal{E}(t, y(t)) \leqslant \mathcal{E}(0, y(0)) + \int_0^t \frac{\Lambda_1}{\Lambda_0}(\mathcal{E}(s, y(s)) + \frac{2\Lambda_0^2}{\alpha})\,\mathrm{d}s$. Applying Gronwall's estimate to $\mathcal{E}(t, y(t)) + 2\Lambda_0^2/\alpha$ we find

$$\forall t \in [0, T]: \quad \mathcal{E}\big(t, y(t)\big) \leqslant \mathrm{e}^{t\Lambda_1/\Lambda_0}\left(\mathcal{E}(0, y_0) + \frac{2\Lambda_0^2}{\alpha}\right) - \frac{2\Lambda_0^2}{\alpha}. \tag{2.14}$$

Inserting this into (E) once again, we obtain the second estimate

$$\begin{aligned} \forall t \in [0, T]: \quad c_1 \int_0^t \big\|\dot{y}(s)\big\|_X \,\mathrm{d}s &\leqslant \mathrm{Diss}_\Psi\big(y; [0, t]\big) \\ &\leqslant \mathrm{e}^{t\Lambda_1/\Lambda_0}\left(\mathcal{E}(0, y_0) + \frac{2\Lambda_0^2}{\alpha}\right). \end{aligned} \tag{2.15}$$

It turns out that these two purely *energetic estimates* apply in very general situations as long as (2.13) holds. They imply a priori estimates for $\|y(t)\|$ via the coercivity of the energy, cf. (2.12), as well as some control on the derivative $\dot{y}(t)$ via (2.15).

However, in the case of a quadratic energy (or in general convex situations, see Section 3.5) we may also derive Lipschitz bounds using the uniform convexity due to A. Using (i) $(\mathrm{E})_{\mathrm{loc}}$ for $y(s)$, (ii) $\Psi(y(t)-y(s)) \leqslant \mathrm{Diss}_\Psi(y,[s,t])$ and (iii) (E), we obtain for $0 \leqslant s < t \leqslant T$,

$$
\begin{aligned}
&\frac{\alpha}{2}\big\|y(t)-y(s)\big\|^2\\
&\quad\leqslant \big\|y(t)-y(s)\big\|_A^2\\
&\quad= \mathcal{E}\big(s,y(t)\big)-\mathcal{E}\big(s,y(s)\big)-\big\langle \mathrm{D}\mathcal{E}\big(s,y(s)\big),y(t)-y(s)\big\rangle\\
&\quad\overset{\text{(i)}}{\leqslant} \mathcal{E}\big(s,y(t)\big)-\mathcal{E}\big(s,y(s)\big)+\Psi\big(y(t)-y(s)\big)\\
&\quad\overset{\text{(ii)}}{\leqslant} \mathcal{E}\big(s,y(t)\big)-\mathcal{E}\big(s,y(s)\big)+\mathrm{Diss}_\Psi\big(y,[s,t]\big)\\
&\quad\overset{\text{(iii)}}{=} \mathcal{E}\big(s,y(t)\big)-\mathcal{E}\big(t,y(t)\big)+\int_s^t \partial_\tau\mathcal{E}\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\\
&\quad= \int_s^t \big[\partial_\tau\mathcal{E}\big(\tau,y(\tau)\big)-\partial_\tau\mathcal{E}\big(\tau,y(t)\big)\big]\,\mathrm{d}\tau\\
&\quad\leqslant \Lambda_1\int_s^t \big\|y(t)-y(\tau)\big\|\,\mathrm{d}\tau.
\end{aligned}
$$

From this, we easily derive the Lipschitz bound (cf. Theorem 3.4),

$$
\forall s,t\in[0,T],\qquad \big\|y(t)-y(s)\big\| \leqslant \frac{\Lambda_1}{\alpha}|t-s| \quad\text{or}\quad \|\dot y\|_{\mathrm{L}^\infty((0,T),Y)} \leqslant \frac{\Lambda_1}{\alpha}.
$$

Similar estimates give the continuity with respect to the data. Let y_1 and y_2 be two solutions with data (y_j^0,ℓ_j). Then, with (VI) we find

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle A(y_1-y_2),y_1-y_2\big\rangle\\
&\quad= \langle Ay_1,\dot y_1-\dot y_2\rangle+\langle Ay_2,\dot y_2-\dot y_1\rangle\\
&\quad= -\langle Ay_1-\ell_1,\dot y_2-\dot y_1\rangle-\langle Ay_2-\ell_2,\dot y_1-\dot y_2\rangle+\langle \ell_1-\ell_2,\dot y_1-\dot y_2\rangle\\
&\quad\leqslant \Psi(\dot y_2)-\Psi(\dot y_1)+\Psi(\dot y_1)-\Psi(\dot y_2)+\|\ell_1-\ell_2\|_*\|\dot y_1-\dot y_2\|\\
&\quad\leqslant \|\ell_1-\ell_2\|_*\big(\|\dot y_1\|+\|\dot y_2\|\big).
\end{aligned}
$$

Thus, for all $t\in[0,T]$ we have the estimate

$$
\begin{aligned}
\alpha\big\|y_1(t)-y_2(t)\big\|^2 &\leqslant \big\|y_1(t)-y_2(t)\big\|_A^2\\
&\leqslant \big\|y_1^0-y_2^0\big\|_A^2+\frac{2\Lambda_1}{\alpha}\int_0^t \big\|\ell_1(s)-\ell_2(s)\big\|_*\,\mathrm{d}s. \qquad (2.16)
\end{aligned}
$$

Alternatively, the former estimate leads to

$$\begin{aligned}&\big\| y_1(t) - y_2(t) \big\|_A^2 \\ &\quad \leqslant \big\| y_1^0 - y_2^0 \big\|_A^2 + 2\|\ell_1 - \ell_2\|_{\mathrm{L}^\infty([0,T],Y^*)} \int_0^t \big(\big\| \dot y_1(s) \big\| + \big\| \dot y_2(s) \big\| \big)\, \mathrm{d}s. \qquad (2.17)\end{aligned}$$

2.3. *Basic existence theory*

There are essentially two different approaches to the existence theory. The first approach uses time discretization and solves a (static) variational inequality or a minimization problem in each time step. This method will be the main focus in this work as it generalizes to complicated nonsmooth and nonconvex situations. However, the method is restricted to symmetric operators in the variational inequality or, what is the same, to associated flow laws for the rate-independent problem.

The second approach is based on the theory of monotone or accretive operators, which is somehow more restrictive as it heavily uses the Hilbert or Banach space structure. However, it allows for more general flow laws, such as nonassociated ones, see Section 6. Here, we show how in the present situation both methods can be applied and finally mention also the so-called Yosida regularization which is often used to treat nonsmooth problems.

2.3.1. *Time-incremental minimization.* This approach will be discussed in full detail in later sections. Here we give a simplified version of the existence proof which shows the same features as in Section 5.1, where a much more general situation is treated. Here we use the simplifying structures of the quadratic energy.

We choose a sequence of partitions $0 = t_0^N < t_1^N < \cdots < t_{N-1}^N < t_N^N = T$ of the interval $[0, T]$ such that the fineness $f_N = \max\{t_j^N - t_{j-1}^N \mid j = 1, \ldots, N\}$ tends to 0. For given initial value $y_0 \in \mathcal{S}(0)$ we solve iteratively

$$y_k \in \operatorname{Arg\,min}\big\{\mathcal{E}(t_k, y) + \Psi(y - y_k) \mid y \in Y\big\}.$$

By convexity of $\mathcal{E}(t, \cdot)$, this minimization problem is equivalent to the static variational inequality:

$$\begin{aligned}&\text{Find } y_k \in Y \text{ such that } \forall \hat v \in Y\colon \\ &\quad \big\langle Ay_k - \ell(t_k), \hat v - (y_k - y_{k-1})\big\rangle - \Psi(y_k - y_{k-1}) + \Psi(\hat v) \geqslant 0.\end{aligned}$$

In fact, most work on evolutionary variational inequalities uses this form of the incremental problem, considers the piecewise linear interpolant of their solutions and shows that their limit exists and satisfies the corresponding variational inequality (VI), given in (2.2). We will instead stay with the minimization formulation and show that the limit function satisfies the energetic formulation (S) and (E), see (2.9). Thus, we also provide a simplified version of the general proof of Theorem 5.2.

THEOREM 2.1. *Let* $\ell \in \mathrm{C}^1([0,T],Y^*)$ *and* $y_0 \in \mathcal{S}(0) = A^{-1}(\ell(0) - C_*)$. *Then, the energetic problem* (S) *and* (E) (*cf.* (2.9)) *and hence also* (VI) *have a unique solution* $y \in \mathrm{C}^{\mathrm{Lip}}([0,T],Y)$.

PROOF. Except for the uniqueness part, we follow the six steps of the proof of Theorem 5.2. We use the norm $\|\cdot\|_A$ on Y and the dual norm on Y^* and indicate this fact by writing Y_A and Y_A^*, respectively.

Step 0. *Uniqueness.* For any two solutions estimate (2.16) with $\ell_1 = \ell_2 = \ell$ gives the estimate $\|y_1(t) - y_2(t)\|_A \leqslant \|y_1(0) - y_2(0)\|_A$, which proves uniqueness.

Step 1. *A priori estimates.* Let $\Pi = \{t_k \mid k = 0,\dots,N\}$ be any partition. Since the functional $y \mapsto \mathcal{E}(t_k,y) + \Psi(y - y_{k-1})$ is strictly convex, it has a unique minimizer y_k in each step and we have

$$\begin{aligned} \forall y \in Y, \quad & \mathcal{E}(t_k,y) + \Psi(y - y_{k-1}) \\ & \geqslant \frac{1}{2}\|y - y_k\|_A^2 + \mathcal{E}(t_k,y_k) + \Psi(y_k - y_{k-1}). \end{aligned} \tag{2.18}$$

Inserting $y = y_{k-1}$ and $y = y_{k+1}$, respectively, we obtain

$$\begin{aligned} &\forall k \in \{1,\dots,N\}, \\ &\quad \mathcal{E}(t_k,y_k) + \Psi(y_k - y_{k-1}) \\ &\qquad \leqslant \mathcal{E}(t_k,y_{k-1}) = \mathcal{E}(t_{k-1},y_{k-1}) + \int_{t_{k-1}}^{t_k} \partial_s\mathcal{E}(s,y_{k-1})\,\mathrm{d}s; \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\forall k \in \{0,\dots,N-1\}, \\ &\quad \frac{1}{2}\|y_{k+1} - y_k\|_A^2 \leqslant \mathcal{E}(t_k,y_{k+1}) + \Psi(y_{k+1} - y_k) - \mathcal{E}(t_k,y_k), \end{aligned} \tag{2.20}$$

where we have estimated $\Psi(y_k - y_{k-1}) \geqslant 0$. Note that the last estimate is claimed also for $k = 0$, which follows from the assumption $y_0 \in \mathcal{S}(0)$. Thus, for $k = 0,\dots,N-1$, we obtain

$$\begin{aligned} &\frac{1}{2}\|y_{k+1} - y_k\|_A^2 \\ &\quad \leqslant \mathcal{E}(t_k,y_{k+1}) + \Psi(y_{k+1} - y_k) - \mathcal{E}(t_k,y_k) \\ &\quad \leqslant \mathcal{E}(t_{k+1},y_{k+1}) - \int_{t_k}^{t_{k+1}} \partial_s\mathcal{E}(s,y_{k+1})\,\mathrm{d}s + \Psi(y_{k+1} - y_k) - \mathcal{E}(t_k,y_k) \\ &\quad \overset{(2.19)}{\leqslant} \mathcal{E}(t_k,y_k) - \int_{t_k}^{t_{k+1}} \partial_s\mathcal{E}(s,y_{k+1})\,\mathrm{d}s - \mathcal{E}(t_k,y_k) \\ &\quad = \int_{t_k}^{t_{k+1}} \big[\partial_s\mathcal{E}(s,y_k) - \partial_s\mathcal{E}(s,y_{k+1})\big]\,\mathrm{d}s \leqslant \Lambda_{1,A}(t_{k+1} - t_k)\|y_{k+1} - y_k\|_A, \end{aligned}$$

where $\Lambda_{1,A} = \|\dot{\ell}\|_{\mathrm{L}^\infty([0,T],Y_A^*)}$. Thus, the piecewise linear interpolant $\hat{y}^\Pi : [0,T] \to Y$ and the piecewise constant interpolant $\bar{y}^\Pi : [0,T] \to Y$ with $y(t) = y_{k-1}$ for $t \in [t_{k-1}, t_k)$ satisfy the a priori bounds

$$\begin{aligned}
&\|\hat{y}^\Pi\|_{\mathrm{C}^0([0,T],Y_A)} \leqslant \|y_0\|_A + 2\Lambda_{1,A}T,\\
&\|\bar{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leqslant \|y_0\|_A + 2\Lambda_{1,A}T,\\
&\|\dot{\hat{y}}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leqslant 2\Lambda_{1,A},\\
&\|\hat{y}^\Pi - \bar{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y_A)} \leqslant 2\Lambda_{1,A} f(\Pi).
\end{aligned}$$

Step 2. *Selection of a subsequence.* Now we choose an arbitrary sequence $(\Pi^m)_{m\in\mathbb{N}}$ of partitions with $f(\Pi^m) \to 0$. Since the function $\hat{y}^{\Pi^m}$ satisfies a uniform Lipschitz bound and since closed balls in the reflexive Banach space Y are weakly compact, we can apply the Arzela–Ascoli theorem which provides a subsequence $(m_l)_{l\in\mathbb{N}}$ and a limit function $y : [0,T] \to Y$ such that $\hat{y}^l = \hat{y}^{\Pi^{m_l}}$ and $\bar{y}^l = \bar{y}^{\Pi^{m_l}}$ satisfy

$$\forall t \in [0,T], \quad \hat{y}^l(t) \rightharpoonup y(t), \qquad \bar{y}^l(t) \rightharpoonup y(t) \quad \text{and} \quad \|\dot{y}\|_{\mathrm{L}^\infty([0,T],Y_A)} \leqslant 2\Lambda_{1,A}.$$

It remains to be shown that y is a solution.

Step 3. *Stability of the limit function.* Estimate (2.18) and the triangle inequality for Ψ imply, for all $y \in Y$,

$$\begin{aligned}
\mathcal{E}(t_k, y_k) &\leqslant \mathcal{E}(t_k, y) + \Psi(y - y_{k-1}) - \Psi(y_k - y_{k-1})\\
&\leqslant \mathcal{E}(t_k, y) + \Psi(y - y_k),
\end{aligned}$$

which means $y_k \in \mathcal{S}(t_k)$. Hence, for the sequence $\hat{y}^l$ we have $\hat{y}^l(t) \in \mathcal{S}(t)$ for each $t \in \Pi^{m_l}$. As $f(\Pi^{m_l}) \to 0$ we find, for each $t_* \in [0,T]$, a sequence t^l with $t^l \to t_*$. Using $\|\dot{\hat{y}}^l\|_{\mathrm{L}^\infty} \leqslant 2\Lambda_{1,A}$ we obtain $\hat{y}^l(t^l) \rightharpoonup y(t_*)$. Moreover, the graph set,

$$\begin{aligned}
\mathcal{S}_{[0,T]} &= \bigcup_{t\in[0,T]} \big(t, \mathcal{S}(t)\big)\\
&= \big\{(t,y) \bigm| y \in \mathcal{S}(t)\big\}\\
&= \big\{(t,y) \bigm| y \in A^{-1}\big(\ell(t) - C_*\big)\big\} \subset [0,T] \times Y,
\end{aligned}$$

is closed with respect to the weak topology of Y, since each $\mathcal{S}(t)$ is strongly closed and convex and since ℓ is strongly continuous. Thus, $(t^l, \hat{y}^l(t^l)) \in \mathcal{S}_{[0,T]}$ implies $(t_*, y(t_*)) \in \mathcal{S}_{[0,T]}$, which means $y(t_*) \in \mathcal{S}(t_*)$ and the stability (S) is proved.

Step 4. *Upper energy estimate.* For $t \in (0, T]$ and $l \in \mathbb{N}$ let j be the largest index with $t_j = \max\{t_n \in \Pi^{m_l} \mid t_n \leqslant t\}$. Then, adding (2.19) from $k = 1$ to j gives

$$\begin{aligned}
&\mathcal{E}\big(t, \bar{y}^l(t)\big) + \mathrm{Diss}_\Psi\big(\bar{y}^l; [0, t]\big) \\
&\quad = \int_{t_j}^{t} \partial_s \mathcal{E}\big(s, \bar{y}^l(t_j)\big)\,\mathrm{d}s + \mathcal{E}\big(t_j, \bar{y}^l(t_j)\big) + \sum_{k=1}^{j} \Psi\big(\bar{y}^l(t_k) - \bar{y}^l(t_{k-1})\big) \\
&\quad \leqslant \int_{t_j}^{t} \partial_s \mathcal{E}\big(s, \bar{y}^l(s)\big)\,\mathrm{d}s + \mathcal{E}(0, y_0) + \int_0^{t_j} \partial_s \mathcal{E}\big(s, \bar{y}^l(s)\big)\,\mathrm{d}s \\
&\quad = \mathcal{E}(0, y_0) - \int_0^t \big\langle \dot{\ell}(s), \bar{y}^l(s)\big\rangle\,\mathrm{d}s.
\end{aligned}$$

The right-hand side of this estimate converges to $\mathcal{E}(0, y_0) - \int_0^t \langle \dot{\ell}(s), y(s)\rangle\,\mathrm{d}s$ by the weak convergence and Lebesgue theorem on dominated convergence. The left-hand side is lower semicontinuous, since $\mathcal{E}(t, \cdot)$ and Diss_Ψ are convex and strongly continuous. Thus, for each $t \in [0, T]$ we conclude $\mathcal{E}(t, y(t)) + \mathrm{Diss}_\Psi(y; [0, t]) \leqslant \mathcal{E}(0, y_0) - \int_0^t \langle \dot{\ell}(s), y(s)\rangle\,\mathrm{d}s$, which is the desired upper energy estimate.

Step 5. *Lower energy estimate.* The lower estimate is a consequence of stability of the limit function y as proved in Step 3. Take an arbitrary partition $\mathcal{T} = \{\tau_j \mid j = 0, \ldots, M\}$ of the interval $[0, t]$. Then, for $j \geqslant 1$ stability of $y(\tau_{j-1})$ gives

$$\begin{aligned}
&\mathcal{E}\big(\tau_j, y(\tau_j)\big) + \Psi\big(y(\tau_j) - y(\tau_{j-1})\big) \\
&\quad = \int_{\tau_{j-1}}^{\tau_j} \partial_s \mathcal{E}\big(s, y(\tau_j)\big)\,\mathrm{d}s + \mathcal{E}\big(\tau_{j-1}, y(\tau_j)\big) + \Psi\big(y(\tau_j) - y(\tau_{j-1})\big) \\
&\quad \geqslant \int_{\tau_{j-1}}^{\tau_j} \partial_s \mathcal{E}\big(s, y(\tau_j)\big)\,\mathrm{d}s + \mathcal{E}\big(\tau_{j-1}, y(\tau_{j-1})\big).
\end{aligned}$$

Adding these estimates from $j = 1$ to M and using the definition of Diss_Ψ, we find

$$\begin{aligned}
\mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_\Psi\big(y; [0, t]\big) &\geqslant \mathcal{E}\big(t, y(t)\big) + \sum_1^M \Psi\big(y(\tau_j) - y(\tau_{j-1})\big) \\
&\geqslant \mathcal{E}(0, y_0) - \sum_1^M \int_{\tau_{j-1}}^{\tau_j} \big\langle \dot{\ell}(s), y(\tau_j)\big\rangle\,\mathrm{d}s \\
&\geqslant \mathcal{E}(0, y_0) - \int_0^t \big\langle \dot{\ell}(s), y(s)\big\rangle\,\mathrm{d}s - f(\mathcal{T})2\Lambda_{1,A}.
\end{aligned}$$

Thus, by making the partition $\mathcal{T}$ as fine as we like, we obtain the lower energy estimate and together with Step 4 the energy balance (E) is established and y is a solution.

Step 6. *Improved convergence.* By Step 0 we know that there exists at most one solution. We conclude that not only the subsequence y^l converges weakly to y, but the whole sequence $\hat{y}^{\Pi^m}$ converges weakly to y. In fact, in Section 4.1 it is shown that the convergence is strong with a convergence like $\sqrt{f(\Pi)}$, see Theorem 4.3. □

2.3.2. *Monotone operators.* The existence theory via monotone operators is based on the concept of multivalued monotone operators on a Hilbert space. A mapping $\mathcal{M} : D(\mathcal{M}) \subset Y \to \mathcal{P}(Y)$, where $\mathcal{P}$ denotes the power set, is called monotone if

$$\forall y_1, y_2 \in D(\mathcal{M}), \forall w_1 \in \mathcal{M}(y_1), w_2 \in \mathcal{M}(y_2), \quad \langle w_1 - w_2 | y_1 - y_2 \rangle \geqslant 0,$$

where $\langle \cdot | \cdot \rangle$ denotes the scalar product in Y. A monotone operator $\mathcal{M}$ is called *maximal monotone*, if its graph $G(\mathcal{M}) = \bigcup_{y \in D(\mathcal{M})} (y, \mathcal{M}(y)) \subset Y \times Y$ does not have a proper extension to a graph of another monotone operator. The following result is contained in many textbooks, see, e.g., [Zei85], Theorem 55A, or [Bre73].

THEOREM 2.2. *Let Y be a real, separable Hilbert space and $\mathcal{M} : D(\mathcal{M}) \subset Y \to \mathcal{P}(Y)$ a maximal monotone operator. Then, for each $T > 0$ the Cauchy problem*

$$0 \in \dot{\hat{y}}(t) + \mathcal{M}\big(\hat{y}(t)\big) - b(t) \text{ for almost every } t \in [0, T] \text{ and } \hat{y}(0) = \hat{y}^0, \tag{2.21}$$

has, for each $\hat{y}^0 \in D(\mathcal{M})$ and each $b \in \mathrm{W}^{1,1}((0, T), Y)$, a unique solution $\hat{y} \in \mathrm{W}^{1,1}((0, T), Y)$. In fact, we have $\hat{y} \in \mathrm{W}^{1,\infty}((0, T), Y)$.

Moreover, two solutions $\hat{y}_1, \hat{y}_2$ associated with data $(\hat{y}^0_j, b_j)$ satisfy the estimate

$$\big\| \hat{y}_1(t) - \hat{y}_2(t) \big\| \leqslant \big\| \hat{y}^0_1 - \hat{y}^0_2 \big\| + \int_0^t \big\| b_1(s) - b_2(s) \big\| \, \mathrm{d}s \quad \text{for } t \in [0, T]. \tag{2.22}$$

In our rate-independent problems b will be related to $\dot{\ell}$ and the values of $\mathcal{M}$ must be closed cones, as the values $\mathcal{M}(\hat{y})$ of any maximal monotone operator are closed and convex and rate independence gives $\mathcal{M}(\hat{y}) = \gamma \mathcal{M}(\hat{y})$ for all $\gamma > 0$. To apply the above theorem to our problems of Section 2.1 we let $\hat{y}(t) = y(t) - A^{-1}\ell(t)$ and $\mathcal{M}(\hat{y}) = -\partial I_{C_*}(-A\hat{y})$. Then (DI) takes the form

$$0 \in \dot{\hat{y}}(t) + \mathcal{M}\big(\hat{y}(t)\big) - A^{-1}\dot{\ell}(t), \quad \hat{y}(0) = y_0 - A^{-1}\ell(0).$$

Clearly, $\hat{y}(0) \in D(\mathcal{M}) = \{\hat{y} \mid \mathcal{M}(\hat{y}) \neq \emptyset\}$ means $-A\hat{y}(0) \in C_*$ which is equivalent to the stability condition $y_0 \in \mathcal{S}(0) = A^{-1}(\ell(0) - C_*)$. Moreover, $\mathcal{M}$ is a maximal monotone operator, since it is the subdifferential of the lower semicontinuous, convex function $\varphi : Y \to \mathbb{R}_\infty; \hat{y} \mapsto I_{C_*}(-A\hat{y})$, i.e., $\varphi = I_{-A^{-1}C_*}$. For this choose the scalar product to be defined by A as usual, namely $\langle y_1 | y_2 \rangle = \langle Ay_1, y_2 \rangle$; then it is easy to check that the Hilbert space subdifferential using the scalar product $\partial\varphi(\hat{y}) = \{w \in Y \mid \forall y \in Y\colon \varphi(y) \geqslant \varphi(\hat{y}) + \langle w | y - \hat{y} \rangle\}$ is equal to $-\mathrm{N}_{C_*}(-A\hat{y})$ as desired.

In principle this approach provides the desired existence result. However, it has the disadvantage that it strongly relies on the linearity of $y \mapsto \mathrm{D}\mathcal{E}(t, y) = Ay - \ell(t)$ and that it uses the time derivative $\dot{\ell}$.

Finally, let us mention that the restriction of y being a Hilbert space can be avoided by using the theory of m-accretive operators on Banach spaces, see [Bar76] or [Vis94], Section XII.4, for a short survey.

2.3.3. *Doubly nonlinear problems.* An approach better adapted to our needs is the theory of doubly nonlinear equations developed in [ColV90]. There, general equations of the type

$$0 \in \mathcal{R}\big(\dot{y}(t)\big) + \Sigma_0\big(y(t)\big) - \ell(t) \quad \text{for a.a. } t \in [0, T];\ y(0) = y_0, \tag{2.23}$$

are studied, where $\mathcal{R}$ and Σ_0 are (possibly multivalued) maximal monotone operators on the Hilbert space Y. Theorem 2.1 treats the case $\Sigma_0 = \partial\mathcal{U}$ while $\mathcal{R}$ is general with linear growth. Theorem 2.2 is dedicated to the case $\mathcal{R} = \partial\Psi$ with strongly monotone Σ_0. Finally, Theorem 2.3 assumes $\mathcal{R} = \partial\Psi$ and $\Sigma_0 = \partial\mathcal{U}$ with minimal assumptions on the potentials Ψ and $\mathcal{U}$. Theorem 2.2 is most suited for our purposes and we repeat a variant of it for the reader's convenience.

THEOREM 2.3. *The Hilbert space Y is densely and compactly embedded into the Hilbert space H. The dissipation potential $\Psi : H \to \mathbb{R}_\infty$ is convex and lower semicontinuous and $\mathcal{R} = \partial\Psi$. The mapping $\Sigma_0 : Y \to Y^*$ is Lipschitz continuous and uniformly monotone, i.e.,*

$$\exists c_1, C_2 > 0, \forall y_1, y_2 \in Y, \quad \big\|\Sigma_0(y_1) - \Sigma_0(y_2)\big\|_{Y^*} \leqslant C_2\|y_1 - y_2\|_Y \quad \textit{and}$$
$$\big\langle \Sigma_0(y_1) - \Sigma_0(y_2), y_1 - y_2\big\rangle \geqslant c_1\|y_1 - y_2\|_Y^2.$$

Then, for every $\ell \in \mathrm{H}^1((0,T), Y^*)$ and every y_0 with $\ell(0) - \Sigma_0(y_0) \in D(\mathcal{L}\Psi)$, there exists a solution $y \in \mathrm{H}^1((0,T), Y)$ of (2.23).

Here $\mathcal{L}\Psi$ denotes the Legendre–Fenchel transform of the convex function Ψ. In the case of a Ψ which is 1-homogeneous, we have $\mathcal{L}\Psi = I_{C_*}$ and thus $D(\mathcal{L}\Psi) = C_*$. Clearly, the above theorem provides the solvability of our subdifferential formulation (SF) given in (2.1) if we let $\Sigma_0(y) = Ay$. However, for this general version involving monotone operators Σ_0 instead of a linear and positive operator A we have to pay by making an additional assumption on $C_* = \partial\Psi(0) \subset H = H^*$. Since Y is compactly embedded in H, we find that $C_* \cap \{\sigma \in Y^* \mid \|\sigma\|_* \leqslant \rho\}$ is compact in Y^* for $\rho > 0$. This relates to the conditions of the closedness of the stable set in the weak topology, which is central in the abstract Section 5.

It is interesting to note that the existence result for monotone operators is mostly obtained by using regularization techniques, which replace the nonsmooth, multivalued problem by a classical Lipschitz continuous ordinary differential equation. The main technique is the *Yosida regularization* which works for all maximal monotone operators (even for m-accretive operators), see [Zei85], Section 55.2. If $M : Y \to \mathcal{P}(Y)$ is maximal monotone, then for all $\varepsilon > 0$ the problem $y + \varepsilon M(y) = b$ has a unique solution, which we denote by $y = R_{\varepsilon,M}(b)$. The Yosida regularization of M is then defined to be the operator

$$M_\varepsilon^{\mathrm{Y}} : Y \to Y; \qquad y \mapsto \frac{1}{\varepsilon}\big(y - R_{\varepsilon,M}(y)\big).$$

Defining M_0 via $M_0(y) = \arg\min\{\|w\| \mid w \in M(w)\}$ we obtain the following standard results: each $M_\varepsilon^{\mathrm{Y}}$ is Lipschitz continuous and maximal monotone, and for $y \in D(M)$ we have $M_\varepsilon^{\mathrm{Y}}(y) \to M_0(y)$ for $\varepsilon \to 0$ whereas $y \notin D(M)$ implies $\|M_\varepsilon^{\mathrm{Y}}(y)\| \to \infty$.

Instead of solving the Cauchy problem (2.21) one then solves the regularized problem $0 = \dot{y}(t) + M_\varepsilon^{\mathrm{Y}}(y(t)) - b(t)$ which by Lipschitz continuity of $M_\varepsilon^{\mathrm{Y}}$ has a unique solution $y_\varepsilon \in \mathrm{C}^{\mathrm{Lip}}([0,T],Y)$. Using the monotonicity properties it is then possible to show that the limit $y(t) = \lim_{\varepsilon\to 0} y_\varepsilon(t)$ exists and solves (2.21).

The point here is that in the case of our interest the operator M is the subdifferential of the characteristic function $\varphi = I_C$ with $C = -A^{-1}C_* \subset Y$. It is easy to see that $R_{\varepsilon,M}$ is independent of ε and is equal to the orthogonal projection $P_C : Y \to C \subset Y$, i.e., $P_C(y) = \arg\min\{\|\hat{y} - y\|_Y \mid \hat{y} \in C\}$. Thus, the Yosida regularization $M_\varepsilon^{\mathrm{Y}}$ takes the form

$$M_\varepsilon^{\mathrm{Y}}(y) = \frac{1}{\varepsilon}\big(y - P_C(y)\big) = \partial\mathcal{D}_\varepsilon(y), \quad \text{where } \mathcal{D}_\varepsilon(y) = \frac{1}{2\varepsilon}\mathrm{dist}(y,C)^2.$$

Thus, the regularized equation

$$0 = \dot{y} + M_\varepsilon^{\mathrm{Y}}(y) - b(t) = \dot{y} + \frac{1}{\varepsilon}\big(y - P_C(y)\big) - b(t)$$

corresponds to the traditional viscoplastic approximation to plasticity. This fact was first noted in [Ort81].

In [ColV90] the doubly nonlinear problems (2.23) are regularized in a two-fold way. The auxiliary problem is

$$0 \in \varepsilon\dot{y}(t) + \mathcal{R}\big(\dot{y}(t)\big) + M_\varepsilon^{\mathrm{Y}}\big(y(t)\big) - \ell(t),$$

where the first term can be understood as a viscous friction term. To solve for $\dot{y}$ we use the operator $\mathbf{R}_\varepsilon : v \mapsto R_{1/\varepsilon,\mathcal{R}}(\frac{1}{\varepsilon}v)$, which leads to the ordinary differential equation $0 = \dot{y} + \mathbf{R}_\varepsilon(M_\varepsilon(y(t)) - \ell(t))$, which is Lipschitz continuous. We will consider related viscous regularizations in Section 5.4. Such regularizations are important if M is no longer strictly monotone and thus solutions can develop jumps.

Note that the viscous regularization $\mathcal{R}_\varepsilon^{\mathrm{vis}} = \varepsilon\mathbf{1} + \mathcal{R}$ for $\mathcal{R} = \partial\Psi$ is adjusted to the fact that $\partial\Psi$ is 1-homogeneous. For $\mathcal{R}(v) = \mathrm{Sign}(v)$ the regularization $\varepsilon\mathbf{1} + \mathcal{R}$ has a Lipschitz continuous inverse, while the Yosida regularization $\mathcal{R}_\varepsilon^{\mathrm{Y}}$ is bounded and hence not invertible, namely $\mathcal{R}_\varepsilon^{\mathrm{Y}}(v) = \min\{\frac{1}{\varepsilon}, \frac{1}{\|v\|}\}v$.

2.3.4. *Sweeping processes.* Finally, we mention some special existence results related to the sweeping-process formulation taken from [Mon93], Chapters 2 and 5. The origins stem from Moreau, see, e.g., [Mor74]. There, for $y \in \mathrm{BV}_+([0,T],Y)$, the BV functions which are continuous from the right, the differential measure $\mathrm{d}y$ and the associated Radon measure $|\mathrm{d}y|$ are defined such that $\mathrm{d}y([s,t]) = \int_s^t \mathrm{d}y = y(t) - y(s)$ and $\mathrm{Diss}_{\|\cdot\|}(y) = \int_{[s,t]} |\mathrm{d}y|$. The new point is that y may have jumps, i.e., t is a jump point, if $\lim_{r\searrow t} y(r) = y(t) \neq y_-(t) = \lim_{s\nearrow t} y(s)$. This implies $|\mathrm{d}y|(\{t\}) = \|y(t) - y_-(t)\|$ and $\mathrm{d}y(\{t\}) = y(t) - y_-(t)$. The important fact is that $\mathrm{d}y$ can be decomposed into a directional part y' and the length $|\mathrm{d}y|$ such that y' is defined $|\mathrm{d}y|$-almost everywhere and takes values in the unit sphere $\{y \in Y \mid \|y\| = 1\}$. One shortly writes $y' = \frac{\mathrm{d}y}{|\mathrm{d}y|}$.

For a given family $(C(t))_{[0,T]}$ of closed convex sets, we can now formulate the sweeping process using the normal cone $\mathrm{N}_{C(t)}(y) \subset Y$, which is defined via the scalar product on Y, as follows

$$\begin{aligned} &y(0) = y_0 \in C(0), \qquad \forall t \in [0,T], \quad y(t) \in C(t), \\ &-\frac{\mathrm{d}y}{|\mathrm{d}y|}(t) \in \mathrm{N}_{C(t)}\big(y(t)\big) \quad \text{for } |\mathrm{d}y|\text{-almost every } t \in [0,T]. \end{aligned}$$

The existence of solutions in $\mathrm{BV}_+([0,T],Y)$ can now be shown using special incremental problems or the Yosida regularization if one of the following conditions hold:

(A) ([Mon93], Chapter 1, Theorem 1.5) $t \mapsto C(t)$ is right-continuous and of bounded variation with respect to the Hausdorff distance on closed sets.

(B) ([Mon93], Chapter 2, Theorem 2.1) $t \mapsto C(t)$ is Hausdorff continuous and each $C(t)$ has nonempty interior (then $y \in \mathrm{C}^0([0,T],Y) \cap \mathrm{BV}([0,T],Y)$), cf. Theorem 2.6.

(C) ([Mon93], Chapter 2, Theorem 2.4) Y is finite dimensional, there exists $\rho > 0$ and $y_0 \in Y$ with $B_\rho(y_0) \subset C(t)$ and $t \mapsto C(t)$ is lower semicontinuous from the right (i.e., $\forall t_0 \in [0,T)$, $\forall O \subset Y$ open with $O \cap C(t_0) \neq \emptyset$ $\exists \varepsilon > 0$: $O \cap C(t) \neq \emptyset$ for all $t \in [t_0, t_0 + \varepsilon]$).

(D) ([Mon93], Chapter 2, Theorem 2.4) Y is finite dimensional, each $C(t)$ has non-empty interior and $t \mapsto C(t)$ is lower semicontinuous (i.e., $\forall t_0 \in [0,T]$, $\forall O \subset Y$ open with $O \cap C(t_0) \neq \emptyset$ $\exists \varepsilon > 0$: $O \cap C(t) \neq \emptyset$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap [0,T]$).

2.4. *Continuity properties of the solution operator*

In Sections 2.2 and 2.3 we have seen that associated with the linear operator $A: Y \to Y^*$ and the dissipation functional $\Psi = \mathcal{L}I_{C_*}$ there is a solution operator

$$\begin{aligned} &\mathcal{H}: \begin{cases} C_* \times \mathrm{C}^{\mathrm{Lip}}\big([0,T],Y^*\big) \to \mathrm{C}^{\mathrm{Lip}}\big([0,T],Y\big), \\ (\sigma_0, \ell) \mapsto y(\cdot), \end{cases} \\ &\quad \text{with } y(0) = y_0 = A^{-1}\big(\ell(0) - \sigma_0\big), \end{aligned}$$

where y solves any of the equivalent formulations in Section 2.1. In fact, the estimates (2.16), (2.17) and (2.22) provided a first result on continuous-dependence for $\mathcal{H}$. Here and below we use the energy norm $\|y\|_A = \langle Ay, y\rangle^{1/2}$ on Y and the dual norm on Y^*, i.e., $\|\sigma\|_* = \langle \sigma, A^{-1}\sigma\rangle^{1/2}$.

In particular, we have established that $\mathcal{H}(\cdot,\ell)$ defines a contraction semigroup, namely for $y_j = \mathcal{H}(\sigma_j, \ell)$ with fixed $\ell \in \mathrm{C}^{\mathrm{Lip}}$ we have

$$\begin{aligned} \big\| y_1(t) - y_2(t) \big\|_A &= \big\| \mathcal{H}(\sigma_1,\ell)(t) - \mathcal{H}(\sigma_2,\ell)(t) \big\|_A \\ &\leqslant \|\sigma_1 - \sigma_2\|_* = \big\| y_1(0) - y_2(0) \big\|_A \end{aligned} \tag{2.24}$$

for all $t \in [0,T]$.

So far, we have shown that $\mathcal{H}(\sigma_0,\cdot)$ maps $\mathrm{C}^{\mathrm{Lip}}([0,T],Y^*)$ into $\mathrm{C}^{\mathrm{Lip}}([0,T],Y)$. However, using the rate-independence, it can be easily seen that $\mathcal{H}(\sigma_0,\cdot)$ also maps $\mathrm{W}^{1,p}([0,T],Y^*)$ into $\mathrm{W}^{1,p}([0,T],Y)$ for any $p \in [1,\infty]$ (where $\mathrm{C}^{\mathrm{Lip}} = \mathrm{W}^{1,\infty}$ since Y is a Hilbert space). For this, just note that $\ell \in \mathrm{W}^{1,p}([0,T],Y^*)$ allows to define the new time variable

$$\tau = \alpha(t) = t + \int_0^t \big\|\dot{\ell}(s)\big\|\,\mathrm{d}s \in [0,T_\circ] \quad \text{with } T_\circ = \alpha(T).$$

If $\beta:[0,T_\circ] \to [0,T]$ is the inverse of α, then $\ell_\circ = \ell \circ \beta \in \mathrm{C}^{\mathrm{Lip}}([0,T_\circ],Y^*)$, since $\ell'_\circ(\tau) = (1+\|\dot{\ell}(\beta(\tau))\|_*)^{-1}\dot{\ell}(\beta(\tau))$ has a norm less than 1 a.e. in $[0,T_\circ]$. With the corresponding solution $y_\circ = \mathcal{H}(\sigma_0,\ell_\circ) \in \mathrm{C}^{\mathrm{Lip}}([0,T_\circ],Y)$ we are then able to obtain the solution $y = \mathcal{H}(\sigma_0,\ell)$ in the form $y = y_\circ \circ \alpha$ and find

$$\begin{aligned}
\|\dot{y}\|_{\mathrm{L}^p([0,T],Y)} &= \big\|y'_\circ\big(\alpha(\cdot)\big)\dot{\alpha}(\cdot)\big\|_{\mathrm{L}^p([0,T],Y)} \\
&\leqslant \big\|y'_\circ\big\|_{\mathrm{L}^\infty([0,T],Y)}\big\|\dot{\alpha}(\cdot)\big\|_{\mathrm{L}^p([0,T])} \\
&\leqslant \big\|y'_\circ\big\|_{\mathrm{L}^\infty([0,T],Y)}\big(T^{1/P} + \big\|\dot{\ell}\big\|_{\mathrm{L}^p([0,T],Y^*)}\big),
\end{aligned}$$

since $\dot{\alpha} = 1 + \|\dot{\ell}\|_*$.

In [Kre99], Theorem 3.6 (see also [BrS96]), we find the following general result, where the last statement follows from (2.22).

PROPOSITION 2.4. *For each $p \in [1,\infty]$ the mapping $\mathcal{H}: C_* \times \mathrm{W}^{1,p}([0,T],Y^*) \to \mathrm{W}^{1,p}([0,T],Y)$ is continuous. Moreover, the mapping $\mathcal{H}: C_* \times \mathrm{W}^{1,1}([0,T],Y^*) \to \mathrm{C}^0([0,T],Y)$ is globally Lipschitz continuous.*

REMARK 2.5. In [Kre99] the continuity results are formulated in terms of the play operator $\mathbf{P}$ and the stop operator $\mathbf{S}$ and written in the Hilbert space setting with $A = \mathbf{1}$ and $\langle y_1|y_2\rangle = \langle Ay_1, y_2\rangle$. In our setting the corresponding definitions are

$$y = \mathbf{P}(\sigma_0,\ell) = \mathcal{H}(\sigma_0,\ell)$$

and

$$\sigma = \mathbf{S}(\sigma_0,\ell) = \ell - Ay = \ell - A\mathcal{H}(\sigma_0,\ell) \in \mathrm{C}^0\big([0,T],C_*\big),$$

such that $\mathbf{S}(\sigma_0,\ell)(0) = \sigma_0$ and $\mathbf{S} + A\mathbf{P} = \mathbf{1}$.

The next result states that $\mathcal{H}$ can be extended to an operator from $C_* \times \mathrm{C}^0([0,T],Y^*)$ into $\mathrm{BV}([0,T],Y) \cap \mathrm{C}^0([0,T],Y)$, if 0 is an interior point of C_*, which is equivalent to a lower bound on Ψ:

$$B^*_\rho(0) = \big\{\sigma \in Y^* \bigm| \|\sigma\|_* \leqslant \rho\big\} \subset C_* \quad \Longleftrightarrow \quad \forall v \in Y, \quad \Psi(v) \geqslant \rho\|v\|_A. \tag{2.25}$$

For the reader's convenience we give the main ingredients of the proof, which is a special case of [Mon93], Chapter 2, Theorem 2.1. The solutions $y = \mathcal{H}(\sigma_0, \ell)$ in the following result will solve the weakened form (2.6) of the dual variational inequality which reads

$$\int_0^T \big\langle \ell(t) - Ay(t) - \tilde{\sigma}(t), \mathrm{d}y(t) \big\rangle \geqslant 0$$
$$\text{for all } \tilde{\sigma} \in \mathrm{C}^0\big([0,T], C_*\big);\ y(0) = A^{-1}\big(\ell(0) - \sigma_0\big). \tag{2.26}$$

See also [KreL02,KreV03] for generalizations to the space of regulated functions, i.e., to functions which may be not continuous but have limits from the left and right at each point.

THEOREM 2.6. *Let C_* and Ψ satisfy* (2.25). *Then, the operator $\mathcal{H}: C_* \times \mathrm{W}^{1,1}([0,T], Y^*) \to \mathrm{W}^{1,1}([0,T], Y)$ can be extended to a continuous operator from $C_* \times \mathrm{C}^0([0,T], Y^*)$ into $\mathrm{C}^0([0,T], Y)$. Moreover, $\mathcal{H}$ maps $C_* \times \mathrm{C}^0([0,T], Y^*)$ into $\mathrm{BV}([0,T], Y) \cap \mathrm{C}^0([0,T], Y)$ and around each $(\sigma_0, \ell) \in C_* \times \mathrm{C}^0([0,T], Y^*)$ there exists a neighborhood $\mathcal{B}$ such that $\mathcal{H}: \mathcal{B} \to \mathrm{C}^0([0,T], Y^*)$ is Hölder continuous with exponent* $1/2$.

PROOF. By (2.24) it suffices to consider the case $\sigma_0 = 0$. Choose any $\hat{\ell} \in \mathrm{C}^0([0,T], Y^*)$, then by uniform continuity there exists $N \in \mathbb{N}$ such that with $t_j = jT/N$, $j = 0, \ldots, N$, we have $\|\hat{\ell}(t) - \hat{\ell}(t_j)\|_* \leqslant \rho/3$ for $t \in [t_j, t_{j+1}]$. Denote by $\mathcal{B}$ the set of all $\ell \in \mathrm{C}^0([0,T], Y^*)$ with $\|\hat{\ell} - \ell\|_\infty \leqslant \rho/6$, then $\|\ell(t) - \ell(t_j)\|_* \leqslant \rho/2$ for $t \in [t_j, t_{j+1}]$.

We show that for $\ell \in \mathcal{B} \cap \mathrm{W}^{1,1}([0,T], Y^*)$ the solutions $y = \mathcal{H}(0, \ell)$ have a uniformly bounded variation. For $t \in [t_j, t_{j+1}]$ we have, by using $(\mathrm{E})_{\mathrm{loc}}$ from (2.8) and (2.25),

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|Ay(t) - \ell(t_j)\big\|_*^2 &= \big\langle Ay - \ell(t_j), \dot{y}\big\rangle \\
&= \big\langle Ay - \ell(t), \dot{y}\big\rangle + \big\langle \ell(t) - \ell(t_j), \dot{y}\big\rangle \\
&\leqslant -\Psi(\dot{y}) + \big\|\ell(t) - \ell(t_j)\big\|_* \|\dot{y}\|_A \\
&\leqslant -\Psi(\dot{y}) + \frac{\rho}{2}\|\dot{y}\|_A \\
&\leqslant -\frac{1}{2}\Psi(\dot{y}).
\end{aligned}$$

Thus, we conclude $\|Ay(t_{j+1}) - \ell(t_j)\|_* \leqslant \delta_j := \|Ay(t_j) - \ell(t_j)\|_*$ and

$$\int_{t_j}^{t_{j+1}} \Psi\big(\dot{y}(s)\big)\,\mathrm{d}s \leqslant \big\|Ay(t_j) - \ell(t_j)\big\|_*^2 - \big\|Ay(t_{j+1}) - \ell(t_j)\big\|_*^2 \leqslant \delta_j^2. \tag{2.27}$$

This implies $\delta_{j+1} \leqslant \delta_j + \|\ell(t_{j+1}) - \ell(t_j)\|_* \leqslant \delta_j + \rho/2$. With $\delta_0 = \|Ay(0) - \ell(0)\|_* = \|\sigma_0\|_*$ we find by induction $\delta_j \leqslant \|\sigma_0\|_* + j\rho/2$. Adding (2.27) over j gives the a priori bound

$$\rho\int_0^T \big\|\dot{y}(t)\big\|_A\,\mathrm{d}t \leqslant \int_0^T \Psi\big(\dot{y}(t)\big)\,\mathrm{d}t \leqslant N\left(\|\sigma_0\|_* + \frac{N\rho}{2}\right)^2 = K_{\mathcal{B}}. \tag{2.28}$$

Now employ the estimate (2.17) to obtain for all $\ell_1, \ell_2 \in \mathcal{B} \cap \mathrm{W}^{1,1}([0,T], Y^*)$ the estimate

$$\|y_1 - y_2\|^2_{\mathrm{C}^0([0,T],Y_A)} \leqslant \big\|A^{-1}\big(\ell_1(0) - \ell_2(0)\big)\big\|^2_A + \frac{2K_{\mathcal{B}}}{\rho}\|\ell_1 - \ell_2\|_{\mathrm{C}^0([0,T],Y^*)}.$$

Thus, by density there is a unique Hölder continuous extension of $\mathcal{H}$ to all of $\mathcal{B}$. □

The a priori bound (2.28) can easily be extended to all small neighborhoods of compact subsets of $\mathrm{C}^0([0,T], Y^*)$. Uniform continuity results for $\mathcal{H}$ can be obtained by further assumptions.

THEOREM 2.7. (A) ([Kre99], Theorem 4.1.) *If C_* is uniformly strictly convex* (*see* [Kre99], *Definition* 2.13), *then there exists a monotone function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(\delta) \to 0$ for $\delta \to 0$, such that the operator $\mathcal{H} : C_* \times \mathrm{C}^0([0,T], Y^*) \to \mathrm{C}^0([0,T], Y)$ satisfies*

$$\begin{aligned}&\big\|\mathcal{H}(\sigma_1, \ell_1) - \mathcal{H}(\sigma_2, \ell_2)\big\|_\infty \\ &\quad \leqslant \max\big\{\big\|\big(\sigma_1 - \ell_1(0)\big) - \big(\sigma_2 - \ell_2(0)\big)\big\|_*, \beta\big(\|\ell_1 - \ell_2\|_\infty\big)\big\}.\end{aligned}$$

(B) ([Kre99], Corollary 5.9, [Des98].) *Let C_* satisfy $B^*_\rho(0) \subset C_* \subset B^*_R(0)$ and assume that for each point $\sigma \in \partial C_*$ there exists a unique outward normal vector $n(\sigma)$ of length* 1 *such that $n : \partial C_* \to Y$ is Lipschitz continuous. Then, for each ball $\mathcal{B}_R(0) = \{\ell \in \mathrm{W}^{1,1}([0,T], Y^*) \mid \|\ell\|_{\mathrm{W}^{1,1}} \leqslant R\}$ there exists a Lipschitz constant L_R such that*

$$\begin{aligned}&\forall \sigma_1, \sigma_2 \in C_*, \forall \ell_1, \ell_2 \in \mathcal{B}_R, \\ &\quad \big\|\mathcal{H}(\sigma_1, \ell_1) - \mathcal{H}(\sigma_2, \ell_2)\big\|_{\mathrm{W}^{1,1}} \leqslant L_R\big(\|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{W}^{1,1}}\big).\end{aligned}$$

See also [Kre99,Kre01] for certain counterexamples which show that without the given conditions the results may fail. In particular, it is not sufficient that $n : \partial C_* \to Y$ is C^1, since global Lipschitz continuity implies the boundedness of the derivative.

In [KraP89] hysteresis problems with *polyhedral characteristics* are studied, i.e., C_* has the form

$$C_* = \big\{\sigma \in Y^* \;\big|\; \forall j = 1, \ldots, K\colon \langle \sigma, n_j\rangle \leqslant \beta_j\big\}, \tag{2.29}$$

where $n_j \in Y$ with $\|n_j\| = 1$ and $\beta_j > 0$ are given. Denoting $X = \mathrm{span}\{n_1, \ldots, n_K\} \subset Y$ and $X^* = AX$ we may decompose Y and Y^* as follows:

$$Y = X \oplus X^\perp \quad \text{with } X^\perp = \big\{y \in Y \;\big|\; \forall j = 1, \ldots, K : \langle An_j, y\rangle = 0\big\}$$

and

$$Y^* = X^* \oplus X^\perp_* \quad \text{with } X^\perp_* = AX^\perp.$$

Thus, C_* takes the form of a polyhedral cylinder $\widehat{C}_* + X_*^\perp$ with $\widehat{C}_* = C_* \cap X^*$ and the Legendre transform gives $\Psi(\dot y) = \Psi(\dot x + \dot{\hat{x}}^\perp) = \widehat{\Psi}(\dot x)$. The important fact is that $0 \in X^*$ is now an interior point of $\widehat{C}_*$. Writing $A = \operatorname{diag}(A_X, A^\perp)$, $y = x + x^\perp$ and $\ell = \ell_X + \ell^\perp$, the Legendre transform gives $\Psi(\dot x + \dot x^\perp) = \widehat{\Psi}(\dot x)$ and the subdifferential equation (SF) (2.1) decouples into

$$0 \in \partial\widehat{\Psi}\big(\dot x(t)\big) + A_X x(t) - \ell_X(t) \quad \text{in } X^*$$

and

$$0 = A^\perp x^\perp(t) - \ell^\perp(t) \quad \text{in } X_*^\perp.$$

Hence, we find $x^\perp$ by solving a static problem, namely $x^\perp(t) = (A^\perp)^{-1}\ell^\perp(t)$. Only, $x : [0, T] \to X$ has to be found by solving a hysteresis problem.

THEOREM 2.8 [KraP89,DesT99]. *If C_* is polyhedral as defined in* (2.29), *then there exist global Lipschitz constants L_{C} and L_{W} such that the following holds* ($y_j = \mathcal{H}(\sigma_j, \ell_j)$)

$$\begin{aligned}
&\forall \sigma_1, \sigma_2 \in \mathrm{C}_*, \forall \ell_1, \ell_2 \in \mathrm{C}^0\big([0, T], Y^*\big),\\
&\quad \|y_1 - y_2\|_{\mathrm{C}^0([0,T],Y)} \leqslant L_{\mathrm{C}}\big(\|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{C}^0([0,T],Y^*)}\big),\\
&\forall \sigma_1, \sigma_2 \in \mathrm{C}_*, \forall \ell_1, \ell_2 \in \mathrm{W}^{1,1}\big([0, T], Y^*\big),\\
&\quad \|y_1 - y_2\|_{\mathrm{W}^{1,1}([0,T],Y)} \leqslant L_{\mathrm{W}}\big(\|\sigma_1 - \sigma_2\|_* + \|\ell_1 - \ell_2\|_{\mathrm{W}^{1,1}([0,T],Y^*)}\big).
\end{aligned}$$

Short and complete proofs of these results can be found in [Kre99], Section 6. Further results concerning the continuity of $\mathcal{H}$ with respect to changes in the set C_* can be found in [Kre96]. If $(C_*^m)_{m\in\mathbb{N}}$ is a sequence of closed convex sets containing $0 \in Y^*$ and converging in the sense of the Hausdorff distance to C_*, then, for each $p \in [1, \infty)$, the convergence of the data $(\sigma_m, \ell_m) \to (\sigma, \ell)$ in $Y^* \times \mathrm{W}^{1,p}([0, T], Y^*)$ implies the convergence of the solutions, namely $\mathcal{H}_{C_*^m}(\sigma_m, \ell_m) \to \mathcal{H}_{C_*}(\sigma, \ell)$ in $\mathrm{W}^{1,p}([0, T], Y)$. If additionally all C_*^m are recession sets, then the result holds as well with $\mathrm{C}^0([0, T], Y^*)$ and $\mathrm{C}^0([0, T], Y)$, respectively. In fact, in the light of Theorem 2.6 it should be sufficient to assume that (2.25) holds uniformly for $(C_*^m)_{m\in\mathbb{N}}$ and that the property of recession sets is no longer needed.

3. Incremental problems and a priori estimates

For general nonlinear problems in discrete and continuum mechanics the assumption of convexity is not reasonable. First, the constitutive laws may be nonmonotone giving rise to nonconvex potentials, and second, even the underlying set of states may have no underlying linear structure any more. Such situations are usually modeled by working on suitable subsets of Banach spaces which are equipped either with the weak or the strong topology. Here we propose an abstract approach which does not need any Banach space theory and thus highlights the fundamental nature of the energy functionals $\mathcal{E}$ and $\mathcal{D}$ even more.

3.1. *Abstract setup of the problem*

To keep the connection with continuum mechanics we consider the set $\mathcal{Y} = \mathcal{F} \times \mathcal{Z}$ as the basic state space, cf. (2.10) for the same splitting in the linearized setting. Whenever possible we will write y instead of (φ, z) to shorten notation. Note that the splitting is done such that changes in z involve dissipation whereas those of φ do not. The state space $\mathcal{Y}$ is equipped with a Hausdorff topology $\mathcal{T} = \mathcal{T}_{\mathcal{F}} \times \mathcal{T}_{\mathcal{Z}}$ and we denote by $y_k \overset{\mathcal{Y}}{\to} y$, $\varphi_k \overset{\mathcal{F}}{\to} \varphi$ and $z_k \overset{\mathcal{Z}}{\to} z$ the corresponding convergence of sequences. Throughout it will be sufficient to consider sequential closedness, compactness and continuity.

The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$, which is a semidistance, which means

$$\begin{aligned} &\text{(i)} \quad \forall z_1, z_2 \in \mathcal{Z}, \quad \mathcal{D}(z_1, z_2) = 0 \iff z_1 = z_2, \\ &\text{(ii)} \quad \forall z_1, z_2, z_3 \in \mathcal{Z}, \quad \mathcal{D}(z_1, z_3) \leqslant \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3). \end{aligned} \tag{A1}$$

Here (i) is the classical positivity of a distance and (ii) the triangle inequality. Note that we allow the value ∞ and that we do not enforce symmetry, i.e., $\mathcal{D}(z_1, z_2) \neq \mathcal{D}(z_2, z_1)$ is allowed, as this is needed in many applications.

One major point of the theory is the interplay between the topology $\mathcal{T}_{\mathcal{Z}}$ and the dissipation distance. To have a typical nontrivial application in mind, one may consider $\mathcal{Z} = \{z \in \mathrm{L}^1(\Omega, \mathbb{R}^k) \mid \|z\|_{\mathrm{L}^\infty} \leqslant 1\}$ equipped with the weak L^1-topology and the dissipation distance $\mathcal{D}(z_1, z_2) = \|z_1 - z_2\|_{\mathrm{L}^1}$.

For a given curve $z : [0, T] \to \mathcal{Z}$ we define the total dissipation on $[s, t]$ via

$$\begin{aligned} &\mathrm{Diss}_{\mathcal{D}}\big(z; [s, t]\big) \\ &\quad = \sup\left\{ \sum_{1}^{N} \mathcal{D}\big(z(\tau_{j-1}), z(\tau_j)\big) \;\middle|\; N \in \mathbb{N}, s = \tau_0 < \tau_1 < \cdots < \tau_N = t \right\}. \end{aligned} \tag{3.1}$$

Further we define the following set of functions:

$$\mathrm{BV}_{\mathcal{D}}\big([0, T], \mathcal{Z}\big) := \big\{ z : [0, T] \to \mathcal{Z} \;\big|\; \mathrm{Diss}_{\mathcal{D}}\big(z; [0, T]\big) < \infty \big\}.$$

The functions are defined everywhere and changing them at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_{\mathcal{D}}\big(z; [r, t]\big) = \mathrm{Diss}_{\mathcal{D}}\big(z; [r, s]\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [s, t]\big) \quad \text{for all } r < s < t.$$

Later on, we will sometimes use the notation $\mathcal{D}(y_0, y_1)$ instead of $\mathcal{D}(z_0, z_1)$ where $y_j = (\varphi_j, z_j)$. This slight abuse of notation will never lead to confusion, since $\mathcal{D}$ as a function on $\mathcal{Y} = \mathcal{F} \times \mathcal{Z}$ still satisfies all assumptions except of the positivity (A1)(i).

The second ingredient is the energy-storage functional $\mathcal{E} : [0, T] \times \mathcal{Y} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Here $t \in [0, T]$ plays the role of a (very slow) process time which changes the underlying system via changing loading conditions. We assume that for all y with $\mathcal{E}(t, y_*) < \infty$ the function $t \mapsto \mathcal{E}(t, y_*)$ is differentiable, namely

$$\text{There exist } c_E^{(1)}, c_E^{(0)} > 0 \text{ such that for all } y_* \in \mathcal{Y}:$$
$$\mathcal{E}(t, y_*) < \infty \quad \Longrightarrow \quad \begin{cases} \partial_t \mathcal{E}(\cdot, y_*) : [0, T] \to \mathbb{R} \text{ is measurable and} \\ \left|\partial_t \mathcal{E}(t, y_*)\right| \leqslant c_E^{(1)}\big(\mathcal{E}(t, y_*) + c_E^{(0)}\big). \end{cases} \tag{A2}$$

From (A2) and Gronwall's inequality we easily derive

$$\mathcal{E}(t, y) + c_E^{(0)} \leqslant \big(\mathcal{E}(s, y) + c_E^{(0)}\big)\, \mathrm{e}^{c_E^{(1)}|t-s|}, \tag{3.2}$$

which implies the Lipschitz continuity of $t \mapsto \mathcal{E}(t, y)$. The notion of *self-controlling models* in [Che01,Che03] corresponds closely to our condition (A2).

DEFINITION 3.1. A curve $y = (\varphi, z) : [0, T] \to \mathcal{Y} = \mathcal{F} \times \mathcal{Z}$ is called an *energetic solution* of the rate-independent system associated with $(\mathcal{D}, \mathcal{E})$, if $t \mapsto \partial_t \mathcal{E}(t, y(t))$ is integrable and if the *global stability* (S) and the *energy equality* (E) hold for all $t \in [0, T]$:

$$\text{(S)} \quad \text{For all } \hat{y} = (\hat{\varphi}, \hat{z}) \in \mathcal{Y}, \quad \text{we have} \quad \mathcal{E}\big(t, y(t)\big) \leqslant \mathcal{E}(t, \hat{y}) + \mathcal{D}\big(z(t), \hat{z}\big);$$

$$\text{(E)} \quad \mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0, t]\big) = \mathcal{E}\big(0, y(0)\big) + \int_0^t \partial_t \mathcal{E}\big(\tau, y(\tau)\big)\, \mathrm{d}\tau.$$

The stability condition (S) can be rephrased by defining the set $\mathcal{S}(t)$ of stable states at time t via

$$\mathcal{S}(t) := \big\{ y \in \mathcal{Y} \;\big|\; \mathcal{E}(t, y) < \infty, \mathcal{E}(t, y) \leqslant \mathcal{E}(t, \hat{y}) + \mathcal{D}(y, \hat{y}) \text{ for all } \hat{y} \in \mathcal{Y} \big\},$$
$$\mathcal{S}_{[0,T]} := \big\{ (t, y) \in [0, T] \times \mathcal{Y} \;\big|\; y \in \mathcal{S}(t) \big\} = \bigcup_{t \in [0,T]} \big(t, \mathcal{S}(t)\big).$$

Then, (S) simply means that $y(t) \in \mathcal{S}(t)$ for all $t \in [0, T]$. The properties of the stable sets turn out to be crucial for deriving existence results.

The definition of solutions of (S) and (E) is such that it implies the two natural requirements for evolutionary problems, namely that *restrictions* and *concatenations* of solutions remain solutions. To be more precise, for any solution $y : [0, T] \to \mathcal{Y}$ and any subinterval $[s, t] \subset [0, T]$, the restriction $y|_{[s,t]}$ solves (S) and (E) with initial datum $y(s)$. Moreover, if $y_1 : [0, t] \to \mathcal{Y}$ and $y_2 : [t, T] \to \mathcal{Y}$ solve (S) and (E) on the respective intervals and if $y_1(t) = y_2(t)$, then the concatenation $y : [0, T] \to \mathcal{Y}$ solves (S) and (E) as well. Thus, the definition implies that if solvability can be shown for all $z_0 \in \mathcal{S}(0)$, then we have a multivalued evolutionary semigroup as explained in Section 1, see also [Bal97].

Rate-independence manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that y is a solution for $(\mathcal{D}, \mathcal{E})$ if and only if the reparametrized curve $\tilde{y} : t \mapsto y(\alpha(t))$, where $\dot{\alpha} > 0$, is a solution for $(\mathcal{D}, \widetilde{\mathcal{E}})$ with $\widetilde{\mathcal{E}}(t, y) = \mathcal{E}(\alpha(t), y)$. In particular, the stability (S) is a static concept and the energy balance (E) is rate-independent, since the dissipation defined via (3.1) is scale invariant like the length of a curve.

Before discussing the question of existence of solutions we want to point out, that the concept of energetic solutions provides a priori bounds on the solutions. For the time-continuous problem these bounds are easy to derive and the main structure becomes more transparent. Of course, similar estimates will be crucial in the time-discrete setting. Using the assumption (A2) the energy balance (E) gives

$$\begin{aligned} &\mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0, t]\big) \\ &\quad \leqslant \mathcal{E}\big(0, y(0)\big) + \int_0^t c_E^{(1)}\big(\mathcal{E}\big(s, y(s)\big) + c_E^{(0)}\big)\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Omitting the dissipation and adding $c_E^{(0)}$ on both sides allows for an application of Gronwall's inequality and we obtain

$$\mathcal{E}\big(t, y(t)\big) \leqslant \big(\mathcal{E}\big(0, y(0)\big) + c_E^{(0)}\big)\,\mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}.$$

Inserting this again into (3.3) we can also estimate the dissipation via

$$\mathrm{Diss}_{\mathcal{D}}\big(z; [0, T]\big) \leqslant \big(\mathcal{E}\big(0, y(0)\big) + c_E^{(0)}\big)\,\mathrm{e}^{c_E^{(1)} T},$$

since $\mathcal{E}(t, y(t)) \geqslant -c_E^{(0)}$ by (A2).

Because of the rate-independence it is easily possible to generalize assumption (A2) to include absolutely continuous loadings instead of C^1-loadings. We may replace (A2) by

$$\exists c_E^{(0)}, \lambda \in \mathrm{L}^1\big([0, T]\big), \quad \big|\partial_t \mathcal{E}(t, y_*)\big| \leqslant \lambda(t)\big(\mathcal{E}(t, y_*) + c_E^{(0)}\big).$$

With $\Lambda(t) = \int_0^t \lambda(\tau)\,\mathrm{d}\tau$ we easily find the estimates $\mathcal{E}(t, y) + c_E^{(0)} \leqslant (\mathcal{E}(s, y) + c_E^{(0)}) \times \mathrm{e}^{|\Lambda(t)-\Lambda(s)|}$, $\mathcal{E}(t, y(t)) \leqslant (\mathcal{E}(0, y(0)) + c_E^{(0)})\mathrm{e}^{\Lambda(t)} - c_E^{(0)}$ and $\mathrm{Diss}_{\mathcal{D}}(z; [0, T]) \leqslant (\mathcal{E}(0, y(0)) + c_E^{(0)})\mathrm{e}^{\Lambda(T)}$.

3.2. *The time-incremental problem*

The most natural approach to solve (S) and (E) is via time discretization using the fact that incremental problems exist which are minimization problems. Using the classical approach for the *direct method in the calculus of variations* (cf. [Dac89]) it is possible to find solutions as minimizers of a lower semicontinuous functional on $\mathcal{Y}$. For this we make the following standard assumptions:

$$\begin{aligned} &\mathcal{E}(\cdot, \cdot) : [0, T] \times \mathcal{Y} \to \mathbb{R}_\infty \text{ has compact sublevels,} \\ &\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is lower semicontinuous.} \end{aligned} \tag{A3}$$

Here the sublevels L_α of $\mathcal{E}$ are defined via $L_\alpha := \{(t, y) \in [0, T] \times \mathcal{Y} \mid \mathcal{E}(t, y) \leqslant \alpha\}$. Compactness of all L_α implies lower semicontinuity, in particular each $\mathcal{E}(t, \cdot) : \mathcal{Y} \to \mathbb{R}_\infty$ will be lower semicontinuous. In the standard case $\mathcal{Y}$ is a closed, convex and bounded subset of a reflexive Banach space (like the Sobolev space $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^m)$ with $p \in (1, \infty)$) equipped with its weak topology $\mathcal{T}$. Then, lower semicontinuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{Y}, \mathcal{T})$ is the same as the classical weak lower semicontinuity in the calculus of variations, see [Dac89].

For the time discretization we choose a partition $(t_k)_k \in \mathrm{Part}^N([0, T])$ and seek for a y_k which approximates the solution y at t_k, i.e., $y_k \approx y(t_k)$. Our energetic approach has the major advantage that the values y_k can be found incrementally via minimization problems. Since the methods of the calculus of variations are especially suited for applications in material modeling this will allow for a rich field of applications.

In our general setting the *incremental problem* (IP) takes the following form:

$$\text{(IP)} \qquad \begin{array}{l} \text{For } y_0 \in \mathcal{S}(0) \subset \mathcal{Y} \text{ find } y_1, \ldots, y_N \in \mathcal{Y} \text{ such that} \\ y_k \in \operatorname{Arg\,min}\{\mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \mid y \in \mathcal{Y}\} \text{ for } k = 1, \ldots, N. \end{array} \tag{3.4}$$

Here "Arg min" denotes the set of all minimizers. The following result shows that (IP) is intrinsically linked to (S) and (E).

3.3. *Energetic a priori bounds*

Without any smallness assumptions on the time steps, the solutions of (IP) satisfy properties which are closely related to (S) and (E).

THEOREM 3.2. *Let* (A1) *and* (A2) *hold. Any solution of the incremental problem* (3.4) *satisfies the following properties.*

(i) *For $k = 0, \ldots, N$ we have that y_k is stable at time t_k, i.e., $y_k \in \mathcal{S}(t_k)$.*

(ii) *For $k = 1, \ldots, N$ we have*

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, y_k)\, \mathrm{d}s &\leqslant e_k - e_{k-1} + \delta_k \\ &\leqslant \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, y_{k-1})\, \mathrm{d}s, \end{aligned}$$

where $e_j = \mathcal{E}(t_j, y_j)$ and $\delta_k = \mathcal{D}(z_{k-1}, z_k)$.

(iii) *If* (A3) *holds, then solutions of* (IP) *exist.*

PROOF. (i) The stability follows from minimization properties of the solutions and the triangle inequality. For all $\hat{y} \in \mathcal{Y}$ we have

$$\begin{aligned} \mathcal{E}(t_k, \hat{y}) + \mathcal{D}(z_k, \hat{z}) &= \mathcal{E}(t_k, \hat{y}) + \mathcal{D}(z_{k-1}, \hat{z}) + \mathcal{D}(z_k, \hat{z}) - \mathcal{D}(z_{k-1}, \hat{z}) \\ &\geqslant \mathcal{E}(t_k, y_k) + \mathcal{D}(z_{k-1}, z_k) + \mathcal{D}(z_k, \hat{z}) - \mathcal{D}(z_{k-1}, \hat{z}) \\ &\geqslant \mathcal{E}(t_k, y_k). \end{aligned}$$

(ii) The first estimate is deduced from $y_{k-1} \in \mathcal{S}(t_{k-1})$ as follows

$$\begin{aligned}
&\mathcal{E}(t_k, y_k) + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, y_{k-1})\\
&\quad = \mathcal{E}(t_{k-1}, y_k) + \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_k)\,\mathrm{d}s + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, y_{k-1})\\
&\quad \geqslant \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_k)\,\mathrm{d}s.
\end{aligned}$$

Since $y_k \in \operatorname{Arg\,min}\{\mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, z) \mid y \in \mathcal{Y}\}$ the second estimate follows via

$$\begin{aligned}
&\mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(z_{k-1}, z_k)\\
&\quad \leqslant \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(z_{k-1}, z_{k-1}) = \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_{k-1})\,\mathrm{d}s.
\end{aligned} \tag{3.5}$$

(iii) The minimizers are constructed inductively. In the kth step y_{k-1} is known and any minimizer y has to satisfy $\mathcal{J}_k(y) := \mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, z) \leqslant \mathcal{E}(t_k, y_{k-1}) = \mathcal{J}_k(y_{k-1})$ since $y = y_{k-1}$ is a candidate. Using $\mathcal{D} \geqslant 0$ it suffices to minimize the lower semicontinuous functional $\mathcal{J}_k$ on the compact sublevel $\mathcal{E}(t_k, \cdot) \leqslant \mathcal{E}(t_k, y_{k-1})$. Hence, Weierstrass' extremum principle provides the existence of a minimizer y_k. □

Now we use assumption (A2) to obtain a priori bounds on the energy and the dissipation for the solution of (IP). Combining (A2), (3.2) and the upper estimate in (ii) of Theorem 3.2 give

$$e_k + \delta_k \leqslant e_{k-1} + \big(c_E^{(0)} + e_{k-1}\big)\big(\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - 1\big) \tag{3.6}$$

$$= \big(c_E^{(0)} + e_{k-1}\big)\,\mathrm{e}^{c_E^{(1)}(t_k - t_{k-1})} - c_E^{(0)}. \tag{3.7}$$

Using $\delta_k \geqslant 0$ and (3.7), induction over k leads to

$$c_E^{(0)} + e_k \leqslant \big(c_E^{(0)} + e_0\big) \prod_{j=1}^{k} \mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} = \big(c_E^{(0)} + e_0\big)\mathrm{e}^{c_E^{(1)} t_k} \quad \text{for } k = 1, \ldots, N. \tag{3.8}$$

For the dissipated energy we find the estimate

$$\begin{aligned}
\sum_{j=1}^{k} \delta_j &\overset{(3.6)}{\leqslant} e_0 - e_k + \sum_{j=1}^{k} \big(c_E^{(0)} + e_{j-1}\big)\big(\mathrm{e}^{c_E^{(1)}(t_j - t_{j-1})} - 1\big)\\
&\overset{(3.8)}{\leqslant} \big(c_E^{(0)} + e_0\big) - \big(c_E^{(0)} + e_k\big) + \big(c_E^{(0)} + e_0\big) \sum_{1}^{k} \big(\mathrm{e}^{c_E^{(1)} t_j} - \mathrm{e}^{c_E^{(1)} t_{j-1}}\big)\\
&\leqslant \big(c_E^{(0)} + e_0\big)\mathrm{e}^{c_E^{(1)} t_k},
\end{aligned}$$

where $c_E^{(0)} + e_k \geqslant 0$ was used in the last step.

For each incremental solution $(y_k)_{k=1,\dots,N}$ of (IP) associated with a partition $\Pi \in \mathrm{Part}^N([0,T])$ we define the piecewise constant interpolant Y^Π with

$$\begin{aligned} &Y^\Pi(T) = y_N \quad \text{and} \\ &Y^\Pi(t) = y_{k-1} \quad \text{for } t \in [t_{k-1}, t_k), \text{ where } k = 1, \dots, N. \end{aligned} \tag{3.9}$$

COROLLARY 3.3. *Assume that* (A1) *and* (A2) *hold and let* $\Pi \in \mathrm{Part}^N([0,T])$. *Then, for any solution* $(y_k)_{k=0,\dots,N}$ *of* (IP) *the interpolant* $Y^\Pi = (\varphi^\Pi, z^\Pi) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ *satisfies the following three properties.*

(1) $(\mathrm{S})_{\mathrm{discr}}$ *For* $t \in \Pi$ *we have* $Y^\Pi(t) \in \mathcal{S}(t)$.

(2) $(\mathrm{E})_{\mathrm{discr}}$ *For* $s, t \in \Pi$ *with* $s < t$ *we have the energy estimate*

$$\mathcal{E}\big(t, Y^\Pi(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z^\Pi; [s,t]\big) \leqslant \mathcal{E}\big(s, Y^\Pi(s)\big) + \int_s^t \partial_\tau \mathcal{E}\big(\tau, Y^\Pi(\tau)\big)\, \mathrm{d}\tau.$$

(3) *For all* $t \in [0,T]$ *we have the a priori estimates* (*with* $E_0 = \mathcal{E}(0, y_0) + c_E^{(0)}$)

$$\mathcal{E}\big(t, Y^\Pi(t)\big) \leqslant E_0\, \mathrm{e}^{c_E^{(1)} t} - c_E^{(0)} \quad \textit{and} \quad \mathrm{Diss}_{\mathcal{D}}\big(z^\Pi; [0,T]\big) \leqslant E_0 \mathrm{e}^{c_E^{(1)} T}. \tag{3.10}$$

3.4. *The condensed and reduced incremental problem*

Recall the incremental problem in the form

$$(\varphi_k, z_k) \in \operatorname{Arg\,min}\big\{\mathcal{E}(t_k, \hat{\varphi}, \hat{z}) + \mathcal{D}(z_{k-1}, \hat{z}) \;\big|\; (\hat{\varphi}, \hat{z}) \in \mathcal{F} \times \mathcal{Z}\big\}. \tag{3.11}$$

In many applications in continuum mechanics a specific feature occurs, namely that $\mathcal{E}(t, \varphi, z)$ and $\mathcal{D}(z_0, z)$ depend only locally on z, in the sense that at $x \in \Omega$ the integral over Ω uses z only through its point value $z(x)$. Hence, z can be eliminated pointwise. We define the *condensed energy density* W^{cond} and the *update function* Z^{update} for the internal variable via

$$\begin{aligned} &W^{\mathrm{cond}}(z_{\mathrm{old}}; x, F) := \min\big\{W(x, F, z) + D(x, z_{\mathrm{old}}, z) \;\big|\; z \in Z\big\}, \\ &Z^{\mathrm{update}}(z_{\mathrm{old}}; x, F) \in \operatorname{Arg\,min}\big\{W(x, F, z) + D(x, z_{\mathrm{old}}, z) \;\big|\; z \in Z\big\}. \end{aligned} \tag{3.12}$$

With this we obtain a functional $\mathcal{E}^{\mathrm{cond}}(z_{\mathrm{old}}; t, \varphi) = \int_\Omega W^{\mathrm{cond}}(z_{\mathrm{old}}; \mathrm{D}\varphi)\, \mathrm{d}x - \langle \ell_{\mathrm{ext}}(t), \varphi \rangle$ and the solution of (3.11) is equivalent to finding $\varphi_k \in \operatorname{Arg\,min}\{\mathcal{E}^{\mathrm{cond}}(z_{k-1}; t_k, \hat{\varphi}) \mid \hat{\varphi} \in \mathcal{F}\}$ and then letting $z_k = Z^{\mathrm{update}}(z_{k-1}; \mathrm{D}\varphi_k)$. For more details we refer to [Mie03a,Mie04b].

The above condensation is very useful for computational purposes and it also allows for an existence theory for (IP) in the case of finite-strain elastoplasticity, see [Mie04b]. However, for the mathematical theory associated with the time-continuous problem (S) and (E) it seems advantageous to reduce the problem to the z-variable alone. The major difficulty in considering the pair $y = (\varphi, z)$ is that $\varphi \in \mathcal{F}$ does not appear in the dissipation. Hence, by (S), $\varphi(t)$ will always be a global minimizer of $\mathcal{E}(t, \cdot, z(t))$. But otherwise we have no control over the temporal oscillations in the approximate functions $\varphi^N : [0, T] \to \mathcal{F}$.

A first possible approach to tackle this difficulty is to introduce the reduced energy functional

$$\mathcal{E}^{\mathrm{red}}(t, z) = \min\{\mathcal{E}(t, \varphi, z) \mid \varphi \in \mathcal{F}\}.$$

However, in general we will lose the exact control, since $\mathcal{E}^{\mathrm{red}}$ is no longer explicit. In particular, the differentiability of $t \mapsto \mathcal{E}^{\mathrm{red}}(t, z)$ is not guaranteed in general. At the moment, there is only one way out, which is not always acceptable: We simply restrict ourselves to problems where the minimizer $\varphi = \Phi(t, z)$ of $\mathcal{E}(t, \cdot, z)$ is unique and depends continuously on (t, z). Then, $\mathcal{E}^{\mathrm{red}}(t, z) = \mathcal{E}(t, \Phi(t, z), z)$ and $\partial_t \mathcal{E}^{\mathrm{red}}(t, z) = \partial_t \mathcal{E}(t, \Phi(t, z), z)$.

The same assumption is needed if we keep the φ-variable. In this second approach the bottleneck is the assumption (A4) (cf. Section 5.1) which allows us to control convergence only in the component z. However, for $y_k = (\varphi_k, z_k) \in \mathcal{S}(t_k)$ we know $\varphi_k = \Phi(t_k, z_k)$ and, from $\mathcal{D}(z_k, z) \to 0$ together with continuity of $\mathcal{E}$ and $\mathcal{D}$, we conclude $\varphi_k \to \varphi = \Phi(t, z)$.

This uniqueness assumption will be used in Sections 7.3 and 7.5, see also [MieTL02, MieRou03,KoMR05]. However, in Section 7.6 this uniqueness can be dispensed with.

3.5. *Lipschitz bounds via convexity*

As we have seen in the quadratic case, we are able to obtain also Lipschitz bounds, if convexity of $\mathcal{E}(t, \cdot)$ is used. Classical convexity theory involves a Banach space $\mathcal{Y}$ as the basic state space. Then, a function $f : \mathcal{Y} \to \mathbb{R}_\infty$ is called *uniformly convex*, if there exists $\alpha > 0$ such that for all $y_0, y_1 \in \mathcal{Y}$ we have

$$f(y_\theta) \leqslant (1-\theta) f(y_0) + \theta f(y_1) - \frac{\alpha}{2}\theta(1-\theta)\|y_1 - y_0\|^2,$$
$$\text{where } y_\theta = (1-\theta) y_0 + \theta y_1.$$

This notion can be generalized to arbitrary metric spaces by using so-called geodesical convexity. So let $(\mathcal{Y}, d)$ be a general metric space. For each $y_0, y_1 \in \mathcal{Y}$ we define the set

$$[y_0, y_1]_\theta = \{y \in \mathcal{Y} \mid d(y, y_0) = \theta d(y_0, y_1) \text{ and } d(y, y_1) = (1-\theta) d(y_0, y_1)\}.$$

We define the following convexity notions for $f : \mathcal{Y} \to \mathbb{R}_\infty$:

f is convex if

$$\forall y_0, y_1 \in \mathcal{Y}, \forall \theta \in (0,1), \exists y \in [y_0, y_1]_\theta: \quad f(y) \leqslant (1-\theta) f(y_0) + \theta f(y_1);$$

f is strictly convex if

$$\forall y_0, y_1 \in \mathcal{Y}, \forall \theta \in (0,1), \exists y \in [y_0, y_1]_\theta: \quad f(y) < (1-\theta) f(y_0) + \theta f(y_1);$$

f is α-convex if

$$\forall y_0, y_1 \in \mathcal{Y}, \forall\theta \in (0,1), \exists y \in [y_0, y_1]_\theta\colon$$
$$f(y) \leqslant (1-\theta)f(y_0) + \theta f(y_1) - \frac{\alpha}{2}\theta(1-\theta)d(y_0,y_1)^2.$$

For strictly convex Banach spaces the sets $[y_0, y_1]_\theta$ are singletons and the above notions coincide with the classical ones. If an α-convex function f is two-times differentiable, then $\mathrm{D}^2 f$ satisfies $\mathrm{D}^2 f(y)[v,v] \geqslant \alpha\|v\|^2$. We again use the notion uniformly convex, if we have α-convexity for some $\alpha > 0$.

The major difficulty in general rate-independent problems is that the dissipation distance $\mathcal{D}$ is not convex, even if $\mathcal{Y}$ is a Banach space. For instance, we may consider a system $\widetilde{\mathcal{E}}, \widehat{\mathcal{D}}$ on $\mathcal{Y}$ which is a Banach space and $\widehat{\mathcal{D}}$ has the form $\widehat{\mathcal{D}}(\hat{y}_0, \hat{y}_1) = \Psi(\hat{y}_1 - \hat{y}_0)$. Doing a coordinate transformation $\hat{y} = \Phi(y)$ we arrive at the transformed dissipation distance $\mathcal{D}\colon (y_0, y_1) \mapsto \widehat{\mathcal{D}}(\Phi(y_0), \Phi(y_1))$, which is no longer convex on $\mathcal{Y} \times \mathcal{Y}$ if Φ is a nonlinear mapping. However, with respect to the transformed metric $d\colon (y_0, y_1) \mapsto \|\Phi(y_1) - \Phi(y_0)\|$ geodesic convexity is preserved.

It should be noted that our dissipation distance is in general different from the metric to be considered. For instance we want to allow for $\mathcal{D}$ assuming the value $+\infty$ and for asymmetry. It is also important to note, that in general the function $d(y_*, \cdot)\colon y \mapsto d(y_*, y)$ is not geodesically convex on $(\mathcal{Y}, d)$, for instance the arc-length distance on $\mathbb{S}^1$. However, on $\mathbb{S}^1$ nonconstant convex functions exist, but they must attain the value $+\infty$.

THEOREM 3.4. *Assume that $\mathcal{E}$ and $\mathcal{D}$ are defined on the metric space $(\mathcal{Y}, d)$ and satisfy* (A2) *and the following two conditions*:

$$\exists\alpha > 0, \forall(t,y) \in [0,T] \times \mathcal{Y}\colon \quad \hat{y} \mapsto \mathcal{E}(t,\hat{y}) + \mathcal{D}(y,\hat{y}) \quad \textit{is } \alpha\textit{-convex}, \tag{3.13}$$
$$\forall t \in [0,T], \forall y_0, y_1 \in \mathcal{Y}\colon \quad \left|\partial_t\mathcal{E}(t,y_0) - \partial_t\mathcal{E}(t,y_1)\right| \leqslant C_3 d(y_0,y_1). \tag{3.14}$$

Then, any solution $y\colon [0,T] \to Y$ of (S) *and* (E) *satisfies the estimate*

$$\forall s,t \in [0,T], \quad d\big(y(s), y(t)\big) \leqslant \frac{C_3}{\alpha}|t-s|.$$

Moreover, the solution $(y_k)_{k=0,1,\ldots,N}$ of (IP) *is unique and satisfies*

$$\forall k = 1,\ldots,N, \quad d(y_{k-1}, y_k) \leqslant 2\frac{C_3}{\alpha}(t_k - t_{k-1}).$$

PROOF. We use the fact that for the minimizer y_* of an α-convex function f we always have $f(y) \geqslant f(y_*) + \frac{\alpha}{2}d(y,y_*)^2$, hence it is unique.

For the first assertion we use $y(s) \in \mathcal{S}(s)$, $\mathcal{D}(y(s), y(t)) \leqslant \mathrm{Diss}_{\mathcal{D}}(y; [s,t])$ and (E):

$$\begin{aligned}
&\frac{\alpha}{2} d\big(y(s), y(t)\big)^2 \\
&\quad \leqslant \mathcal{E}\big(s, y(t)\big) + \mathcal{D}\big(y(s), y(t)\big) - \mathcal{E}\big(s, y(s)\big) \\
&\quad \leqslant \mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(y; [s,t]\big) - \mathcal{E}\big(s, y(s)\big) - \int_s^t \partial_r \mathcal{E}\big(r, y(t)\big)\, \mathrm{d}r \\
&\quad \leqslant \int_s^t \big[\partial_r \mathcal{E}\big(r, y(r)\big) - \partial_r \mathcal{E}\big(r, y(t)\big)\big]\, \mathrm{d}r.
\end{aligned}$$

For fixed $t > 0$ let $\delta(s) = d(y(s), y(t))$, then $\delta(s)^2 \leqslant \frac{2C_3}{\alpha} \int_s^t \delta(r)\, \mathrm{d}r$ for $s \in [0,t]$. Now, let $h(\tau) = \int_{t-\tau}^t \delta(r)\, \mathrm{d}r$ for $\tau \geqslant 0$, then $h(0) = 0$ and $h'(\tau) \leqslant (2C_3 h(\tau)/\alpha)^{1/2}$. This implies $h(\tau) \leqslant C_3 \tau^2/(2\alpha)$ and hence $\delta(t-\tau) \leqslant C_3 \tau/\alpha$, which is the desired result.

The second assertion follows in the same way by using Theorem 3.2(i) and (ii), namely

$$\begin{aligned}
\frac{\alpha}{2} d(y_k, y_{k-1})^2 &\leqslant \int_{t_{k-1}}^{t_k} \big[\partial_r \mathcal{E}(r, y_{k-1}) - \mathcal{E}(r, y_k)\big]\, \mathrm{d}r \\
&\leqslant C_3 (t_k - t_{k-1}) d(y_k, y_{k-1}),
\end{aligned}$$

which proves the claim. □

With this result we supply Lipschitz continuity results for the solution of (S) and (E) and of (IP). They will be useful in the construction of solutions.

EXAMPLE 3.5. We illustrate the concept with a simple example on $\mathcal{Y} = \mathbb{R}$. Take the energy functional $\mathcal{E}(t, y) = (y - \ell(t))^2/2$ and the dissipation metric

$$\Psi(y, \dot{y}) = h(y)|\dot{y}|, \quad \text{where } h(z) = \begin{cases} 1 + a - a\,\mathrm{sign}(z) & \text{for } |z| \geqslant 1, \\ 1 + a - az & \text{for } |z| \leqslant 1, \end{cases}$$

where $a > 0$. It is easy to see that the associated dissipation distance is given by $\mathcal{D}(y_0, y_1) = |H(y_1) - H(y_0)|$ with $H(y) = \int_0^y h(s)\, \mathrm{d}s$.

Thus, using the classical distance $d(y_0, y_1) = |y_1 - y_0|$ on $\mathbb{R}$, we see that $y \mapsto \mathcal{E}(t, y) + \mathcal{D}(y_0, y)$ is (geodesically) α-convex, for $\alpha = \inf\{\mathcal{E}''(t, y) + \partial_y^2 \mathcal{D}(y_0, y) \mid y \in \mathbb{R}\}$. Thus, for $y_0 \geqslant 1$ we obtain $\alpha = 1$ and for $y_0 < 1$ we have $\alpha = 1 - a$. In particular, $y_0 < 1$ and $a > 1$ imply that convexity is lost and uniqueness of minimizers and Lipschitz bounds are no longer valid.

3.6. *A simplified incremental problem*

If $\mathcal{Y}$ is a Banach space Y and the dissipation distance $\mathcal{D}$ is implicitly defined through a dissipation potential $\Psi : Y \times Y \to [0, \infty]$ via

$$\begin{aligned}&\mathcal{D}(y_0, y_1)\\&\quad := \inf\left\{ \int_0^1 \Psi\big(y(t), \dot{y}(t)\big)\,\mathrm{d}t \;\middle|\; y \in \mathrm{C}^1\big([0,1], Y\big),\, y(0) = y_0,\, y(1) = y_1 \right\},\end{aligned}$$

then it is desirable to approach the differential inclusion

$$0 \in \partial_v \Psi(y, \dot{y}) + \mathrm{D}\mathcal{E}(t, y) \subset Y^*$$

via an incremental problem which avoids $\mathcal{D}$ and uses Ψ instead. Under suitable conditions we have

$$\mathcal{D}(y_0, y_0 + \varepsilon v) = \varepsilon \Psi(y_0, v) + \frac{\varepsilon^2}{2} \mathrm{D}_z \Psi(y_0, v)[v] + \mathrm{o}\big(\varepsilon^2\big).$$

Thus, if some convexity is available then we may hope that the increments are small and thus it should suffice to approximate $\mathcal{D}(y_0, y_1)$ by $\Psi(y_0, y_1 - y_0)$. This leads to the following incremental problem:

$$(\widetilde{\mathrm{IP}}) \qquad \begin{aligned}&\text{For } y_0 \in \mathcal{S}(0) \text{ find } y_1, \dots, y_N \in \mathcal{Y} \text{ such that}\\&y_k \in \operatorname{Arg\,min}\{\mathcal{E}(t_k, y) + \Psi(y_{k-1}, y - y_{k-1}) \mid y \in Y\}.\end{aligned} \tag{3.15}$$

The function $\Psi(y, \cdot)$ is always convex, hence the solutions y_k are unique as soon as $\mathcal{E}(t_k, \cdot)$ is strictly convex. Thus, $(\widetilde{\mathrm{IP}})$ is solvable, but we find no counterpart to Theorem 3.2 concerning stability and energy inequalities. The problem is that we do not have a counterpart to the triangle inequality. To obtain useful bounds we need the dependence of $\Psi(z, v)$ on z to be sufficiently mild.

In general we need the following smallness assumption:

$$\begin{aligned}&\exists \psi_* > 0, \exists \alpha > \psi_* \text{ such that } \mathcal{E}(t, \cdot) \text{ is } \alpha\text{-convex and}\\&\forall v, y_1, y_2 \in Y\colon\ |\Psi(y_1, v) - \Psi(y_2, v)| \leqslant \psi_* d(y_1, y_2)\|v\|.\end{aligned} \tag{3.16}$$

Then the incremental solutions as well as time-continuous solutions satisfy an a priori Lipschitz bound. From $(\widetilde{\mathrm{IP}})$ and α-convexity we obtain

$$\begin{aligned}\forall y \in Y\colon \quad &\mathcal{E}(t_j, y_j) + \Psi(y_{j-1}, y_j - y_{j-1}) + \frac{\alpha}{2} d(y, y_j)^2\\&\leqslant \mathcal{E}(t_j, y) + \Psi(y_{j-1}, y - y_{j-1}).\end{aligned}$$

As shorthand we introduce $\delta_j = d(y_j, y_{j-1})$. Fixing k we use the above estimate with $j = k$ and $y = y_{k-1}$ and add this to the estimate obtained with $j = k - 1$ and $y = y_k$. This gives (a):

$$\begin{aligned}
\alpha\delta_k^2 &\overset{\text{(a)}}{\leqslant} \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_k, y_k) + \mathcal{E}(t_{k-1}, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) \\
&\quad + 0 - \Psi(y_{k-1}, y_k - y_{k-1}) + \Psi(y_{k-2}, y_k - y_{k-2}) \\
&\quad - \Psi(y_{k-2}, y_{k-1} - y_{k-2}) \\
&\overset{\text{(b)}}{\leqslant} \int_{t_{k-1}}^{t_k} \partial_r\mathcal{E}(r, y_{k-1}) - \partial_r\mathcal{E}(r, y_k)\,\mathrm{d}r \\
&\quad - \Psi(y_{k-1}, y_k - y_{k-1}) + \Psi(y_{k-2}, y_k - y_{k-1}) \\
&\overset{\text{(c)}}{\leqslant} C_3(t_k - t_{k-1})\delta_k + \psi_*\delta_{k-1}\delta_k.
\end{aligned}$$

For (b) we use the fact that $\Psi(y_{k-2}, \cdot)$ satisfies the triangle inequality and (c) follows from (3.14) and (3.16). After dividing by $\alpha\delta_k$ we find the recurrence relation

$$\delta_k \leqslant \frac{C_3}{\alpha}(t_k - t_{k-1}) + \frac{\psi_*}{\alpha}\delta_{k-1},$$

which provides the desired a priori bound on δ_k, since $\psi_*/\alpha < 1$.

In Example 3.5 the condition $\psi_*/\alpha < 1$ takes the from $\psi_* = \|h'\|_\infty = |a| < 1$ since $\alpha = 1$.

4. Convex energies

In this section we treat the case of general, uniformly convex energies $\mathcal{E}$ on a reflexive Banach space Y (which in most cases will, in fact, be a Hilbert space). Since in this section the distinction between the elastic variable $\varphi \in \mathcal{F}$ and the dissipative variable $z \in \mathcal{Z}$ is not of importance, we will use exclusively the variable y and write $\mathcal{D}(y_0, y_1)$ instead of $\mathcal{D}(z_0, z_1)$. The point is, of course, that $\mathcal{D}(y_0, y_1) = 0$ does not imply $y_0 = y_1$. Similarly, $\Psi(y, v) = 0$ will not imply $v = 0$.

In most parts we will follow [MieT04] and use the classical convexity with respect to the linear Banach space structure, since the general theory of geodesic convexity is not developed enough. However, in Section 4.3 we will address the question of more general concepts of convexity. By strict convexity on a Banach space Y the energetic formulation (S) and (E) is equivalent to the variational inequality (VI) or to the subdifferential formulation (SF):

$$\text{(VI)} \qquad \forall v \in Y, \quad \big\langle \mathrm{D}\mathcal{E}\big(t, y(t)\big), y - \dot y(t)\big\rangle + \Psi\big(y(t), v\big) - \Psi\big(y(t), \dot y(t)\big) \geqslant 0,$$

$$\text{(SF)} \qquad 0 \in \partial_v\Psi\big(y(t), \dot y(t)\big) + \mathrm{D}\mathcal{E}\big(t, y(t)\big) \subset Y^*.$$

In fact, in (SF) it suffices to use the subdifferential $\partial\mathcal{E}(t, y(t))$ in the case of a nondifferentiable, convex energy.

However, in this section we will always assume smoothness, namely

$$\begin{aligned}
&\mathcal{E} \in \mathrm{C}^3([0,T]\times Y,\mathbb{R}) \text{ satisfies (A1) and for each } E_0>0\\
&\text{there exist constants } C_{ty}, C_{yyy}, C_{tyy}>0, \text{ such that}\\
&\quad \mathcal{E}(0,y)\leqslant E_0\\
&\quad\Longrightarrow\quad \begin{cases} \big\|\mathrm{D}\mathcal{E}(t,y)\big\|,\ \big\|\mathrm{D}^2\mathcal{E}(t,y)\big\|,\ \big\|\partial_t\mathrm{D}\mathcal{E}(t,y)\big\|\leqslant C_{ty},\\ \big\|\mathrm{D}^3\mathcal{E}(t,y)\big\|\leqslant C_{yyy}, \qquad \big\|\partial_t\mathrm{D}^2\mathcal{E}(t,y)\big\|\leqslant C_{tyy}.\end{cases}
\end{aligned} \tag{C1}$$

In the quadratic case of Section 2 both constants C_{tyy} and C_{yyy} are equal to 0.

The second major assumption is of course uniform convexity, i.e.,

$$\exists\alpha>0, \forall v,y\in Y:\quad \big\langle \mathrm{D}^2\mathcal{E}(t,y)v,v\big\rangle \geqslant \alpha\|v\|^2. \tag{C2}$$

Because of C^2-regularity this condition is equivalent to α-convexity as defined in Section 3.5, namely

$$\mathcal{E}\big(t,(1-\theta)y_0+\theta y_1\big)\leqslant(1-\theta)\mathcal{E}(t,y_0)+\theta\mathcal{E}(t,y_1)-\frac{\alpha}{2}(1-\theta)\theta\|y_1-y_0\|^2.$$

In Section 4.1 we will treat the case of a translation invariant dissipation metric, i.e., $\mathrm{D}_y\Psi\equiv 0$, and in Section 4.3 we will consider partial results in cases where Ψ really depends on y. In any case $\Psi: Y\times Y\to[0,\infty]$ will always be such that $\Psi(z,\cdot): Y\to[0,\infty]$ is 1-homogeneous, convex and lower semicontinuous. The value $+\infty$ is allowed as well as $\Psi(z,v)=0$ for $v\neq 0$. Recall that typical applications in continuum mechanics relate to spaces $Y=\mathrm{L}^2(\Omega)$ and $\Psi(y,v)=\|v\|_{\mathrm{L}^1(\Omega)}=\int_\Omega|v(x)|\,\mathrm{d}x$. Thus, we do not need any of the assumptions $\Psi(z,v)\geqslant\rho_1\|v\|$ or $\Psi(y,v)\leqslant\rho_2\|v\|$ with $\rho_j>0$ (which would mean equivalently for $C_*(y)=\partial\Psi(z,0)$ the inclusions $B^*_{\rho_1}(0)\subset C_*(y)$ and $C_*(z)\subset B^*_{\rho_2}(0)$, respectively).

4.1. *Translation invariant dissipation potentials*

Like in [MieT04] we consider the case $\mathrm{D}_y\Psi\equiv 0$, i.e., $\Psi(y,v)=\Psi(v)$,

$$\Psi: Y\to[0,\infty] \quad \text{is convex, 1-homogeneous and lower semicontinuous.} \tag{C3}$$

As a consequence the dissipation distance $\mathcal{D}: Y\times Y\to[0,\infty]$ has a simple form and the closed, convex sets $C_*(y)=\partial\Psi(y,0)$ are constant

$$\mathcal{D}(y_0,y_1)=\Psi(y_1-y_0)\quad\text{and}\quad C_*=\partial\Psi(0)\subset Y^*.$$

By smoothness and convexity of $\mathcal{E}$ the stable sets can be characterized as

$$\mathcal{S}(t)=\big\{y\in Y\ \big|\ -\mathrm{D}\mathcal{E}(t,y)\in C_*\big\}.$$

Note that in general these sets are not convex unless $\mathcal{E}(t,\cdot)$ is quadratic, see Example 5.12. In particular, $\mathcal{S}(t)$ will in general be strongly closed but not weakly closed.

The major advantage in assuming that Ψ is translation invariant is that in the variational inequality (VI) we can compare two solutions. For instance, if y_1 and y_2 are solutions of

$$\text{(VI)} \qquad \forall v \in Y, \quad \big\langle \mathrm{D}\mathcal{E}\big(t, y(t)\big), v - \dot{y}(t)\big\rangle + \Psi(v) - \Psi\big(\dot{y}(t)\big) \geqslant 0, \tag{4.1}$$

then we may test with $v_j = \dot{y}_{3-j}(t)$ and add the two inequalities to obtain

$$\big\langle \mathrm{D}\mathcal{E}\big(t, y_1(t)\big) - \mathrm{D}\mathcal{E}\big(t, y_2(t)\big), \dot{y}_1(t) - \dot{y}_2(t)\big\rangle \leqslant 0, \tag{4.2}$$

which generalizes the basic monotonicity estimate employed in Section 2.2 to show that in the quadratic case the hysteresis operator defines a contraction semigroup, see (2.16).

We will first use this estimate to prove an existence result and then show that similar methods allow us to establish existence via proving strong convergence of the solutions obtained by the time-incremental method.

PROPOSITION 4.1. *If assumptions* (C1)–(C3) *hold, then the variational inequality* (4.1) *has for each* $y_0 \in \mathcal{S}(0)$ *at most one solution with* $y(0) = y_0$.

PROOF. Let y_1 and y_2 be two solutions. By Theorem 3.4 we know that each solution must be Lipschitz continuous, hence $\dot{y}_j(t)$ exist a.e. in $[0,T]$ and satisfies $\|\dot{y}_j(t)\| \leqslant K$. With $\mathrm{D}\mathcal{E}_j = \mathrm{D}\mathcal{E}(t, y_j(t))$ define

$$\gamma(t) = \big\langle \mathrm{D}\mathcal{E}_1 - \mathrm{D}\mathcal{E}_2, y_1(t) - y_2(t)\big\rangle \geqslant \alpha \big\| y_1(t) - y_2(t)\big\|^2,$$

where we used α-convexity of $\mathcal{E}$. Moreover, we have

$$\dot{\gamma}(t) = \big\langle \partial_t \mathrm{D}\mathcal{E}(t, y_1) - \partial_t \mathrm{D}\mathcal{E}(t, y_2), y_1 - y_2\big\rangle + \big\langle r_1^*, \dot{y}_1\big\rangle + \big\langle r_2^*, \dot{z}_2\big\rangle,$$

where $r_j^* = 2(\mathrm{D}\mathcal{E}_j - \mathrm{D}\mathcal{E}_{3-j}) + b_j^*$ with $b_j^* = \mathrm{D}^2\mathcal{E}(t, y_j)[y_j - y_{3-j}] - \mathrm{D}\mathcal{E}_j + \mathrm{D}\mathcal{E}_{3-j}$. Using (C1) we find $\|b_j^*\|_* \leqslant C_{yyy}\|y_1 - y_2\|^2$ and $\|\partial_t \mathrm{D}\mathcal{E}(t, y_1) - \partial_t \mathrm{D}\mathcal{E}(t, y_2)\| \leqslant C_{tyy} \times \|y_1 - y_2\|$ which leads to

$$\begin{aligned} \dot{\gamma} &\leqslant C_{tyy}\|y_1 - y_2\|^2 + C_{yyy}\|y_1 - y_2\|^2\big(\mathrm{Lip}(y_1) + \mathrm{Lip}(y_2)\big) \\ &\quad + 2\langle \mathrm{D}\mathcal{E}_1 - \mathrm{D}\mathcal{E}_2, \dot{y}_1 - \dot{y}_2\rangle \\ &\leqslant (C_{tyy} + C_{yyy}2K)\|y_1 - y_2\|^2 + 0 \\ &\leqslant \frac{(C_{tyy} + C_{yyy}2K)\gamma}{\alpha}, \end{aligned}$$

where we have used (4.2) in the second estimate. Thus, Gronwall's estimate implies the desired result if $\gamma(0) = 0$. $\square$

Another way to establish uniqueness without the above strong smoothness condition (C1) and the α-convexity in (C2) is possible, if the stable sets $\mathcal{S}(t)$ are convex, see [MieT04], Section 5, for sufficient conditions.

THEOREM 4.2. *If $\mathcal{E}$ has the form $\mathcal{E}(t,y) = \mathcal{U}(y) - \langle \ell(t), y\rangle$ where $\mathcal{U}$ is strictly convex and if the stable sets $\mathcal{S}(t)$ are convex for all $t \in [0,T]$, then for each initial condition $y_0 \in \mathcal{S}(0)$ there is at most one solution to* (S) *and* (E).

PROOF. Let $y_j : [0,T] \to Y$ be two solutions with $y_j(0) = y_0$ and $\tilde{y}(t) = \frac{1}{2}(y_1(t) + y_2(t))$. By the convexity of the stable sets we know that $\tilde{y}(t) \in \mathcal{S}(t)$ and thus $\tilde{y}$ satisfies (S).

Now assume $y_2(t) \neq y_1(t)$ for some $t > 0$. Using strict convexity of $\mathcal{U}$, the energy balance (E) of (2.9) and the linearity of $\partial_t \mathcal{E}$, we obtain

$$\begin{aligned}
&\mathcal{E}\big(t, \tilde{y}(t)\big) + \mathrm{Diss}_\Psi\big(\tilde{y}; [0,t]\big) \\
&\quad < \frac{1}{2}\big[\mathcal{E}\big(t, y_2(t)\big) + \mathcal{E}\big(t, y_1(t)\big) + \mathrm{Diss}_\Psi\big(y_2; [0,t]\big) + \mathrm{Diss}_\Psi\big(y_1; [0,t]\big)\big] \\
&\quad = \frac{1}{2}\big[\mathcal{E}\big(0, y_2(0)\big) + \mathcal{E}\big(0, y_1(0)\big)\big] - \int_0^t \frac{1}{2}\big[\langle \dot{\ell}, y_2\rangle + \langle \dot{\ell}, y_1\rangle\big]\,\mathrm{d}s \\
&\quad = \mathcal{E}(0, y_0) + \int_0^t \partial_t \mathcal{E}\big(s, \tilde{y}(s)\big)\,\mathrm{d}s.
\end{aligned}$$

However, as in Proposition 5.7 it can be shown that (S) implies the opposite energy inequality, i.e., $\mathcal{E}(t, \tilde{y}(t)) + \mathrm{Diss}_\Psi(\tilde{y}; [0,t]) \geqslant \mathcal{E}(0, y_0) + \int_0^t \partial_t \mathcal{E}(s, \tilde{y}(s))\,\mathrm{d}s$. This produces a contradiction and we conclude $y_1 \equiv y_2$. □

The following result was first established in [MieT04], Section 7. For the readers convenience we repeat the main steps, since the proof of Proposition 7.2 contains several wrong signs. The proof is an adaptation of the strong convergence result in [HanR95,HanR99] for the quadratic case.

THEOREM 4.3. *Assumptions* (C1)–(C3) *hold. Then, the variational inequality* (4.1) *has for each $y_0 \in \mathcal{S}(0)$ a unique solution $y \in \mathrm{C}^{\mathrm{Lip}}([0,T], Y)$ with $y(0) = y_0$ which depends Lipschitz continuously on the initial value $y_0 \in \mathcal{S}(0)$.*

Moreover, the solutions $Y^\Pi : [0,T] \to Y$ (cf. (3.9)) *of the incremental problem* (IP) *in* (3.4) *associated with a partition Π are unique and converge strongly to the unique solution such that*

$$\big\| Y^\Pi - y \big\|_{\mathrm{L}^\infty([0,T],Y)} \leqslant C\sqrt{f(\Pi)},$$

where $f(\Pi) = \min\{t_i - t_{i-1} \mid i = 1, \ldots, N\}$ is the fineness of the partition and where C is independent of the solution and of the partition.

The uniqueness part and the Lipschitz continuity in the initial condition was established already in Proposition 4.1. For the existence and the strong convergence the following proposition is crucial.

PROPOSITION 4.4. *Let $\mathcal{E}^1$ and $\mathcal{E}^2$ satisfy assumptions* (C1)–(C3) *and let $y_0 \in \mathcal{S}(0)$ be given. Set*

$$\rho = \sup\bigl\{ \bigl\| \mathrm{D}\mathcal{E}^1(t,y) - \mathrm{D}\mathcal{E}^2(t,y) \bigr\|_* \bigm| \mathcal{E}(t,y) \leqslant \bigl(\mathcal{E}(0,y_0) + c_E^{(0)}\bigr)\mathrm{e}^{c_E^{(1)} t} - c_E^{(0)} \bigr\}.$$

Then, there exists a constant $C_0 > 0$ such that for any partition Π the associated solutions Y_1^Π and Y_2^Π satisfy

$$\bigl\| Y_1^\Pi - Y_2^\Pi \bigr\|_{\mathrm{L}^\infty([0,T],Y)} \leqslant C_0 \sqrt{\rho}.$$

PROOF. We introduce the notation $\sigma^j(t,y) = \mathrm{D}\mathcal{E}^j(t,y)$, $e_k = y_k^1 - y_k^2$ and the difference operator $\tau_k \zeta = \zeta_k - \zeta_{k-1}$, where ζ stands for t, y^j, $\sigma^j(t, y_k^l)$ or e.

The incremental solutions $(y_k^j)_{k=0,\dots,N}$ are defined via the variational inequality

$$\forall v \in Y, \quad \bigl\langle \sigma^j\bigl(t_k, y_k^j\bigr), v - \tau_k z^j \bigr\rangle + \Psi(v) - \Psi\bigl(\tau_k y^j\bigr) \geqslant 0.$$

Choosing $v = \tau_k y^{3-j}$ and adding the equations for $j = 1$ and 2 gives the discrete counterpart to (4.2), namely

$$\bigl\langle \sigma^1\bigl(t_k, z_k^1\bigr) - \sigma^2\bigl(t_k, z_k^2\bigr), \tau_k e \bigr\rangle \leqslant 0. \tag{4.3}$$

As in the uniqueness proof we introduce an energetic quantity γ_k which controls the error $e_k = y_k^1 - y_k^2$ because of α-convexity

$$\begin{aligned}
\gamma_k &= \bigl\langle \sigma^1\bigl(t_k, y_k^1\bigr) - \sigma^1\bigl(t_k, y_k^2\bigr), e_k \bigr\rangle \\
&= \bigl\langle \mathrm{D}\mathcal{E}^1\bigl(t_k, y_k^1\bigr) - \mathrm{D}\mathcal{E}^1\bigl(t_k, y_k^2\bigr), y_k^1 - y_k^2 \bigr\rangle \\
&\geqslant \alpha \bigl\| y_k^1 - y_k^2 \bigr\|^2 = \alpha \|e_k\|^2.
\end{aligned}$$

The increment $\tau_k \gamma = \gamma_k - \gamma_{k-1}$ can be estimated via (4.3) as follows

$$\begin{aligned}
\tau_k \gamma &= \bigl\langle \sigma^1\bigl(t_k, y_k^1\bigr) - \sigma^1\bigl(t_k, y_k^2\bigr), \tau_k e \bigr\rangle + \bigl\langle \tau_k\bigl(\sigma^1\bigl(t_k, y_k^1\bigr) - \sigma^1\bigl(t_k, y_k^2\bigr)\bigr), e_{k-1} \bigr\rangle \\
&= 2\bigl\langle \sigma^1\bigl(t_k, y_k^1\bigr) - \sigma^2\bigl(t_k, y_k^2\bigr), \tau_k e \bigr\rangle + \beta_k \leqslant \beta_k,
\end{aligned}$$

where β_k takes the form

$$\begin{aligned}
\beta_k &= \langle A_k e_k - A_{k-1} e_{k-1}, e_{k-1} \rangle - \langle A_k e_k, \tau_k e \rangle - 2\bigl\langle \sigma^1\bigl(t_k, y_k^2\bigr) - \sigma^2\bigl(t_k, y_k^2\bigr), \tau_k e \bigr\rangle \\
&= -\langle A_k \tau_k e, \tau_k e \rangle + \bigl\langle (A_k - A_{k-1}) e_{k-1}, e_{k-1} \bigr\rangle \\
&\quad - 2\bigl\langle \sigma^1\bigl(t_k, y_k^2\bigr) - \sigma^2\bigl(t_k, y_k^2\bigr), \tau_k e \bigr\rangle.
\end{aligned}$$

The symmetric operators $A_k \in \mathrm{Lin}(Y, Y^*)$ are defined via $A_k = \int_0^1 \mathrm{D}^2\mathcal{E}^1(t_k, y_k^2 + \theta e_k)\,\mathrm{d}\theta$ and satisfy $A_k e_k = \sigma^1(t_k, y_k^1) - \sigma^1(t_k, y_k^2)$. By convexity and three-times differentiability we obtain

$$\langle A_k y, y\rangle \geqslant 0$$

and

$$\|A_k - A_{k-1}\| \leqslant C_{tyy}\tau_k t + C_{yyy}\big(\big\|\tau_k y^1\big\| + \big\|\tau_k y^2\big\|\big) \quad \text{where } \tau_k t = t_k - t_{k-1}.$$

Together with $\|\tau_k e\| \leqslant \|\tau_k z^1\| + \|\tau_k z^2\| \leqslant 2C_K\tau_k t$ (see Theorem 3.4) we find

$$\begin{aligned}\tau_k\gamma &\leqslant 0 + (C_{tyy} + C_{yyy}2C_K)\tau_k t\|e_{k-1}\|^2 + \big\|\sigma^1\big(t_k, y_k^2\big) - \sigma^2\big(t_k, y_k^2\big)\big\|2C_K\tau_k t\\ &\leqslant \left(\frac{C_{tyy} + 2C_K C_{yyy}}{\alpha}\gamma_{k-1} + \rho 2C_K\right)\tau_k t\\ &= (C_1\gamma_{k-1} + C_2\rho)(t_k - t_{k-1}).\end{aligned}$$

By induction over k we find

$$\begin{aligned}\gamma_k &\leqslant C_2\rho\sum_{n=1}^{k}(t_n - t_{n-1})\prod_{j=n+1}^{k}\big[1 + C_1(t_j - t_{j-1})\big]\\ &\leqslant C_2\rho\sum_{n=1}^{k}(t_n - t_{n-1})\mathrm{e}^{C_1(t_k - t_n)} \leqslant C_2\rho\int_0^{t_k}\mathrm{e}^{C_1(t_k - s)}\,\mathrm{d}s\\ &= \frac{C_2\rho}{C_1}\big(\mathrm{e}^{C_1 t_k} - 1\big).\end{aligned}$$

Together with $\|y_k^1 - y_k^2\|^2 \leqslant \frac{1}{\alpha}\gamma_k$ this is the desired result $\|y_k^1 - y_k^2\|^2 \leqslant \rho C_2\mathrm{e}^{C_1 T}/(\alpha C_2)$. □

PROOF OF THEOREM 4.3. By convexity and the a priori assumptions we know that for all partitions the solutions of (IP) exist and lie in the sublevel $\mathcal{E}(t, y) \leqslant (\mathcal{E}(0, y_0) + c_E^{(0)}) \times \mathrm{e}^{c_E^{(1)} t} - c_E^{(0)}$.

We start with an arbitrary partition Π of $[0, T]$ and define the sequence of partitions Π_m by $\Pi_1 = \Pi$ and by successive dividing each subinterval into two equal intervals of half the length, in particular, $f(\Pi_m) = 2^{1-m} f(\Pi)$. Denote by $Y^m : [0, T] \to Y$ the solution associated with the partition Π_m. For comparing Y^m and Y^{m+1} we want to apply Proposition 4.4. For this we define $\mathcal{E}^1$ and $\mathcal{E}^2$ as follows. For $t_k \in \Pi_{m+1}$ define $\hat{t}_k \in \Pi_m$ via $\hat{t}_k = \max\{s_j \in \Pi_m \mid s_j \leqslant t_k\}$ and let

$$\mathcal{E}^1(t_k, y) = \mathcal{E}\big(\hat{t}_k, y\big) \quad \text{and} \quad \mathcal{E}^2(t_k, y) = \mathcal{E}(t_k, y) \quad \text{for } t_k \in \Pi_{m+1}.$$

For $t \notin \Pi_{m+1}$ we may define $\mathcal{E}^j$ by piecewise linear interpolation. The construction was done such that the incremental solution for $\mathcal{E}^2$ gives exactly Y^{m+1}, whereas Y^m is exactly the incremental solution obtained with $\mathcal{E}^1$ on the partition Π_{m+1}, since $\hat{t}_{2j} = \hat{t}_{2j+1}$ leads to the fact that the unique incremental solution for $\mathcal{E}^1$ does only move on every second step.

Now $|t_k - \hat{t}_k| \leqslant f(\Pi_{m+1}) = 2^{-m} f(\Pi)$ implies $\|\mathrm{D}\mathcal{E}^1(t, y) - \mathrm{D}\mathcal{E}^2(t, y)\| \leqslant C_{ty} 2^{-m} f(\Pi)$ on the relevant sublevel. Thus, we conclude

$$\left\| Y^{m+1} - Y^m \right\|_{\mathrm{L}^\infty([0,T],Y)} \leqslant C_0 \big(C_{ty} 2^{-m} f(\Pi)\big)^{1/2} = C_* \sqrt{f(\Pi)}\, 2^{-m/2},$$

and $(Y^m)_{m\in\mathbb{N}}$ form a Cauchy sequence in $\mathrm{L}^\infty([0,T], Y)$ with a limit $y : [0,T] \to Y$. Note that the total distance between y and $Y^1 = Y^\Pi$ is less than or equal to $\sum_1^\infty C_* \sqrt{f(\Pi)} \times 2^{-m/2} \leqslant 3C_* \sqrt{f(\Pi)}$ as desired.

It remains to show that y is a solution of (S) and (E) which is equivalent to (4.1). Using the a priori Lipschitz estimate of Theorem 3.4 shows that all Y^m are uniformly Lipschitz when restricted to Π_m. Hence, y satisfies $\|y(t) - y(s)\| \leqslant C_{ty}|t - s|/\alpha$. Moreover, using the stability of Y^m at the points in Π_m, the strong convergence and the strong closedness of $\mathcal{S}_{[0,T]} = \{(t, y) \mid y \in \mathcal{S}(t)\}$ we easily conclude $y(t) \in \mathcal{S}(t)$ for all $t \in [0, T]$. Finally we are able to pass to the limit in the discrete energy estimates (ii) obtained in Theorem 3.2 and find that y also satisfies (E). This concludes the proof of Theorem 4.3. □

4.2. *Quasi-variational inequalities*

In [BrKS04] the evolution quasi-variational inequality

$$\begin{aligned}
&\text{(i)} && \ell(t) - Ay(t) \in C_*\big(g\big(t, \ell(t), y(t)\big)\big) \quad \text{for all } t \in [0, T], \\
&\text{(ii)} && \ell(0) - Ay(0) = \sigma_0 \in C_*\big(g\big(0, \ell(0), y(0)\big)\big), \\
&\text{(iii)} && \big\langle \ell(t) - Ay(t) - \hat{\sigma}, \dot{y}(t)\big\rangle \geqslant 0 \quad \text{for } \hat{\sigma} \in C_*\big(g\big(t, \ell(t), y(t)\big)\big) \\
& && \quad \text{for a.a. } t \in [0, T],
\end{aligned} \tag{4.4}$$

with Y being a Hilbert space and $\langle Ay_1, y_2\rangle = \langle y_1 | y_2\rangle$ is considered. Here $g : [0, T] \times Y^* \times Y \to \mathcal{R}$ is a continuous map and $\mathcal{R}$ is a closed, convex subset of a Banach space R. Moreover, for each $g \in \mathcal{R}$ the set $C_*(g)$ is closed and convex and satisfies $B^*_{\rho_1}(0) \subset C_*(g) \subset B^*_{\rho_2}(0) \subset Y^*$. Under suitable conditions, which we explain below, an existence and uniqueness result is derived which corresponds to Theorem 4.3.

To compare the results in [BrKS04] with the results presented so far, we translate it into our subdifferential framework. With $\mathcal{E}(t, y) = \frac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle$ and $\Psi(g, \cdot) = \mathcal{L}I_{C_*(g)}$, (4.4) takes the form

$$\begin{aligned}
&0 \in \partial\Psi\big(g\big(t, \ell(t), y(t)\big), \dot{y}(t)\big) + \mathrm{D}\mathcal{E}\big(t, y(t)\big) \quad \text{a.e. on } [0, T]; \\
&\quad y(0) = A^{-1}\big(\ell(0) - \sigma_0\big).
\end{aligned} \tag{4.5}$$

The above assumptions imply that the dissipation metric $\Psi(g,\cdot)$ satisfies the estimates

$$\forall r \in \mathcal{R}, \forall v \in Y, \quad \rho_1\|v\| \leqslant \Psi(r,v) \leqslant \rho_2\|v\|. \tag{4.6}$$

The results in [BrKS04] are formulated in terms of the Minkowski functional $\mathcal{M}_{C_*(r)} : Y^* \to [0,\infty)$ of the sets $C_*(r)$, namely

$$M(r,\sigma) = \mathcal{M}_{C_*(r)}(\sigma) = \inf\left\{ s > 0 \;\middle|\; \frac{1}{s}\sigma \in C_*(r) \right\}.$$

By the Legendre–Fenchel transform $\mathcal{L}$ we have, for $\mathbb{B}(r,\sigma) = \frac{1}{2}M(r,\sigma)^2$, the identity

$$\frac{1}{2}\Psi(r,v)^2 = \mathcal{L}\big[\mathbb{B}(r,\cdot)\big](v) = \mathcal{L}\left[\frac{1}{2}M(r,\cdot)^2\right](v).$$

The main assumptions on $C_*(r)$ or $\Psi(r,\cdot)$ are now phrased in terms of $\mathbb{B}$:

$$\begin{aligned}
&\text{(a)} \quad \mathbb{B} \in \mathrm{C}^1\big(\mathcal{R} \times Y^*; \mathbb{R}\big) \quad \text{with} \\
&\qquad J(r,\sigma) = \mathrm{D}_\sigma \mathbb{B}(r,\sigma) \in Y \quad \text{and} \quad K(r,\sigma) = \mathrm{D}_r \mathbb{B}(r,\sigma) \in R^*; \\
&\text{(b)} \quad \forall r \in \mathbb{R}, \forall \sigma \in C_*(r): \quad \big\|K(r,\sigma)\big\|_{R^*} \leqslant K_0; \\
&\text{(c)} \quad \forall r_1, r_2 \in \mathcal{R}, \forall \sigma_1 \in C_*(r_1), \forall \sigma_2 \in C_*(r_2): \\
&\qquad\quad \big\|J(r_1,\sigma_1) - J(r_2,\sigma_2)\big\|_Y \leqslant C_J \big\|(r_1,\sigma_1) - (r_2,\sigma_2)\big\|_{R\times Y^*}, \\
&\qquad\quad \big\|K(r_1,\sigma_1) - K(r_2,\sigma_2)\big\|_{R^*} \leqslant C_K \big\|(r_1,\sigma_1) - (r_2,\sigma_2)\big\|_{R\times Y^*}.
\end{aligned} \tag{4.7}$$

With these assumptions for the linear, time-dependent problem

$$\begin{aligned}
&0 \in \partial\Psi\big(r(t), \dot{y}(t)\big) + Ay(t) - \ell(t) \quad \text{a.e. on } [0,T]; \\
&\quad y(0) = A^{-1}\big(\ell(0) - \sigma_0\big),
\end{aligned} \tag{4.8}$$

the following existence result and Lipschitz estimate are derived.

PROPOSITION 4.5. *If assumptions* (4.6) *and* (4.7) *hold, then, for each* $\ell \in \mathrm{W}^{1,1}([0,T], Y^*)$, $r \in \mathrm{W}^{1,1}([0,T],\mathcal{R})$ *and* $\sigma_0 \in \mathcal{C}_*(r(0))$, (4.8) *has a unique solution. Moreover, if* y_1, y_2 *are solutions of* (4.8) *associated with* (r_1,ℓ_1) *and* (r_2,ℓ_2), *respectively, then for a.a.* $t \in [0,T]$ *we have the estimate*

$$\begin{aligned}
&\frac{1}{\rho_2}\big\|\dot{y}_1(t) - \dot{y}_2(t)\big\| + \frac{\mathrm{d}}{\mathrm{d}t}\big|\mathbb{B}_1(t) - \mathbb{B}_2(t)\big| \\
&\quad\leqslant \frac{1}{\rho_1}\big\|\dot{\ell}_1(t) - \dot{\ell}_2(t)\big\|_* + K_0\big\|\dot{r}_1(t) - \dot{r}_2(t)\big\|_R \\
&\qquad + \big(2C_J\big\|\dot{\ell}_1(t)\big\|_* + (C_K + \rho_2 C_J K_0)\big\|\dot{r}_1(t)\big\|_R\big) \\
&\qquad\quad \times \big(\big\|r_1(t) - r_2(t)\big\|_R + \big\|\ell_1(t) - \ell_2(t)\big\|_*\big),
\end{aligned} \tag{4.9}$$

where $\mathbb{B}_j(t) = \mathbb{B}(r_j(t), \ell_j(t) - Ay_j(t))$.

The crucial new idea here is the introduction of the new quantity $\mathbb{B}$ which takes values in $[0, \frac{1}{2}]$, since $M(r, \sigma) \leqslant 1$ on $C_*(r)$. In fact, $\mathbb{B}$ measures the distance to the yield surface $\partial C_*(r(t))$, namely $\sigma \in \partial C_*(r) \Leftrightarrow \mathbb{B}(r, \sigma) = 1/2$. Moreover, the identity

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}\big(r(t), \ell(t) - Ay(t)\big) \\ &\quad = \big\langle K\big(r(t), \Sigma(t)\big), \dot{r}(t)\big\rangle + \big\langle J\big(r(t), \Sigma(t)\big), \dot{\ell}(t) - A\dot{y}(t)\big\rangle, \end{aligned}$$

where $\Sigma(t) = \ell(t) - Ay(t)$, together with the flow law $\dot{y}(t) = \lambda J(r(t), \Sigma(t))$ (where $\lambda \geqslant 0$ for $\mathbb{B} = 1/2$ and $\lambda = 0$ else) allows us to derive (4.9) in a direct manner.

Finally the original nonlinear problem is solved by a contraction argument. For this, the mapping $g : [0, T] \times Y^* \times Y \to \mathcal{R}$ needs to be suitable:

$$\begin{aligned} &\text{(a)} \quad g \in \mathrm{C}^1\big([0, T] \times Y^* \times Y, \mathcal{R}\big); \\ &\text{(b)} \quad \forall (t, \ell, y): \\ &\qquad \big\|\mathrm{D}_\ell g(t, \ell, y)\big\|_{Y^* \to R} \leqslant C_\ell, \qquad \big\|\mathrm{D}_y g(t, \ell, y)\big\|_{Y \to R} \leqslant C_y; \\ &\text{(c)} \quad \partial_t g(t, \cdot), \mathrm{D}_\ell g(t, \cdot), \mathrm{D}_y g(t, \cdot) \quad \text{are uniformly Lipschitz continuous.} \end{aligned} \tag{4.10}$$

THEOREM 4.6. *If assumptions* (4.6), (4.7) *and* (4.10) *hold and*

$$\rho_2 K_0 C_y < 1, \tag{4.11}$$

then, for each $\ell \in \mathrm{W}^{1,1}([0, T], Y^*)$ *and each* $\sigma_0 \in Y^*$ *with* $\sigma_0 \in C_*(g(0, \ell(0), A^{-1}(\ell(0) - \sigma_0)))$, (4.5) (*or equivalently,* (4.4)) *has a unique solution* $y \in \mathrm{W}^{1,1}([0, T], Y)$. *Moreover, the solutions depend Lipschitz continuously on the data in the following way. For each* $\rho_3 > 0$ *there exists a constant* $C_3 > 0$ *such that for every* ℓ_1 *and* ℓ_2 *with* $\|\ell_j\|_{\mathrm{W}^{1,1}} \leqslant \rho_3$ *and every* σ_0^j *with* $\sigma_0^j \in C_*(g(0, \ell_j(0), A^{-1}(\ell_0 - \sigma_0^j)))$ *the unique solutions* y_1 *and* y_2 *satisfy*

$$\|y_1 - y_2\|_{\mathrm{W}^{1,1}([0,T],Y)} \leqslant C_3\big(\big\|\sigma_0^1 - \sigma_0^2\big\|_* + \|\ell_1 - \ell_2\|_{\mathrm{W}^{1,1}([0,T],Y^*)}\big).$$

In [BrKS04], Section 8, one finds a counterexample with $Y = \mathbb{R}^2$ showing that dropping the Lipschitz continuity (4.7)(c) for J leads to nonuniqueness. Similar smallness conditions for the Lipschitz constant were obtained in [KuM98] cf. also [MieR05] and $\psi_*/\alpha < 1$ in Section 3.6.

In [RoS05], the quasivariational sweeping process,

$$-\dot{y} \in \partial I_{C_*(t,y)}(y) \subset H \quad \text{for } y \in \mathrm{W}^{1,1}\big([0, T], H\big), \tag{4.12}$$

is considered in an ordered Hilbert space $(H, \leqslant)$. The assumptions of the time and state dependent set C_* are the following.

(i) There exist functions $\mathbb{Y}_*(t,y)$ and $\mathbb{Y}^*(t,y)$ such that

$$C_*(t,y) = \big[\mathbb{Y}_*(t,y), \mathbb{Y}^*(t,y)\big] := \big\{\tilde{y} \in H \mid \mathbb{Y}_*(t,y) \leqslant \tilde{y} \leqslant \mathbb{Y}^*(t,y)\big\}.$$

(ii) For each $t \in [0,T]$ the functions $-\mathbb{Y}_*(t,\cdot): H \to H$ and $-\mathbb{Y}^*(t,\cdot): H \to H$ are
  (ii.1) maximal (for graph inclusion within monotone operators),
  (ii.2) T-monotone, i.e., $\langle -\mathbb{Y}^*_*(t,y_1) - \mathbb{Y}^*_*(t,y_2) | (y_1-y_2)^+\rangle \geqslant 0$ for all y_1, y_2,
  (ii.3) nondecreasing, i.e., $y_1 \leqslant y_2 \Rightarrow -\mathbb{Y}^*_*(t,y_1) \leqslant \mathbb{Y}^*_*(t,y_2)$.

Here $\mathbb{Y}^*_*$ means either $\mathbb{Y}_*$ or $\mathbb{Y}^*$.

In this general setting uniqueness of solutions can be established. An existence result is obtained for $H = \mathrm{L}^2(\Omega)$ with the usual ordering of functions and under additional assumptions on the functions Y_* and Y^*.

4.3. *General dissipation metrics*

This subsection is speculative, in the sense that we propose a certain philosophy which is under investigation in [MieR05].

The general subdifferential formulation has the form

$$0 \in \partial_v \Psi(y,\dot{y}) + \mathrm{D}\mathcal{E}(t,y) \subset Y^*, \tag{4.13}$$

where $\partial_v \Psi(y,\dot{y})$ denotes the subdifferential of Ψ with respect to the second variable $v = \dot{y}$ at fixed y. This formulation includes the theory in [MieT04] where $\mathrm{D}_z\Psi \equiv 0$ as well as the theory in [BrKS04] with $\mathrm{D}\mathcal{E}(t,y) = Ay - \ell(t)$ if we let $\mathcal{R} = R = Y$ and $g(t,\ell,y) = y$. In the above two subsections the results heavily rely on convexity assumptions of $\mathcal{E}$. However, existence and uniqueness results for such problems should not depend on the coordinate system we use to describe the problem. However, under coordinate transformations convexity properties are usually destroyed. Denote by $y = \Phi(\hat{y})$ a smooth coordinate transformation from $\widehat{Y}$ into Y ($\Phi \in \mathrm{C}^2(\widehat{Y}, Y)$). Then, the transformed energy $\widehat{\mathcal{E}}$ and the transformed dissipation metric $\widehat{\Psi}$ are

$$\widehat{\mathcal{E}}(t,\hat{y}) = \mathcal{E}\big(t, \Phi(\hat{y})\big) \quad \text{and} \quad \widehat{\Psi}(\hat{y},\hat{v}) = \Psi\big(\Phi(\hat{y}), \mathrm{D}\Phi(\hat{y})\hat{v}\big).$$

Using the adjoint operator $\mathrm{D}\Phi(\hat{y})^*: Y^* \to \widehat{Y}^*$, the transformed derivatives read $\partial_{\hat{v}}\widehat{\Psi}(\hat{y},\hat{v}) = \mathrm{D}\Phi(\hat{y})^*\,\partial\Psi(\Phi(\hat{y}), \mathrm{D}\Phi(\hat{y})\hat{v})$ and $\mathrm{D}\widehat{\mathcal{E}}(t,\hat{y}) = \mathrm{D}\Phi(\hat{y})^*\mathrm{D}\mathcal{E}(t,\Phi(\hat{y}))$, such that (4.13) is equivalent to the transformed equation $0 \in \partial\widehat{\Psi}(\hat{y},\dot{\hat{y}}) + \mathrm{D}\widehat{\mathcal{E}}(t,\hat{y}) \subset \widehat{Y}^*$.

For proving existence and uniqueness for (4.13) with a method like in [BrKS04] we would need the smallness condition $\|\mathrm{D}_z\Psi(y,v)\| \leqslant \delta \ll 1$ for all $y \in Y$ and v with $\|v\| \leqslant 1$ which implies (4.11) since by Legendre transform one finds $K(y,\sigma) = -\Psi(y, J(y,\sigma))\mathrm{D}_y\Psi(y, J(y,\sigma))$. However, this condition in not invariant under coordinate changes since

$$\begin{aligned}\mathrm{D}_y\widehat{\Psi}(\hat{y},\hat{v})[\hat{w}] = {} & \mathrm{D}_y\Psi\big(\Phi(\hat{y}), \mathrm{D}\Phi(\hat{y})\hat{v}\big)\big[\mathrm{D}\phi(\hat{y})\hat{w}\big] \\ & + \mathrm{D}_v\Psi\big(\Phi(\hat{y}), \mathrm{D}\Phi(\hat{y})\hat{v}\big)\big[\mathrm{D}^2\phi(\hat{y})[\hat{v},\hat{w}]\big]\end{aligned}$$

needs the second derivative of Φ as well as the derivative of Ψ with respect to v (whose existence we have to assume henceforth). In [MieR05] we argue that the smallness of $\|\mathrm{D}_y\Psi(y,v)\|$ can be replaced by a one-sided convexity condition which takes the form

$$\begin{aligned}&\exists\delta>0, \forall(t,y)\in\mathcal{S}_{[0,T]}, \forall v\in Y:\\ &\quad\big\langle \mathrm{D}^2\mathcal{E}(t,y)v,v\big\rangle+\mathrm{D}_y\Psi(y,v)[v]\geqslant\alpha\Psi(y,v)^2.\end{aligned}\tag{4.14}$$

Note that both sides in the estimate are 2-homogeneous, but the left-hand side in the estimate is not quadratic, since $\mathrm{D}_y\Psi(y,\alpha v)[\alpha v]=\alpha|\alpha|\mathrm{D}_y\Psi(y,v)[v]$.

Nevertheless the new condition is nothing else than the local version of the geodesic convexity if we use $d=\mathcal{D}$. In fact, using the expansion $\mathcal{D}(y,y+v)=\Psi(y,v)+\frac{1}{2}\mathrm{D}_y\Psi(y,v)[v]+\mathrm{o}(\|v\|^2)$, our smoothness assumptions on $\mathcal{E}$ and (4.14) we find

$$\begin{aligned}&\mathcal{E}(t,y+sv)+\mathcal{D}(y,y+sv)\\ &\quad\geqslant\mathcal{E}(t,y)+\big(s\mathrm{D}\mathcal{E}(t,y)[v]+\Psi(y,sv)\big)+\alpha s^2+\mathrm{o}\big(s^2\big)_{s\to0}.\end{aligned}$$

The left-hand side in the estimate of (4.14) is not invariant under coordinate changes. However, taking it together with the stability condition the additional term involving the second derivative of Φ has a positive sign, namely

$$\begin{aligned}&\big\langle\mathrm{D}^2\widehat{\mathcal{E}}(t,\hat{y})\hat{v},\hat{v}\big\rangle+\mathrm{D}_y\widehat{\Psi}(\hat{y},\hat{v})[\hat{v}]\\ &\quad=\big\langle\mathrm{D}^2\mathcal{E}\big(t,\Phi(\hat{y})\big)\mathrm{D}\Phi(\hat{y})\hat{v},\mathrm{D}\Phi(\hat{y})\hat{v}\big\rangle+\mathrm{D}_y\Psi\big(\Phi(\hat{y}),\mathrm{D}\Phi(\hat{y})\hat{v}\big)\big[\mathrm{D}\Phi(\hat{y})\hat{v}\big]\\ &\qquad+\frac{1}{2}\mathrm{D}\mathcal{E}\big(t,\Phi(\hat{y})\big)\big[\mathrm{D}^2\Phi(\hat{y})[\hat{v},\hat{v}]\big]+\frac{1}{2}\mathrm{D}_v\Psi\big(\Phi(\hat{y})\big)\big[\mathrm{D}^2\Phi(\hat{y})[\hat{v},\hat{v}]\big]\\ &\quad\overset{(\mathrm{S})}{\geqslant}\alpha\Psi\big(\Phi(\hat{y}),\mathrm{D}\Phi(\hat{y})\hat{v}\big)^2+0=\alpha\widehat{\Psi}^2(\hat{y},\hat{v})^2,\end{aligned}$$

where for (S) we have used $y=\Phi(\hat{y})\in\mathcal{S}(t)$ which implies $0\in\partial_v\Psi(y,0)+\mathrm{D}\mathcal{E}(t,y)$.

A second condition appearing already in [MieT04], Appendix C, is the so-called *structure condition* on the dissipation metric $\Psi:Y\times Y\to[0,\infty]$:

$$\begin{aligned}&\forall R>0, \exists C_R\geqslant0, \forall y_1,y_2 \text{ with } \|y_j\|\leqslant R:\\ &\quad 0\in\partial_v\Psi(y_j,v_j)+\sigma_j\quad\text{for } j=1,2\\ &\qquad\Longrightarrow\quad\langle\sigma_1-\sigma_2,v_1-v_2\rangle\leqslant C_R\|y_1-y_2\|^2\big(\|v_1\|+\|v_2\|\big).\end{aligned}\tag{4.15}$$

In contrast to the above condition, (4.15) is not invariant under coordinate changes.

EXAMPLE 4.7. We take $Y=\mathbb{R}^2$, $\Psi(y,v)=|v_1|+|v_2|$ and $y=\Phi(\hat{y})=(y_1+y_2^2,y_2)$. Then, Ψ satisfies the structure condition with $C_R=0$, since $0\in\partial\Psi(v_j)+\sigma_j$ is equivalent to $v_j \in \partial(\mathcal{L}\Psi)(-\sigma_j)$ and $\mathcal{L}\Psi = I_{C_*}$ is convex. However, $\widehat{\Psi}$ does not satisfy (4.15). To see this, use the explicit form

$$\partial_{\hat v}\widehat{\Psi}(\hat y, \hat v) = \mathrm{Sign}\Bigl(\Bigl\langle\begin{pmatrix}1\\ 2\hat y_2\end{pmatrix}, \hat v\Bigr\rangle\Bigr)(1, 2\hat y_2) + \mathrm{Sign}(\hat v_2)(0,1).$$

For $\hat y = (0,1)$ and $\hat v = (1,-1)$, we find $\partial\widehat{\Psi}(\hat y, \hat v) = \{(-1,-3)\}$. For $\hat y_* = (0, 1-\varepsilon)$ with $0 < \varepsilon < 1/2$ and $\hat v_* = (1,-2)$, we find $\partial\widehat{\Psi}(\hat y_*, \hat v_*) = \{(-1, -3+2\varepsilon)\}$. Thus, we find $\hat\sigma = (1,3)$ and $\sigma_* = (1, 3-2\varepsilon)$ and arrive at

$$\langle \hat\sigma - \hat\sigma_*, \hat v - \hat v_*\rangle = \bigl\langle (0, 2\varepsilon), (0,1)\bigr\rangle = 2\varepsilon.$$

Since $\|\hat y - \hat y_*\| = |\varepsilon|$ we see that (4.15) does not hold. The problem here is not the missing differentiability of Ψ but rather the fact, that Ψ^2 is not uniformly convex.

Based on assumption (4.14) and the natural smoothness assumptions on $\mathcal{E}$ and Ψ (e.g., $\mathcal{E} \in \mathrm{C}^3$, $\Psi^2 \in \mathrm{C}^2$ and Ψ^2 uniformly convex) analogues of Theorems 4.3 and 4.6 are derived in [MieR05].

A related notion of *convex composite systems* was introduced in [Che03]. There the notion of monotone operators is generalized to systems which are monotone after a suitable diffeomorphism is applied. See Section 6.2 for some details of this theory.

4.4. *Higher temporal regularity and improved convergence*

In the general convex case we know from the a priori estimates in Section 3.5 that the solutions are Lipschitz continuous. Under suitable conditions this regularity can be improved somewhat, but the best we can hope for, is that the time derivative $\dot y$ lies in $\mathrm{BV}([0,T], Y)$.

EXAMPLE 4.8. Let $Y = \mathbb{R}$ and $\mathcal{E}(t,y) = \frac12 (y - \ell(t))^2$, $\Psi(y, \dot y) = |\dot y|$ and $y(0) = 0$. For $\ell(t) = a\sin(t)$ with $a > 1$ define $t_1 \in (0, \frac{\pi}{2})$ via $\sin t_1 = 1/a$ and $t_2 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ via $\sin t_2 = 1 - 2/a$. Then the unique solution reads

$$y(t) = \begin{cases} 0 & \text{for } t \in [0, t_1], \\ a\sin(t) - 1 & \text{for } t \in [t_1, \frac{\pi}{2}] \text{ and} \\ & \quad t - 2\pi k \in \bigl[\pi + t_2, \frac{5\pi}{2}\bigr], \\ a - 1 & \text{for } t - 2\pi k \in \bigl[\frac{\pi}{2}, t_2\bigr], \\ a\sin(t) + 1 & \text{for } t - 2\pi k \in \bigl[t_2, \frac{3\pi}{2}\bigr], \\ -a + 1 & \text{for } t - 2\pi k \in \bigl[\frac{3\pi}{2}, \pi + t_2\bigr], \end{cases} \qquad \text{for } k \in \mathbb{N}_0.$$

Hence, the derivative $\dot y$ has jumps whenever $\Sigma(t) = \ell(t) - y(t)$ hits the yields surface (i.e., the boundary of $C_* = [-1, 1]$), namely at t_1 and $t_2 + \pi\mathbb{N}$. Note that the derivative $\dot y$ is continuous when y leaves the yield surface.

Using this observation the following result was derived in [MieT04], Theorem 7.8.

THEOREM 4.9. *If* $\mathcal{E} \in \mathrm{C}^3([0,T] \times Y, \mathbb{R})$ *and* $\Psi = \mathcal{L}I_{C_*}$, *where* C_* *satisfies* $B^*_{\rho_1}(0) \subset C_* \subset B^*_{\rho_2}(0)$ *for* $\rho_2 \geqslant \rho_1 > 0$ *and the boundary* ∂C_* *is of class* C^2, *then any solution of the variational inequality* (4.1) *satisfies* $\dot{y} \in \mathrm{BV}([0,T], Y)$.

In [Kre99], Theorem 7.2, a more abstract approach is used to prove a similar regularity result. It is based on the local Lipschitz continuity results as given in Theorems 2.7 and 2.8.

THEOREM 4.10. *If the hysteresis operator* $\mathcal{H}: C_* \times \mathrm{W}^{1,1}([0,T], Y^*) \to \mathrm{W}^{1,1}([0,T], Y)$ *is Lipschitz continuous on every bounded subset of* $C_* \times \mathrm{W}^{1,1}([0,T], Y^*)$, *then for every* $\sigma_0 \in \mathcal{C}_*$ *and every* $\ell \in \mathrm{W}^{1,1}([0,T], Y^*)$ *with* $\dot{\ell} \in \mathrm{BV}([0,T], Y^*)$ *the solution* $y = \mathcal{H}(\sigma_0, \ell)$ *satisfies* $\dot{y} \in \mathrm{BV}([0,T], Y)$.

PROOF. The idea of the proof is simple. For $h > 0$ consider the inputs ℓ_h with $\ell_h(t) = \ell(0)$ for $t \in [0,h]$ and $\ell(t) = \ell(t-h)$ for $t \in [h,T]$. Because of the rate independence the unique solution $\mathcal{H}(\sigma_0, \ell_h)$ is y_h which is obtained from y in the same way as ℓ_h from ℓ. Since the functions $(\ell_h)_{h\in[0,T]}$ are bounded in $\mathrm{W}^{1,1}([0,T], Y^*)$, we obtain a Lipschitz constant L such that

$$\begin{aligned}\int_h^T \big\|\dot{y}(t) - \dot{y}(t-h)\big\|\,\mathrm{d}t &\leqslant \|y - y_h\|_{\mathrm{W}^{1,1}} \leqslant L\|\ell - \ell_h\|_{\mathrm{W}^{1,1}} \\ &\leqslant C\int_0^T \big\|\ell(t) - \ell_h(t)\big\|_*\,\mathrm{d}t \leqslant Ch\big\|\dot{\ell}\big\|_{\mathrm{BV}([0,T],Y^*)}.\end{aligned}$$

This implies $\|\dot{y}\|_{\mathrm{BV}([0,T],Y)} \leqslant C\|\dot{\ell}\|_{\mathrm{BV}([0,T],Y^*)}$, see [Kre99] for details. □

In numerical approaches to elastoplasticity or other hysteresis problems such higher temporal regularity can be used to improve the convergence rates of the incremental problem. We do this for the linear variational inequality (2.2), namely

$$\big\langle Ay(t) - \ell(t), v - \dot{y}(t)\big\rangle + \Psi(v) - \Psi\big(\dot{y}(t)\big) \geqslant 0 \quad \text{for a.a. } t \in [0,T], \tag{4.16}$$

under the same assumptions as in Section 2. Instead of the fully implicit Euler scheme (IP), which was used above, one considers the more general semiimplicit scheme as follows:

$$(\mathrm{IP})_\vartheta \qquad \begin{array}{l}\text{Find } y_k \in Y \text{ such that } \forall \hat{v} \in Y, \\ \langle Ay_k^\vartheta - \ell(t_k^\vartheta), \hat{v} - (y_k - y_{k-1})\rangle - \Psi(y_k - y_{k-1}) + \Psi(\hat{v}) \geqslant 0,\end{array}$$

where $t_k^\vartheta = (1-\vartheta)t_{k-1} + \vartheta t_k$ and $y_k^\vartheta = (1-\vartheta)y_{k-1} + \vartheta y_k$. For $\vartheta = 1$ this is the old fully implicit scheme and for $\vartheta = 1/2$ it is the midpoint rule (also called the Crank–Nicholson scheme).

Under the proved assumption of Lipschitz continuity (i.e., $\|\dot{y}\|_{\mathrm{L}^\infty} < \infty$), it is shown in [HanR95], that the convergence of the discrete solution to the exact solutions behaves like

$$\|y^\Pi - y\|_{\mathrm{L}^\infty} \leqslant C\big(f(\Pi)\big)^s$$

with $s = 1/2$ if $\vartheta \in [\frac{1}{2}, 1]$, see Theorem 4.3 for the case $\vartheta = 1$. Under the unproved assumption $y \in \mathrm{W}^{2,2}([0,T], Y)$ this convergence was improved to the order $s = 1$ in [AlbeC00], Remark 4.1. For the Crank–Nicholson scheme, i.e., $\vartheta = 1/2$, [HanR95] obtained $s = 1$ if $y \in \mathrm{W}^{3,1}([0,T], Y)$ and this was improved to $s = 2$ in [AlbeC00], Remark 4.3.

Following the method in [AlbeC00] we provide a short proof of the convergence rate $s = 1$ under the assumption $\dot{y} \in \mathrm{BV}([0,T], Y)$, which is the best we can expect for any true hysteretic behavior. We also add the convergence estimate for the derivative as derived there.

THEOREM 4.11. *Let $y \in \mathrm{C}^{\mathrm{Lip}}([0,T], Y)$ be a solution of the variation inequality* (4.16) *with $\dot{y} \in \mathrm{BV}([0,T], Y)$. Then there exists a constant $C > 0$ such that for each partition Π the piecewise linear interpolant $\hat{y}^\Pi$ of the solution of* $(\mathrm{IP})_{\vartheta=1}$ *satisfies*

$$\|y - \hat{y}^\Pi\|_{\mathrm{L}^\infty([0,T],Y)} \leqslant f(\Pi)\left[1 + \frac{f(\Pi)}{2}\right]\mathrm{Var}_A\big(\dot{y}; [0,T]\big),$$

$$\|\dot{y} - \dot{\hat{y}}^\Pi\|_{\mathrm{L}^2([0,T],Y)} \leqslant 2\frac{f(\Pi)}{\sqrt{f_*(\Pi)}}\,\mathrm{Var}_A\big(\dot{y}; [0,T]\big),$$

where $f_*(\Pi) = \min\{t_k - t_{k-1} \mid k = 1, \ldots, N\} \leqslant f(\Pi)$.

PROOF. Like in Section 2 we use the energetic scalar product $\langle v|w\rangle = \langle Av, w\rangle$ and the associated norm $\|v\|_A$.

Define the error function e via $e(t) = y(t) - \hat{y}^\Pi(t)$ and set $I_k = [t_{k-1}, t_k]$, $\tau_k = t_k - t_{k-1}$, $\ell_k = \ell(t_k)$, $e_k = e(t_k)$ and $\dot{e}_k = \dot{y}(t_k) - \frac{1}{\tau_k}(y_k - y_{k-1})$. Since $\hat{y}^\Pi$ is piecewise linear, we have

$$\begin{aligned}\|e_k - e_{k-1} - \tau_k\dot{e}_k\|_A &= \left\|\int_{I_k} \dot{y}(s) - \dot{y}(t_k)\,\mathrm{d}s\right\|_A \leqslant \int_{I_k} \big\|\dot{y}(s) - \dot{y}(t_k)\big\|_A\,\mathrm{d}s \\ &\leqslant \int_{I_k} \mathrm{Var}_A\big(\dot{y}; [s, t_k]\big)\,\mathrm{d}s \leqslant \tau_k\,\mathrm{Var}_A(\dot{y}; I_k). \end{aligned} \tag{4.17}$$

Since y satisfies (4.16) and y_k satisfies $(\mathrm{IP})_{\vartheta=1}$, we obtain, by using the test functions $v = \frac{1}{\tau_k}(y_k - y_{k-1})$ and $\hat{v} = \tau_k\dot{y}(t_k)$, respectively,

$$0 \leqslant \left\langle Ay(t_k) - \ell_k, \frac{1}{\tau_k}(y_k - y_{k-1}) - \dot{y}(t_k)\right\rangle + \Psi\left(\frac{1}{\tau_k}(y_k - y_{k-1})\right) - \Psi\big(\dot{y}(t_k)\big),$$

$$0 \leqslant \big\langle Ay_k - \ell_k, \tau_k\dot{y}(t_k) - (y_k - y_{k-1})\big\rangle + \Psi\big(\tau_k\dot{y}(t_k)\big) - \Psi(y_k - y_{k-1}).$$

Dividing the second equation by τ_k and adding it to the first one, we find

$$0 \leqslant \left\langle A\big(y(t_k) - y_k\big), \frac{1}{\tau_k}(y_k - y_{k-1}) - \dot{y}(t_k) \right\rangle = \langle e_k | -\dot{e}_k \rangle = -\langle e_k | \dot{e}_k \rangle.$$

Now the discrete error can be estimated via

$$\begin{aligned} \|e_k\|_A^2 &= \|e_{k-1}\|_A^2 + 2\langle e_k - e_{k-1} | e_k \rangle - \|e_k - e_{k-1}\|_A^2 \\ &\leqslant \|e_{k-1}\|_A^2 + 2\langle e_k - e_{k-1} - \tau_k \dot{e}_k | e_k \rangle - \|e_k - e_{k-1}\|_A^2 \\ &\leqslant \|e_{k-1}\|_A^2 - \|e_k - e_{k-1}\|_A^2 + \tau_k \operatorname{Var}_A(\dot{y}; I_k) \|e_k\|_A. \end{aligned} \tag{4.18}$$

Let m be such that $\max\{\|e_k\|_A \mid k = 1, \dots, N\}$ is attained at $k = m$. Then, adding the above estimate from $k = 1$ to m, using $e_0 = 0$ and neglecting the terms $\|e_k - e_{k-1}\|_A^2$ gives

$$\|e_m\|_A^2 \leqslant \sum_{k=1}^{m} \tau_k \operatorname{Var}_A(\dot{y}; I_k) \|e_m\|_A \leqslant f(\Pi) \operatorname{Var}_A\big(\dot{y}; [0, T]\big) \|e_m\|_A,$$

which implies $\|e_k\|_A \leqslant f(\Pi) \operatorname{Var}_A(\dot{y}; [0, T])$ for all k. Moreover, again using the fact that $\hat{y}^\Pi$ is piecewise linear gives for $t \in (t_{k-1}, t_k)$,

$$e(t) = \frac{t - t_{k-1}}{\tau_k} e_k + \frac{t_k - t}{\tau_k} e_{k-1} + \int_{I_k} \alpha_k(t, s)\big(\dot{y}(s) - \dot{y}(t_k)\big)\, \mathrm{d}s$$
$$\text{with } \alpha_k(t, s) = \begin{cases} \frac{t_k - t}{\tau_k} & \text{if } s < t, \\ -\frac{t - t_{k-1}}{\tau_k} & \text{if } s > t, \end{cases}$$

since $\int_{I_k} \alpha_k(t, s)\, \mathrm{d}s = 0$. By $\int_{I_k} |\alpha_k(t, s)|\, \mathrm{d}s \leqslant \tau_k/2$ this implies

$$\left\| e(t) - \left(\frac{t - t_{k-1}}{\tau_k} e_k + \frac{t_k - t}{\tau_k} e_{k-1} \right) \right\|_A \leqslant \frac{\tau_k}{2} \operatorname{Var}_A(\dot{y}; I_k)$$

and the first estimate is established.

Summing (4.18) once again we find, with $e_0 = 0$ and $f_*(\Pi) = \min \tau_k$,

$$\begin{aligned} f_*(\Pi) \sum_1^N \frac{1}{\tau_k} \|e_k - e_{k-1}\|_A^2 &\leqslant \sum_1^N \|e_k - e_{k-1}\|_A^2 \\ &\leqslant \sum_1^N \tau_k \operatorname{Var}_A(\dot{y}; I_k) \|e_k\|_A \\ &\leqslant \big(f(\Pi) \operatorname{Var}_A\big(\dot{y}; [0, T]\big)\big)^2. \end{aligned}$$

We define Y_Π to be the piecewise linear interpolant of the exact solution y. Then we have $\dot{Y}_\Pi - \dot{\hat{y}}^\Pi = \frac{1}{\tau_k}(e_k - e_{k-1})$ on (t_{k-1}, t_k) and the left-hand side in the above estimate is $\|\dot{Y}_\Pi - \dot{\hat{y}}^\Pi\|^2_{\mathrm{L}^2([0,T],Y)}$. Let $v = y - Y_\Pi$. Then $v(t_k) = 0$ for all k and $\mathrm{Var}_A(\dot{v}; I_k) \leqslant \mathrm{Var}_A(\dot{y}; I_k)$. Hence,

$$\int_{I_k} \|\dot{v}(s)\|_A^2 \,\mathrm{d}s = \big[\langle v(s) | \dot{v}(s)\rangle\big]_{t_{k-1}}^{t_k} - \int_{I_k} \langle v|\mathrm{d}\dot{v}\rangle \leqslant \|v\|_{\mathrm{L}^\infty(I_k,Y)} \mathrm{Var}_A(\dot{v}; I_k).$$

As above we have $\|v\|_{\mathrm{L}^\infty(I_k,Y)} \leqslant \frac{\tau_k}{2} \mathrm{Var}_A(\dot{v}; I_k)$ and conclude

$$\begin{aligned}
\|\dot{y} - \dot{Y}_\Pi\|^2_{\mathrm{L}^2([0,T],Y)} = \|\dot{v}\|^2_{\mathrm{L}^2([0,T],Y)} &\leqslant \frac{f(\Pi)}{2} \mathrm{Var}\big(\dot{v}; [0,T]\big)^2 \\
&\leqslant f(\Pi)\, \mathrm{Var}\big(\dot{y}; [0,T]\big)^2.
\end{aligned}$$

Together with the triangle inequality and $f_*(\Pi) \leqslant f(\Pi)$ we obtain the desired result. □

The strong convergence of the derivative like $\sqrt{f(\Pi)}$ can also be shown using interpolation between the linear convergence in L^∞ and a uniform bound on $\mathrm{Var}_A(\hat{y}^\Pi; [0,T])$.

5. Nonconvex and nonsmooth problems

We recall the three major conditions (A1)–(A3) from Sections 3.1 and 3.2:

$$\begin{aligned}
&\text{(i) } \forall z_1, z_2 \in \mathcal{Z}\colon\ \mathcal{D}(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2, \\
&\text{(ii) } \forall z_1, z_2, z_3 \in \mathcal{Z}\colon\ \mathcal{D}(z_1, z_3) \leqslant \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3).
\end{aligned} \tag{A1}$$

There exist $c_E^{(1)}, c_E^{(0)} > 0$ such that for all $y_* \in \mathcal{Y}$:

$$\mathcal{E}(t, y_*) < \infty \quad \Longrightarrow \quad \begin{cases} \partial_t \mathcal{E}(\cdot, y_*) : [0,T] \to \mathbb{R} \text{ is measurable and} \\ \big|\partial_t \mathcal{E}(t, y_*)\big| \leqslant c_E^{(1)}\big(\mathcal{E}(t, y_*) + c_E^{(0)}\big). \end{cases} \tag{A2}$$

$$\begin{aligned}
&\forall t \in [0,T]\colon\ \mathcal{E}(t, \cdot) : \mathcal{Y} \to \mathbb{R}_\infty \text{ has compact sublevels,} \\
&\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is lower semicontinuous.}
\end{aligned} \tag{A3}$$

The existence theory developed below will build on the incremental problem (IP) and the a priori estimates derived in Section 3.

5.1. *Existence results*

The general strategy for constructing solutions to (S) and (E) is to choose a sequence of partitions Π^m with fineness f_m tending to 0, to extract a convergent subsequence of $(Y^l)_l$

of $(Y^{\Pi^m})_{m\in\mathbb{N}}$ and then to show that the limit $Y:[0,T]\to\mathcal{Y}$ solves (S) and (E). A major problem arises from the fact that the temporal behavior of the elastic component φ of $y=(\varphi,z)$ cannot be controlled, which is in contrast to the inelastic component z whose variation is controlled via the dissipation.

For the dissipative part it is possible to extract a suitable limit function if the dissipation is strong enough. We need the following assumption for any sequence $(z_k)_k$ and any z in $\mathcal{Z}$:

$$\min\big\{\mathcal{D}(z_k,z),\mathcal{D}(z,z_k)\big\}\to 0 \quad \text{for } k\to\infty \quad\Longrightarrow\quad z_k\overset{\mathcal{Z}}{\to} z \quad \text{for } k\to\infty. \tag{A4}$$

The following version of Helly's selection principle is a special case of [MaiM05], Theorem 3.2. The classical result of Helly relates to real-valued monotone functions. Versions for functions with values in Hilbert and Banach spaces can be found in [Mon93,BarP86].

THEOREM 5.1. *Let $\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ satisfy* (A1) *and* (A4). *Moreover, let $\mathcal{K}$ be a compact subset of $\mathcal{Z}$. Then, for every sequence $(Z^l)_{l\in\mathbb{N}}$ with $Z^l:[0,T]\to\mathcal{K}$ and bounded dissipation, i.e.,*

$$\sup_{l\in\mathbb{N}}\mathrm{Diss}_{\mathcal{D}}\big(Z^l;[0,T]\big)\leqslant C<\infty,$$

there exist a subsequence $(Z^{l_n})_{n\in\mathbb{N}}$, a function $z^\infty:[0,T]\to\mathcal{K}$ and a function $\delta^\infty:[0,T]\to[0,C]$ such that the following holds:

(a) $\delta_{l_n}(t):=\mathrm{Diss}_{\mathcal{D}}(Z^{l_n},[0,t])\to\delta^\infty(t)$ *for all* $t\in[0,T]$,

(b) $Z_{l_n}(t)\overset{\mathcal{Z}}{\to} z^\infty(t)$ *for all* $t\in[0,T]$,

(c) $\mathrm{Diss}_{\mathcal{D}}(z^\infty,[t_0,t_1])\leqslant\lim_{t\searrow t_1}\delta_\infty(t)-\lim_{s\nearrow t_0}\delta_\infty(s)$ *for all* $0\leqslant t_0<t_1\leqslant T$.

Like in the theory of BV functions in Banach spaces, all functions $z\in\mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$ are continuous except at the discontinuity points of $t\mapsto\mathrm{Diss}_{\mathcal{D}}(z;[0,t])$. Moreover, for all $t\in[0,T]$ the right-hand and left-hand limits $z_+(t)$ and $z_-(t)$ exist, see Section 3 in [MaiM05].

For the main existence result we need two more conditions. One condition relates to the power of the external forces $\partial_t\mathcal{E}(t,y)$ which we assume to satisfy not only (A2) but also a uniform continuity property:

$$\begin{aligned}&\text{Condition (A2) holds and}\\ &\forall E^*>0,\forall\varepsilon>0,\exists\delta>0:\\ &\mathcal{E}(t,y)\leqslant E^* \text{ and } |t-s|\leqslant\delta\Rightarrow|\partial_t\mathcal{E}(t,y)-\partial_t\mathcal{E}(s,y)|<\varepsilon.\end{aligned}\tag{A5}$$

The above conditions for the topology $\mathcal{T}$ on $\mathcal{Y}$ and the associated lower semicontinuities of $\mathcal{E}$ and $\mathcal{D}$ appear very natural and are standard from the point of view of the calculus of variations. Condition (A5) concerns only the power of the external forces, which is determined by the prescribed loading data, and thus is uncritical.

THEOREM 5.2. *Assume that $\mathcal{E}$ and $\mathcal{D}$ satisfy assumptions* (A1) *and* (A3)–(A5). *Moreover, let one of the following two conditions* (A6) *or* (A7) *be satisfied*:

$$\mathcal{D}: \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is continuous.} \tag{A6}$$

$$\begin{aligned} &\text{The set } \mathcal{S}_{[0,T]} \text{ of stable states is closed in } [0,T] \times \mathcal{Y} \text{ and} \\ &\forall E_0 > 0\colon\ \partial_t \mathcal{E} : \{(t,y) \mid \mathcal{E}(t,y) \leqslant E_0\} \to \mathbb{R} \text{ is continuous.} \end{aligned} \tag{A7}$$

Then, for each $y_0 \in \mathcal{S}(0)$ there exists a solution $y = (\varphi, z) : [0, T] \to \mathcal{Y}$ of (S) *and* (E).

Moreover, if $\Pi^l \in \mathrm{Part}_{f_l}^{N_l}([0,T])$ is a sequence of partitions with fineness f_l tending to 0 *and Y^{Π_l} is the interpolant of any solution of the associated incremental problem* (IP), *then there exist a subsequence $y_k = Y^{\Pi_{l_k}}$ and a solution $y = (\varphi, z) : [0, T] \to \mathcal{Y}$ of* (S) *and* (E) *such that the following holds*:

(i) $\forall t \in [0,T]$: $z_k(t) \overset{\mathcal{Z}}{\to} z(t)$;
(ii) $\forall t \in [0,T]$: $\mathrm{Diss}_{\mathcal{D}}(z_k; [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(z; [0,t])$;
(iii) $\forall t \in [0,T]$: $\mathcal{E}(t, y_k(t)) \to \mathcal{E}(t, y(t))$;
(iv) $\partial_t \mathcal{E}(\cdot, y_k(\cdot)) \to \partial_t \mathcal{E}(\cdot, y(\cdot))$ *in* $\mathrm{L}^1((0,T))$.

REMARK 5.3. It is easy to see that conditions (A4) and (A6) can be weakened by assuming that the conditions are valid on each sublevel of $\mathcal{E}$.

REMARK 5.4. In Step 3 of the proof of Theorem 5.2, we will show that condition (A6) implies that $\mathcal{S}_{[0,T]}$ is closed. This central condition will be discussed in more detail in Section 5.2.

REMARK 5.5. Theorem 5.2 does not claim the convergence of the elastic component $\varphi^n(t)$ to $\varphi(t)$. In fact, since we do not have any control on the temporal oscillations of φ, we have no selection criterion. We will construct the limit $\varphi(t)$ by choosing a suitable t-dependent subsequence of $\varphi^k(t)$. As a consequence, we do not obtain information on the continuity or on the measurability of the limit $\varphi : [0,T] \to \mathcal{F}$.

However, sometimes the φ-component is uniquely determined via z in the following sense (cf. [MieRou03], equation (3.18))

$$y_1 = (\varphi_1, z_1),\, y_2 = (\varphi_2, z_2) \in \mathcal{S}(t) \quad \text{and} \quad z_1 = z_2 \quad \Longrightarrow \quad \varphi_1 = \varphi_2. \tag{5.1}$$

Then, the φ-component can be controlled more precisely. In [MaiM05] the slightly stronger assumption

$$y_k = (\varphi_k, z_k) \in \mathcal{S}(t) \quad \text{and} \quad z_k \overset{\mathcal{Z}}{\to} z \quad \Longrightarrow \quad \varphi_k \overset{\mathcal{F}}{\to} \varphi \tag{5.2}$$

is used to conclude the stronger result $y^n(t) \overset{\mathcal{Y}}{\to} y(t)$ for all $t \in [0,T]$ in Theorem 5.2(i) as well as continuity of $t \mapsto y(t) \in \mathcal{Y}$ for all t except at the (at most countable) jump points of $\mathrm{Diss}_{\mathcal{D}}(z; [0,T])$.

The proof consists of several steps and uses the two following auxiliary results. The first result concerns the continuity of the power of the external forces as a function on $\mathcal{Y}$, i.e., of $y \mapsto \partial_t \mathcal{E}(t, y)$. Very often it is assumed that the loading acts linearly on φ. This gives the term $\partial_t \mathcal{E}(t, y) = -\langle \dot{\ell}(t), \varphi \rangle$ which is automatically weakly continuous. However, in the case of time-dependent Dirichlet conditions this is more difficult, since we need to control the stresses due to the boundary condition. This problem was first solved in [DalFT05] by showing that the stresses in fact converge weakly if we know that the functions φ^n as well as the energy converge. The following result is an abstract and much simpler version of this fact.

PROPOSITION 5.6. *If $\mathcal{E}$ satisfies* (A3) *and* (A5), *then for all $t \in (0, T)$ the following implication holds*:

$$\left.\begin{array}{l} y_m \overset{\mathcal{Y}}{\rightharpoonup} y \text{ and} \\ \mathcal{E}(t, y_m) \to \mathcal{E}(t, y) < \infty \end{array}\right\} \quad \Longrightarrow \quad \partial_t \mathcal{E}(t, y_m) \to \partial_t \mathcal{E}(t, y). \tag{5.3}$$

PROOF. Let $E_0, h_0 > 0$ be such that $t \pm h_0 \in [0, T]$ and $\mathcal{E}(t, y_m), \mathcal{E}(t, y) \leqslant E_0$ for sufficiently large m. Then, condition (A5) implies the existence of a modulus of continuity $\omega_0 : [0, h_0] \to [0, \infty)$ (i.e., ω_0 is monotone increasing and $\omega_0(h) \to 0$ for $h \searrow 0$) such that for $h \in (0, h_0)$ we have

$$\left| \frac{1}{h}\big(\mathcal{E}(t \pm h, y_m) - \mathcal{E}(t, y_m)\big) \mp \partial_t \mathcal{E}(t, y_m) \right| \leqslant \omega_0(h), \tag{5.4}$$

since the difference quotient can be replaced by a derivative at an intermediate value. The same estimate also holds for y. By $h > 0$, the lower semicontinuity of $\mathcal{E}(t, \cdot)$ from (A3) and the assumed convergence of the energy, we find

$$\liminf_{m\to\infty} \frac{1}{h}\big(\mathcal{E}(t \pm h, y_m) - \mathcal{E}(t, y_m)\big) \geqslant \frac{1}{h}\big(\mathcal{E}(t \pm h, y) - \mathcal{E}(t, y)\big).$$

Combining the case "+" with (5.4) we find

$$\begin{aligned} \liminf_{m\to\infty} \partial_t \mathcal{E}(t, y_m) &\geqslant \liminf_{m\to\infty} \frac{1}{h}\big(\mathcal{E}(t + h, y_m) - \mathcal{E}(t, y_m)\big) - \omega_0(h) \\ &\geqslant \frac{1}{h}\big(\mathcal{E}(t + h, y) - \mathcal{E}(t, y)\big) - \omega_0(h) \geqslant \partial_t \mathcal{E}(t, y) - 2\omega_0(h). \end{aligned}$$

Similarly, the case "−" gives $\limsup_{m\to\infty} \partial_t \mathcal{E}(t, y_m) \leqslant \partial_t \mathcal{E}(t, y) + 2\omega_0(h)$. Since h can be made arbitrarily small, the result is proved. □

The second result shows that the stability property (S) already implies a lower energy estimate, as can be seen in the proof of Theorem 3.2(ii). Thus, it will be sufficient to keep track of the upper energy estimate only, see $(\mathrm{E})_{\mathrm{discr}}$ in Corollary 3.3. This was observed first in [MieTL02].

PROPOSITION 5.7. *Assume that* (A1) *and* (A5) *hold. Let* $y = (\varphi, z) : [0, T] \to \mathcal{Y}$ *be given such that* $\mathrm{Diss}_{\mathcal{D}}(z; [0, T]) < \infty$, $t \to \mathcal{E}(t, y(t))$ *is bounded,* $\partial_t \mathcal{E}(\cdot, y(\cdot)) \in \mathrm{L}^{\infty}((0, T))$, *and* $y(t) \in \mathcal{S}(t)$ *for all* $t \in [0, T]$. *Then, for all* $0 \leqslant r < s \leqslant T$ *we have the lower energy inequality*

$$\mathcal{E}\big(s, y(s)\big) + \mathrm{Diss}_{\mathcal{D}}\big(y; [r, s]\big) \geqslant \mathcal{E}\big(r, y(r)\big) + \int_r^s \partial_t \mathcal{E}\big(t, y(t)\big)\,\mathrm{d}t. \tag{5.5}$$

PROOF. Since $\theta : t \mapsto \partial_t \mathcal{E}(t, y(t))$ is integrable there exists a sequence of partitions $\Pi^m \in \mathrm{Part}_{\delta_m}^{N_m}([r, s])$ with $\delta_m \to 0$ such that the Lebesgue integral can be approximated by the corresponding Riemann sums, namely $\int_r^s \theta(t)\,\mathrm{d}t = \lim_{m\to\infty} \sum_{j=1}^{N_m} \theta(t_j^m)(t_j^m - t_{j-1}^m)$. We refer to [Mai05] or [FM05] for this result, or to [DalFT05] for a more general version.

In each subinterval $[t_{j-1}^m, t_j^m]$ we use the stability (S), see the proof of Theorem 3.2(ii), and (A5) to obtain

$$\begin{aligned}
&\mathcal{E}\big(t_j^m, y\big(t_j^m\big)\big) + \mathcal{D}\big(y\big(t_{j-1}^m\big), y\big(t_j^m\big)\big)\\
&\quad\overset{\text{(S)}}{\geqslant} \mathcal{E}\big(t_{j-1}^m, y\big(t_{j-1}^m\big)\big) + \int_{t_{j-1}^m}^{t_j^m} \partial_s \mathcal{E}\big(s, y\big(t_j^m\big)\big)\,\mathrm{d}s\\
&\quad\overset{\text{(A5)}}{\geqslant} \big[\theta\big(t_j^m\big) - \varepsilon\big]\big(t_j^m - t_{j-1}^m\big),
\end{aligned}$$

where $\varepsilon > 0$ can be made as small as we like by choosing m sufficiently large and hence δ_m sufficiently small. Adding over $j = 1, \ldots, N_m$ and taking the limit $m \to \infty$ gives the desired result, since $\varepsilon > 0$ is arbitrary. □

PROOF OF THEOREM 5.2. For a simplified version of this proof we refer to the proof of Theorem 2.1, where the same steps are followed but for the much simpler case of a quadratic energy on a Banach space Y.

We first prove the result under the assumption that (A6) is satisfied. The differences in the proof for the case when (A7) holds are given afterwards.

Step 1. *A priori estimates.* We choose an arbitrary sequence of partitions Π^m whose fineness f_m tends to 0. According to Section 3.2 the time-incremental minimization problems (IP) are solvable and the piecewise constant interpolants $Y^m : [0, T] \to \mathcal{Y}$ satisfy the a priori estimates

$$\mathrm{Diss}_{\mathcal{D}}\big(Z^m; [0, T]\big) \leqslant C \quad \text{and} \quad \forall t \in [0, T], \quad \mathcal{E}\big(t, Y^m(t)\big) \leqslant C,$$

where C is given explicitly in Corollary 3.3(3).

Step 2. *Selection of subsequences.* Our version of Helly's selection principle in Theorem 5.1 allows us to select a subsequence of $(Z^m)_{m\in\mathbb{N}}$ which converges pointwise and which makes the dissipation converge as well. Moreover, the functions $\Theta^m : t \mapsto$

$\partial_t\mathcal{E}(t, Y^m(t))$ form a bounded sequence in $\mathrm{L}^\infty((0,T))$. Thus, by choosing a further subsequence $(Y^{m_k})_{k\in\mathbb{N}}$ we may assume the following convergence properties for $k\to\infty$, where we write $y_k = (\varphi_k, z_k)$ as shorthand for Y^{m_k} and θ_k for Θ^{m_k},

$$\forall t\in[0,T],\quad \delta_k(t) := \mathrm{Diss}_{\mathcal{D}}\big(z_k;[0,t]\big) \to \delta(t) \quad \text{and} \quad z_k(t) \overset{\mathcal{Z}}{\to} z(t);$$
$$\theta_k \overset{*}{\rightharpoonup} \theta \quad \text{in } \mathrm{L}^\infty\big((0,T)\big).$$

Note that the limit functions δ, z and θ exist. We further define the function $\theta_{\sup}: t\mapsto \limsup_{k\to\infty}\theta_k(t)$ such that $\theta_{\sup}\in\mathrm{L}^\infty((0,T))$ and $\theta\leqslant\theta_{\sup}$ by Fatou's lemma.

To define $\varphi(t)$, fix $t\in[0,T]$ and we choose a t-dependent subsequence $k=K_n^t$ such that

$$\theta_{K_n^t}(t)\to\theta_{\sup}(t) \quad \text{and} \quad \varphi_{K_n^t}(t)\overset{\mathcal{F}}{\to}\varphi(t).$$

Here we use the a priori bound $\mathcal{E}(t, y_k(t))\leqslant C$ and the compactness of the sublevels assumed in (A3). Hence, $y(t) = (\varphi(t), z(t))$ is defined.

Step 3. *Stability of the limit function.* We first show that (A6) implies the closedness of $\mathcal{S}_{[0,T]}$. For a sequence $(t_l, y_l)_{l\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with limit (t,y) consider any test state $\hat{y}$. Since $\mathcal{D}$ is continuous and $\mathcal{E}$ lower semicontinuous, we have

$$\mathcal{E}(t,y)\leqslant\liminf_{l\to\infty}\mathcal{E}(t_l,y_l)\overset{\text{(S)}}{\leqslant}\liminf_{l\to\infty}\mathcal{E}(t_l,\hat{y})+\mathcal{D}(y_l,\hat{y}) = \mathcal{E}(t,\hat{y})+\mathcal{D}(y,\hat{y}),$$

which is the desired stability of y. Hence, $\mathcal{S}_{[0,T]}$ is closed.

Using the closedness of $\mathcal{S}_{[0,T]}$ it is easy to show that the limit function $y: t\mapsto(\varphi(t), z(t))$ is stable. For $t\in[0,T]$ fixed define $\tau_k^t = \min\{\tau\in\Pi^{m_k}\mid \tau\geqslant t\}$, then $y_k(t) = y_k(\tau_k^t)\in\mathcal{S}(\tau_k^t)$ by the definition of the interpolant $y_k = Y^{m_n}$. Thus

$$\big(\tau^t_{K_n^t}, y_{K_n^t}(t)\big)\in\mathcal{S}_{[0,T]},\quad \tau_k^t\to t,\quad \text{and} \quad y_{K_n^t}(t)\overset{\mathcal{Y}}{\to} y(t).$$

Hence, the closedness gives $(t, y(t))\in\mathcal{S}_{[0,T]}$, i.e., $y(t)\in\mathcal{S}(t)$.

Step 4. *Upper energy estimate.* We define the functions

$$e_k(t) := \mathcal{E}\big(t, y_k(t)\big),$$
$$\delta_k(t) := \mathrm{Diss}_{\mathcal{D}}\big(z_k;[0,t]\big),$$
$$w_k(t) := \int_0^t \partial_t\mathcal{E}\big(s, y_k(s)\big)\,\mathrm{d}s = \int_0^t \theta_k(s)\,\mathrm{d}s.$$

Corollary 3.3 and the boundedness of $\partial_t\mathcal{E}$ by a constant C_1 (use (A2) and Step 1) give

$$e_k(t)+\delta_k(t)\leqslant w_k(t)+C_1 f_{m_k}. \tag{5.6}$$

Since $\mathcal{E}$ is lower semicontinuous and δ_k and θ_k converge according to Step 2, the limit $k = K_n^t \to \infty$ for $n \to \infty$ leads to

$$\begin{aligned}\mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0,t]\big) &\leqslant e(t) + \delta(t) \\ &\leqslant e(0) + \int_0^t \theta(s)\,\mathrm{d}s \leqslant e(0) + \int_0^t \theta_{\sup}(s)\,\mathrm{d}s,\end{aligned}$$

where $\mathcal{E}(t, y(t)) \leqslant e(t) = \liminf_{k\to\infty} e_k(t)$. In fact, we have $e(t) = \lim_{n\to\infty} \mathcal{E}(t, y_{K_n^t}(t))$ since

$$\mathcal{E}\big(t, y(t)\big) \overset{\text{(A6)}}{=} \lim_{n\to\infty} \mathcal{E}\big(t, y(t)\big) + \mathcal{D}\big(z_{K_n^t}(t), z(t)\big) \overset{\text{(S)}}{\geqslant} \limsup_{n\to\infty} \mathcal{E}\big(t, y_{K_n^t}(t)\big) \geqslant e(t).$$

Thus, together with $y_{K_n^t}(t) \overset{\mathcal{Y}}{\to} y(t)$ the assumptions of Proposition 5.6 are satisfied and we conclude

$$\theta_{\sup}(t) = \lim_{n\to\infty} \theta_{N_n^t}(t) = \lim_{n\to\infty} \partial_t \mathcal{E}\big(t, y_{K_n^t}(t)\big) = \partial_t \mathcal{E}\big(t, y(t)\big).$$

Together with the above, this is the desired upper energy estimate.

Step 5. *Lower energy estimate.* As we have established that $\theta_{\sup} = \partial_t \mathcal{E}(\cdot, y(\cdot))$ lies in $\mathrm{L}^\infty((0,T))$ we are able to apply Proposition 5.7 and obtain the lower energy estimate

$$\mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0,t]\big) \geqslant \mathcal{E}\big(0, y(0)\big) + \int_0^t \partial_t \mathcal{E}\big(s, y(s)\big)\,\mathrm{d}s.$$

Step 6. *Improved convergence.* Steps 1–5 show that the constructed limit $y : [0,T] \to \mathcal{Y}$ is a solution. In the last step we show that the convergences (i)–(iv) stated at the end of the theorem hold. Part (i) is already shown. The lower and upper energy estimate imply

$$\begin{aligned}e(0) + \int_0^t \theta_{\sup}\,\mathrm{d}s \leqslant e(t) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0,t]\big) &\leqslant e(t) + \delta(t) \\ &\leqslant e(0) + \int_0^t \theta\,\mathrm{d}s \leqslant e(0) + \int_0^t \theta_{\sup}\,\mathrm{d}s.\end{aligned}$$

Hence, all inequalities are in fact equalities and we conclude $\mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \delta(t)$ and $\theta = \theta_{\sup}$ a.e. in $[0,T]$. The first identity is (ii) and the second identity implies (iv), cf. [FM05], Proposition A2. Finally note that the energy $\mathcal{E}(t, y_k(t))$ convergences not only on the t-dependent subsequence $k = N_n^t$, but along the whole sequence. This follows since we have shown that $e_k(t) + \delta_k(t)$ always has a limit and δ_k is convergent.

This completes the proof of Theorem 5.2 in the case that (A6) holds. Now assume that (A7) holds instead.

Steps 1–3 work identical. In Step 4 the identity $\theta_{\sup}(t) = \partial_t \mathcal{E}(t, y(t))$ follows directly from the continuity of $\partial_t \mathcal{E}$ assumed in (A7). Thus, the upper and lower energy estimates follow as above and Steps 4 and 5 are done.

In Step 6 the convergence of the energy is not yet established. However, with (5.6) and $\delta_k(t) \to \delta(t) = \mathrm{Diss}_{\mathcal{D}}(z; [0,t])$ we again find, by the lower semicontinuity of $\mathcal{E}$,

$$\begin{aligned}\mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(z; [0,t]\big) &\leqslant \liminf_{k\to\infty} e_k(t) + \delta(t) \\ &\leqslant \limsup_{k\to\infty} e_k(t) + \delta(t) \\ &\leqslant \mathcal{E}\big(0, y(0)\big) + \int_0^t \theta_{\sup}(s)\,\mathrm{d}s.\end{aligned}$$

Together with the lower energy estimate this proves $e_k(t) \to \mathcal{E}(t, y(t))$ as desired. The remaining parts of Step 6 are the same.

Thus, Theorem 5.2 is proved. □

We formulate now a special version of Theorem 5.2, which is based on Banach spaces and which can be easily applied to several models in continuum mechanics.

THEOREM 5.8. *Let Y_1 and Y be Banach spaces. Suppose that Y_1 is compactly embedded in Y and that $\{y \in Y_1 \mid \|y\|_{Y_1} \leqslant 1\}$ is closed in Y. The dissipation distance $\mathcal{D}: Y \times Y \to \mathbb{R}$ is the Y norm, i.e., $\mathcal{D}(y_1, y_2) = \|y_1 - y_2\|_Y$. Furthermore, the functional $\mathcal{E}: [0,T] \times Y \to [\mathcal{E}_{\min}, \infty]$ has the following properties.*

(a) *$\mathcal{E}$ is lower semicontinuous on $[0,T] \times Y$ (with respect to the norm topology of Y).*

(b) *For some real numbers $c_1 > 0$, C_2 and $\alpha > 0$ we have*

$$\mathcal{E}(t, y) \geqslant c_1 \|y\|_{Y_1}^{\alpha} - C_2 \quad (\textit{i.e., } \mathcal{E}(t,y) = \infty \textit{ for } y \in Y \setminus Y_1). \tag{5.7}$$

(c) *Condition (A5) is satisfied.*

Then, for each $y_0 \in \mathcal{S}(0)$ there exists at least one solution $y \in \mathrm{BV}_{\mathcal{D}}([0,T], Y) \cap B([0,T], Y_1)$ of (S) and (E) with $y(0) = y_0$ and all the conclusions of Theorem 5.2 also hold.

Here $B([0,T], Y_1)$ denotes the set of mappings y such that $t \mapsto \|y(t)\|_{Y_1}$ is bounded.

The result is an easy consequence if we choose $\mathcal{Y} = Y$ equipped with its norm topology. Obviously, $\mathcal{D}$ is continuous and satisfies (A2) and (A4). Moreover, the lower semicontinuity of $\mathcal{E}$ and its coerciveness in the compactly embedded space Y_1 show that $\mathcal{E}$ has compact sublevels. Thus, (A1)–(A6) hold and Theorem 5.2 is applicable.

A possible application of this result is the partial differential inclusion

$$\begin{aligned}0 \in{}& \kappa(x)\,\mathrm{Sign}\big(\dot y(t,x)\big) \\ &- \mathrm{div}\big[a(x) \mathrm{D}_x y(t,x)\big] \\ &+ \mathrm{D}_y F\big(x, y(t,x)\big) - \ell(t,x) \quad \text{in } \Omega, \\ z(t,x) ={}& 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega.\end{aligned}$$

To this end, take $Y = \mathrm{L}^1(\Omega)$, $Y_1 = \mathrm{H}^1_0(\Omega)$ and define $\mathcal{D}$ and $\mathcal{E}$ via

$$\mathcal{D}(y_0, y_1) = \int_\Omega \kappa(x)\big|y_1(x) - y_0(x)\big|\,\mathrm{d}x$$

and

$$\mathcal{E}(t, y) = \int_\Omega \frac{a(x)}{2}\big|\mathrm{D}_x y(x)\big|^2 + F\big(x, y(x)\big) - \ell(t, x)y(x)\,\mathrm{d}x.$$

If we assume that $F : \overline{\Omega} \times \mathbb{R} \to [0, \infty]$ is continuous, then (a) and (b) hold with $\alpha = 2$. Moreover, with $\ell \in \mathrm{C}^{\mathrm{Lip}}([0, T], \mathrm{H}^{-1}(\Omega))$ we obtain $|\partial_t \mathcal{E}(t, y)| = |\langle \partial_t \ell(t), y\rangle| \leqslant C\|y\|_{\mathrm{H}^1} \leqslant c_E^{(1)}(\mathcal{E}(t, y) + c_E^{(0)})$ for suitable $c_E^{(1)}, c_E^{(0)} > 0$.

5.2. *Closedness of the stable set*

The major assumptions of our existence result in Theorem 5.2 are the compactness of the sublevels of $\mathcal{E}$ and the closedness of $\mathcal{S}_{[0,T]}$. Before deriving abstract results in this direction we give two simple nontrivial applications of the theorem and thus highlight that the choice of the topology $\mathcal{T}$ is crucial. For both examples, let $Y = \mathrm{L}^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ be open and bounded, and choose the dissipation distance $\mathcal{D}(y_0, y_1) = \|y_1 - y_0\|_{\mathcal{Y}} = \int_\Omega |y_1(x) - y_0(x)|\,\mathrm{d}x$.

For the first example, consider

$$\mathcal{E}_1(t, y) = \int_\Omega a(x)\big|y(x)\big|^\alpha - g(t, x)y(x)\,\mathrm{d}x,$$

where $a(x) \geqslant a_0 > 0$, $\alpha > 1$ and $g \in \mathrm{C}^1([0, T], \mathrm{L}^\infty(\Omega))$. Since $\mathcal{E}_1(t, \cdot)$ is convex and lower semicontinuous, the sublevels of $\mathcal{E}$ are closed, convex set which are contained an L^α-ball. Hence, taking $\mathcal{T}$ to be the weak topology on $Y = \mathrm{L}^1(\Omega)$, the compactness condition (A4) holds. Note that using the norm topology of $\mathrm{L}^1(\Omega)$ would not supply the desired compactness. The stable sets for $\mathcal{E}_1$ are given by

$$\begin{aligned}&\mathcal{S}_1(t)\\ &\quad = \left\{ y \in \mathrm{L}^1(\Omega) \;\middle|\; \big|y(x)\big|^{\alpha-2}y(x) \in \left[\frac{g(t, x) - 1}{a(x)\alpha}, \frac{g(t, x) + 1}{a(x)\alpha}\right] \text{for a.a. } x \in \Omega \right\},\end{aligned}$$

which are closed with respect to $\mathcal{T}$ since they are convex and closed in the norm topology. Hence, with $\mathcal{T}$ as weak topology in $\mathcal{Y} = \mathrm{L}^1(\Omega)$ all conditions of Theorem 5.2 can be satisfied.

For the second example, consider the nonconvex energy functional

$$\mathcal{E}_2(t, y) = \int_\Omega \frac{1}{2}\big|\mathrm{D}y(x)\big|^2 + f\big(t, x, y(x)\big)\,\mathrm{d}x \quad \text{for } y \in \mathrm{H}^1(\Omega) \quad \text{and} \quad +\infty \text{ else},$$

where $f:[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ and $\partial_t f$ are continuous and bounded. Because of the gradient term the sublevels of $\mathcal{E}$ are already compact in the norm topology of $Y=\mathrm{L}^1(\Omega)$, since they are closed and contained in a Y_1-ball, where $Y_1=H^1(\Omega)$ is compactly embedded in Y. With these properties, it can be shown that all conditions of Theorem 5.8 are satisfied.

The proof of the first abstract result is contained in Step 3 of the proof of Theorem 5.2. We repeat the result here for convenience.

PROPOSITION 5.9. *Let* (A2) *hold. Assume that $\mathcal{E}$ is lower semicontinuous on $[0,T]\times\mathcal{Y}$ and that $\mathcal{D}$ is continuous on $\mathcal{Z}\times\mathcal{Z}$. Then, $\mathcal{E}:\mathcal{S}_{[0,T]}\to\mathbb{R}_\infty$ is continuous as well and the set $\mathcal{S}_{[0,T]}$ is closed.*

PROOF. For $(s,y_s),(t,y_t)\in\mathcal{S}_{[0,T]}$, we have by stability

$$-C_{\mathcal{E}}|t-s|-\mathcal{D}(y_s,y_t)\leqslant\mathcal{E}(t,y_t)-\mathcal{E}(s,y_s)\leqslant C_{\mathcal{E}}|t-s|+\mathcal{D}(y_t,y_s).$$

This estimate together with the continuity of $\mathcal{D}$ implies the continuity of $\mathcal{E}$.

Now, consider a sequence $(t_k,y_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with $t_k\to t^*$ and $y_k\overset{\mathcal{Y}}{\to}y^*$. It remains to show that $y^*\in\mathcal{S}(t^*)$. For an arbitrary $y\in\mathcal{Y}$, we have $\mathcal{E}(t_k,y_k)\leqslant\mathcal{E}(t_k,y)+\mathcal{D}(y_k,y)$ for all $k\in\mathbb{N}$. Taking the limit $k\to\infty$ the continuities yield $\mathcal{E}(t^*,y^*)\leqslant\mathcal{E}(t^*,y)+\mathcal{D}(y^*,y)$. Since $y\in\mathcal{Y}$ is arbitrary, it follows that $y^*\in\mathcal{S}(t^*)$. □

The next result is a strengthened version of the previous one.

PROPOSITION 5.10. *Let* (A2) *hold. Assume that for each sequence $(t_k,y_k)_{k\in\mathbb{N}}$ with $(t_k,y_k)\in\mathcal{S}_{[0,T]}$, $t_k\to t^*$ and $y_k\overset{\mathcal{Y}}{\to}y^*$ in $\mathcal{Y}$ the following condition holds*

$$\forall y\in\mathcal{Y},\quad \limsup_{k\to\infty}\big[\mathcal{E}(t_k,y_k)-\mathcal{D}(y_k,y)\big]\geqslant\mathcal{E}\big(t^*,y^*\big)-\mathcal{D}\big(y^*,y\big). \tag{5.8}$$

Then, the set $\mathcal{S}_{[0,T]}$ is closed.

PROOF. Let $y\in\mathcal{Y}$ be arbitrary. We have to show that $\mathcal{E}(t^*,y^*)\leqslant\mathcal{E}(t^*,y)+\mathcal{D}(y^*,y)$. Since $(t_k,y_k)\in\mathcal{S}_{[0,T]}$, we have the following estimates

$$\begin{aligned}\mathcal{E}\big(t^*,y^*\big)&=\mathcal{E}\big(t^*,y^*\big)-\mathcal{E}(t_k,y_k)+\mathcal{E}(t_k,y_k)\\&\leqslant\mathcal{E}\big(t^*,y^*\big)-\mathcal{E}(t_k,y_k)+\mathcal{E}(t_k,y)+\mathcal{D}(y_k,y)\\&=\mathcal{E}\big(t^*,y\big)+\mathcal{D}\big(y^*,y\big)+\big(\mathcal{E}(t_k,y)-\mathcal{E}\big(t^*,y\big)\big)\\&\quad-\big[\mathcal{E}(t_k,y_k)-\mathcal{D}(y_k,y)-\mathcal{E}\big(t^*,y^*\big)+\mathcal{D}\big(y^*,y\big)\big].\end{aligned}$$

Taking the $\liminf k\to\infty$, using (A2) (i.e., $|\partial_t\mathcal{E}|\leqslant C_{\mathcal{E}}$) and condition (5.8) we obtain the desired result. □

For an application to the delamination problem in Section 7.5 we use the following result, which uses continuity of $\mathcal{E}$ and some approximation property for $\mathcal{D}$. This approximation property is weaker than the continuity assumed in Proposition 5.9. A similar idea, but not in such an abstract setting, is used in [FL03], Theorem 2.1, and [DalFT05], where the corresponding result is named *jump transfer lemma*.

PROPOSITION 5.11. *Let* (A1)–(A4) *hold and assume that* $\mathcal{E}$ *and* $\mathcal{D}$ *satisfy the following condition*:

$$\begin{aligned}&\textit{For all } (t,\hat{y}),(t_k,y_k)\in\mathcal{S}_{[0,T]} \textit{ with } (t_k,y_k)\overset{\mathcal{Y}}{\to}(t,y),\\ &\textit{there exists } \hat{y}_k\in\mathcal{Y} \textit{ such that } \hat{y}_k\overset{\mathcal{Y}}{\to}\hat{y} \textit{ and}\\ &\liminf_{k\to\infty}\mathcal{E}(t_k,\hat{y}_k)+\mathcal{D}(z_k,\hat{z}_k)\leqslant\mathcal{E}(t,\hat{y})+\mathcal{D}(z,\hat{z}).\end{aligned}\tag{5.9}$$

Then, the set $\mathcal{S}_{[0,T]}$ *is closed.*

REMARK. For the case that $\mathcal{Z}$ is a Banach space and $\mathcal{D}(z,\hat{z})=\Delta(\hat{z}-z)$ with $c_1\|z\|\leqslant\Delta(z)\leqslant c_2\|z\|$, we simply choose $\hat{y}_k=(\varphi_k,\hat{z}-z+z_k)$. Then $\mathcal{D}(z_k,\hat{z}_k)=\Delta(\hat{z}-z)=\mathcal{D}(z,\hat{z})$, and the assumption holds if $\mathcal{E}$ is continuous.

PROOF OF PROPOSITION 5.11. Take any sequence $(t_k,y_k)\in\mathcal{S}_{[0,T]}$ with $(t_k,y_k)\overset{\mathcal{Y}}{\to}(t,y)$. We have to show that $y\in\mathcal{S}(t)$. For arbitrary $\hat{y}\in\mathcal{S}(t)$ we choose $\hat{y}_k$ according to condition (5.9). Using the lower semicontinuity of $\mathcal{E}$ and $y_k\in\mathcal{S}(t_k)$ we obtain

$$\mathcal{E}(t,y)\leqslant\liminf_{k\to\infty}\mathcal{E}(t_k,y_k)\leqslant\liminf_{k\to\infty}\mathcal{E}(t_k,\hat{y}_k)+\mathcal{D}(y_k,\hat{y}_k)\leqslant\mathcal{E}(t,\hat{y})+\mathcal{D}(y,\hat{y}),$$

which is the desired stability result. □

If $\mathcal{Y}$ is a Banach space Y, then it is often easy to show that $\mathcal{D}$ is continuous with respect to the strong topology. However, compactness is often only obtained in the weak topology. Hence, it is desirable to know, under which conditions we can show convexity of the stable sets $\mathcal{S}(t)$. The most important case involves a quadratic energy $\mathcal{E}(t,y)=\langle Ay,y\rangle-\langle\ell(t),y(t)\rangle$ and a translationally invariant dissipation metric $\Psi=\mathcal{L}I_{C_*}$. As we have seen in Section 2 we have $\mathcal{S}(t)=A^{-1}(\ell(t)-C_*)$. Under suitable conditions on a general dissipation distance $\Psi:Y\times Y\to[0,\infty]$ (like (4.11)) it is still possible to show the characterization

$$\mathcal{S}(t)=\big\{y\in Y\;\big|\;0\in\partial\Psi(y,0)+Ay-\ell(t)\big\},$$

and in some cases the convexity may be established from this. However, in general the stable sets are not convex and fortunately this condition is not needed in Section 4 where we always prove strong convergence.

EXAMPLE 5.12. Let $Y=\mathbb{R}\times H$, where H is a Hilbert space. Let $y=(a,h)\in X$ and

$$\mathcal{E}(t,y)=\frac{1}{4}\big(a^2+\|h\|^2\big)^2-\gamma(t)a,\qquad\Delta(y)=\sqrt{a^2+\|h\|^2}.$$

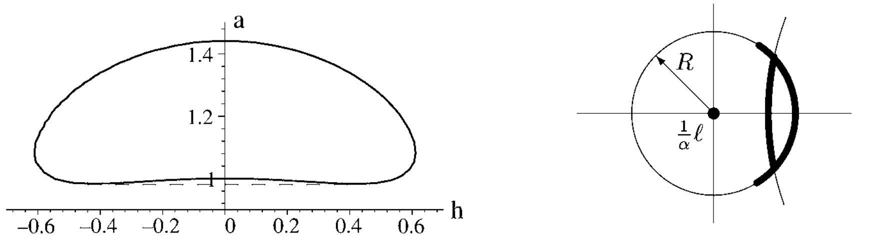

Fig. 1. Visualizations of nonconvex stable sets.
Left: Example 5.12 with $H = \mathbb{R}$. Right: Example 5.13 with $Y = \mathbb{R}^2$.

Then for $\gamma(t_1) = 2$, it can be shown that $\mathcal{S}(t_1)$ is not convex and not weakly closed. In fact, for any h_* with $\|h_*\| = (3 \cdot 5^3/2^{16})^{1/6}$ we have $((3^5/2^8)^{1/3}, h_*) \in \mathcal{S}(t_1)$ but $((3^5/2^8)^{1/3}, 0) \notin \mathcal{S}(t_1)$, see [MieT04], Example 5.5, and on the left in Figure 1.

EXAMPLE 5.13. In this example $\mathcal{E}$ is quadratic plus a characteristic function. Let Y be a Hilbert space with $\dim Y \geqslant 2$ and

$$\mathcal{E}(t, y) = \frac{\alpha}{2}\|y\|^2 + I_{B_R(0)}(y) - \langle \ell(t), y \rangle, \qquad \Psi(v) = \|v\|.$$

Then y with $\|y\| < R$ is stable if and only if $\|\alpha y - \ell(t)\| \leqslant 1$. For y with $\|y\| = R$ the boundary of $B_R(0)$ enlarges the stable set. Stability holds if there exists $\gamma \in [\alpha, \infty)$ such that $\|\gamma y - \ell(t)\| \leqslant 1$. Thus, in the case $\|\ell(t)\| \leqslant \sqrt{1 + \alpha^2 R^2}$ we have the convex stable set $\mathcal{S}(t) = \{z \in E\colon \|\alpha z - \ell(t)\| \leqslant 1\} \cap B_R(0)$, which is the intersection of two balls. In the case $\|\ell(t)\| > \sqrt{1 + \alpha^2 R^2}$, we have

$$\begin{aligned}\mathcal{S}(t) = {} & \big\{ y \in B_R(0) \mid \big\|\alpha y - \ell(t)\big\| \leqslant 1 \big\} \\ & \cup \big\{ y \mid \|y\| = R, \big\| \big(\big\|\ell(t)\big\|^2 - 1\big)^{1/2} y - R\ell(t) \big\| \leqslant R \big\}\end{aligned}$$

which contains a nonconvex part of the boundary of the sphere, see on the right of Figure 1.

5.3. *An example of nonconvergence*

Here we provide an example where the incremental problem (IP) is solvable and the associated interpolants converge to a limit $z^\infty : [0, \tau] \to \mathcal{Z}$. However, the limit is not a solution despite the fact that the energetic problem (S) and (E) has many solutions.

Let $(\ell_1, \|\cdot\|_1)$ and $(c_0, \|\cdot\|_\infty)$ be the Banach spaces of absolutely summable sequences and sequences converging to 0, respectively. Consider $\mathcal{Z} = \{z = (z^{(j)})_{j\in\mathbb{N}} \in \ell_1 \mid \|z\|_1 \leqslant 1\}$,

$$\mathcal{E}(t, z) = -\sum_{j=1}^{\infty} z^{(j)} - \langle \ell(t), z \rangle \quad \text{and} \quad \mathcal{D}(z_0, z_1) = \|z_1 - z_0\|_1,$$

where $\ell \in \mathrm{C}^1([0,3], c_0)$ is given via

$$\ell(t) = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k \varphi\big(2^k t\big) e_k, \quad \text{where } e_k = (0, \ldots, 0, 1, 0, \ldots) \in c_0$$

and $\varphi \in \mathrm{C}^1(\mathbb{R})$ with $\operatorname{supp}(\varphi) = [\frac{1}{2}, 1]$ and $\varphi(t) \in [0,1]$. Hence $\|\ell(t)\|_\infty \leqslant ct^2$ and for each $t \in [0,3]$ there exist $k \in \mathbb{N}$ and $\lambda \in [0,1]$ with $\ell(t) = \lambda e_k$.

The stable sets can be easily computed, since $\mathcal{E}(t,\cdot)$ is linear:

$$\begin{aligned}
&\ell(t) = 0 \quad \Longrightarrow \quad \mathcal{S}(t) = \mathcal{Z},\\
&\ell(t) = \lambda e_k \quad \text{with } \lambda \in (0,1) \quad \Longrightarrow \quad \mathcal{S}(t) = \big\{ z \in \mathcal{Z} \bigm| z^{(k)} = 1 - \big\| z - z^{(k)} e_k \big\|_1 \big\}.
\end{aligned}$$

For the incremental problem (IP) we prescribe the initial condition $z_0 = 0$ and $t_0 = 0$. In the first step we have to minimize

$$z \mapsto \mathcal{E}(t_1, z) + \mathcal{D}(0, z_0) = -\big\langle \hat{e} + \ell(t_1), z \big\rangle + \|z\|_1,$$

where $\hat{e} = (1,1,1,\ldots) \in \ell_\infty$. If $\ell(t_1) = 0$, then any $z_1 \in \mathcal{Z}$ with $z_1^{(j)} \geqslant 0$ for all $j \in \mathbb{N}$ is a minimizer. If $\ell(t_1) = \lambda_1 e_{n(1)}$ with $\lambda_1 \in (0,1)$, then the unique minimizer is $z_1 = e_{n(1)}$. Generically, for small time increments $t_1 - t_0$ the second case occurs and $n(1) \to \infty$ for $t_1 \searrow 0$.

In the second step, $\ell(t_2) = \lambda_2 e_{n(2)}$ with $n(2) \leqslant n(1)$ and $\lambda_2 \in [0,1]$, and we have to minimize

$$z \mapsto \mathcal{E}(t_2, z) + \mathcal{D}(z_1, z) = -\langle \hat{e} + \lambda_2 e_{n(2)}, z \rangle + \|z - e_{n(1)}\|_1.$$

It is easy to see that $z_2 = z_1 = e_{n(1)}$ remains the unique global minimizer, since for $n(2) < n(1)$ we have

$$\begin{aligned}
&\mathcal{E}(t_2, e_{n(2)}) + \mathcal{D}(e_{n(1)}, e_{n(2)})\\
&\quad = -(1 + \lambda_2) + 2\\
&\quad > \mathcal{E}(t_2, e_{n(1)}) + \mathcal{D}(e_{n(1)}, e_{n(1)}) = -1 + 0.
\end{aligned}$$

Finally, for all further steps we find $z_k = e_{n(1)}$. Thus, for all partitions Π the piecewise constant interpolant $z^\Pi : [0,T] \to \mathcal{Z}$ has the form

$$z^\Pi(t) = 0 \quad \text{for } t \in [0, t_1) \quad \text{and} \quad z^\Pi(t) = e_{n(1)} \quad \text{for } t \in [t_1, 3],$$

where $n(1)$ is determined via $\ell(t_1) = \lambda_1 e_{n(1)}$ and hence for $f(\Pi) \to 0$ we find $n(1) \to \infty$.

To study convergence, we fix the topology on $\mathcal{Z}$ as the weak* topology on $\ell_1 = c_0^*$. Then, $\mathcal{Z}$ is a compact space, but $\mathcal{E}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is not weakly* lower semicontinuous.

Even worse, the stable sets $\mathcal{S}(t)$ are not weakly* closed for $\ell(t)=\lambda e_k$ with $\lambda\in(0,1)$. However, we find

$$z^{\Pi_m}(t)\overset{*}{\rightharpoonup}0\quad\text{for all } t\in[0,3].$$

Thus, the limit function $z^\infty:[0,3]\to\mathcal{Z}$ with $z^\infty(t)=0$ is well defined. Obviously, z^∞ solves (E) but the stability (S) fails for all t with $\ell(t)\neq 0$.

Nevertheless, (S) and (E) has many solutions. Choose any $z_*\in\mathcal{Z}$ with $\|z_*\|_1=1$ and $z_*^{(j)}\geqslant 0$ for all $j\in\mathbb{N}$. Define $z:[0,3]\to\mathcal{Z}$ with $z(0)=0$ and $z(t)=z_*$ for $t>0$. Then (S) holds since $z(t)\in\mathcal{S}(t)$ for each $t\in[0,3]$. Moreover, (E) holds since $\mathcal{E}(0,z(0))=0$ and for $t>0$ we have

$$\mathcal{E}\big(t,z(t)\big)=-1-\big\langle\ell(t),z_*\big\rangle,\qquad \mathrm{Diss}_{\mathcal{D}}\big(z;[0,t]\big)=1,$$

$$\int_0^t\partial_s\mathcal{E}\big(s,z(s)\big)\,\mathrm{d}s=-\int_0^t\big\langle\dot\ell(s),z_*\big\rangle\,\mathrm{d}s=-\big\langle\ell(t),z_*\big\rangle.$$

5.4. *Formulations which resolve jumps*

A major disadvantage of the global energetic formulation using (S) and (E) is that the stability condition (S) is a *global* stability condition. Thus, jumps from y_- to y_+ can occur despite the fact that any continuous path $\tilde y:[0,1]\to\mathcal{Y}$ from y_0 to y_1 would have to pass a potential barrier higher than $\mathcal{E}(t,y_0)$, i.e., there is always an $s\in(0,1)$ with $\mathcal{E}(t,\tilde y(s))+\mathcal{D}(y_0,\tilde y(s))>\mathcal{E}(t,y_0)$. However, considering *continuous paths* we need to specify a topology with respect to which we ask for continuity. This topology may be different from $\mathcal{T}$, which was used for the existence theory, it should rather be modeled on physical grounds, or it should be chosen for mathematical convenience. In particular, it is desirable to use semidistances $d:\mathcal{Y}\times\mathcal{Y}\to[0,\infty]$ such that the choice $d=\mathcal{D}$ is possible.

It was first proposed in [Mie03a] to study a version of the incremental problem, where global minimization is replaced by a local one, namely inside a ball in the d-distance of small radius $\delta>0$,

$$(\mathrm{IP})_\delta\qquad y_k\in\operatorname{Arg\,min}\big\{\mathcal{E}(t_k,y)+\mathcal{D}(z_{k-1},z)\;\big|\;y\in\mathcal{Y},\,d(y_{k-1},y)\leqslant\delta\big\}.\tag{5.10}$$

Of course, it will be essential that the additional parameter δ tends to 0 slower than the fineness $f(\Pi)$ of the partition, e.g., $\delta=f(\Pi)^{1/2}$. Then, the solutions of $(\mathrm{IP})_\delta$ will display standard rate-independent behavior in many regions but will have in between phases where the solution performs a fast jump.

Following [Vis01] we define a rate-independent version of *Φ-minimal paths* as follows. The set of arc-length parametrized paths is defined via

$$\mathcal{A}(y_0)=\big\{(t,y)\in\mathrm{C}^0\big([0,\mathcal{T}],\mathbb{R}\times\mathcal{Y}\big)\;\big|\;t(0)=0,\,y(0)=y_0,\,t'(\tau)\geqslant 0\text{ a.e.},\\ t(\tau)+\mathrm{Diss}_d\big(y;[0,\tau]\big)=\tau\text{ for all }\tau\big\},$$

where Diss_d is the dissipation associated with the new metric d. Thus, the curves in $\mathcal{A}(z_0)$ are parametrized by the arc-length variable τ instead of the usual process time t. In particular, we find $t'(\tau) \in [0,1]$ a.e. and $d(y(\tau_1), y(\tau_2)) \leqslant |\tau_1 - \tau_2|$. On $\mathcal{A}(y_0)$ we define the mapping $\Phi : \mathcal{A}(y_0) \to \mathrm{L}^\infty([0,\mathcal{T}])$ via

$$\Phi[t,y](\tau) = \mathcal{E}\big(t(\tau), y(\tau)\big) + \mathrm{Diss}_{\mathcal{D}}\big(y;[0,\tau]\big) - \int_0^\tau \frac{\partial}{\partial t}\mathcal{E}\big(t(\tau), y(\tau)\big)t'(\tau)\,\mathrm{d}\tau,$$

then the path $(t,y) \in \mathcal{A}(y_0)$ is called Φ-minimal, if $(t,y) \leqslant_\Phi (\hat{t},\hat{y})$ for all $(\hat{t},\hat{y}) \in \mathcal{A}(y_0)$, where the relation $\leqslant_\Phi$ is defined as follows. For two paths $(t,y), (\hat{t},\hat{y}) \in \mathcal{A}(y_0)$ define the time of "equality" via $\tilde{\tau}^0_{y,\hat{y}} = \inf\{\tau \in [0,\mathcal{T}] \mid (t(\tau), y(\tau)) \neq (\hat{t}(\tau), \hat{y}(\tau))\}$, then

$$(t,y) \overset{\Phi}{\leqslant} \big(\hat{t},\hat{y}\big) \quad \Longleftrightarrow \quad \forall \tau_2 > \tilde{\tau}^0_{y,\hat{y}}, \exists \tau_1 \in (\tilde{\tau}_0, \tau_2), \quad \Phi[t,y](\tau_1) \leqslant \Phi\big[\hat{t},\hat{y}\big](\tau_1). \tag{5.11}$$

This formulation can be weakened and localized as follows. Define

$$M(t,y) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \inf\big\{\mathcal{E}(t,y) + \mathcal{D}(y,\hat{y}) \,\big|\, d(y,\hat{y}) \leqslant \varepsilon\big\}.$$

Then $(t,y) \in \mathcal{A}(y_0)$ is called *locally Φ-minimal* if

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi[t,y](\tau) \leqslant M\big(t(\tau), y(\tau)\big) \quad \text{for a.a. } t \in [0,\mathcal{T}]. \tag{5.12}$$

The two above formulations are still derivative free in the sense that the underlying space $\mathcal{Y}$ does not need to have a differentiable structure, such that derivatives of $y : [0,\mathcal{T}] \to \mathcal{Y}$ need not be defined. Only the energetic, real-valued quantities $\mathcal{E}$, $\mathrm{Diss}_{\mathcal{D}}$ and Diss_d need to be absolutely continuous.

If the state space $\mathcal{Y}$ has a differentiable structure, then we may assume that the dissipation distance $\mathcal{D}$ and the semidistance d are generated by local metrics $\Psi : \mathrm{T}\mathcal{Y} \to [0,\infty]$ and $\eta : \mathrm{T}\mathcal{Y} \to [0,\infty]$, respectively. Moreover, we consider now solutions which are absolutely continuous. Then, the condition $(t,y) \in \mathcal{A}(y_0)$ implies $t'(\tau) + \eta(y(\tau), y'(\tau)) = 1$ for a.a. $\tau \in [0,\mathcal{T}]$. If additionally $\mathcal{E}$ is differentiable in y, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi[t,y](\tau) = \big\langle \mathrm{D}\mathcal{E}(t,y), y' \big\rangle + \Psi\big(y, y'\big), \\
&M(t,y) = \inf\big\{\big\langle \mathrm{D}\mathcal{E}(t,y), v \big\rangle + \Psi(y,v) \,\big|\, \eta(y,v) \leqslant 1\big\}.
\end{aligned}$$

Thus, condition (5.12) can be reformulated via the combined functional $\Psi_\eta : \mathrm{T}\mathcal{Y} \to [0,\infty]$,

$$\Psi_\eta(y,v) = \begin{cases} \Psi(y,v) & \text{for } \eta(y,v) \leqslant 1, \\ \infty & \text{else.} \end{cases}$$

We obtain the following differentiable version of (5.12):

$$
\begin{aligned}
&0 \in \partial_v \Psi_\eta\big(y(\tau), y'(\tau)\big) \\
&\qquad + \mathrm{D}_y \mathcal{E}\big(t(\tau), y(\tau)\big) \in \mathrm{T}^*_{y(\tau)}\mathcal{Y}, \\
&0 \leqslant t'(\tau) = 1 - \eta\big(y(\tau), y'(\tau)\big),
\end{aligned}
\tag{5.13}
$$

for a.a. $\tau \in [0, \mathcal{T}]$.

In [EfM04] this local formulation is investigated for the case that $\mathcal{Y}$ is a finite-dimensional Banach space Y and that both dissipation metrics are translation invariant and nondegenerate, i.e., there exists a $c > 0$ such that $\Psi(v), \eta(v) \geqslant c\|v\|$ for all $v \in Y$. It is shown that the piecewise linear interpolants of the solutions of the localized incremental problem $(\mathrm{IP})_\delta$ converge, after arc-length parametrization, to a solution of the first equation in (5.13). In general, the limit function will not have arc-length parametrization, but it can be reparametrized to provide a full solution of (5.13). Using a Young measure argument it can be shown that the limit is always in arc-length parametrization if the two metrics Ψ and η satisfy a certain compatibility condition (which holds for instance for $\eta = \Psi$).

Moreover, it is shown in [EfM04], that (5.13) appears as a limit problem if the following viscously regularized problem is considered:

$$
0 \in \partial\Psi_\varepsilon\big(\dot{y}(t)\big) + \mathrm{D}\mathcal{E}\big(t, y(t)\big) \in Y^* \quad \text{for a.a. } t \in [0, T], \tag{5.14}
$$

where $\Psi_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\eta(v)^2$. Existence of solutions $z_\varepsilon \in \mathrm{H}^1([0, T], Y)$ follows under mild assumptions on $\mathcal{E}$, since now Ψ_ε grows quadratically, see [ColV90]. Reparametrizing these solutions as above, one can show that the limits for $\varepsilon \to 0$ exist and satisfy (5.13).

A similar arc-length reparametrization was used in [And95] for the surface friction problem studied in Section 6.3. There, the differential inclusion $0 \in \mathcal{R}(y(t), \dot{y}(t)) + Ay(t) - \ell(t)$ is solved for by using a delay in the form $0 \in \mathcal{R}(y(t-\varepsilon), \dot{y}(t)) + Ay(t) - \ell(t)$ which produces a unique solution y^ε. It is then shown that the reparametrized solutions contain a subsequence which converges to a generalized solutions which, in the original time t (not reparametrized) may have jumps.

5.5. *Time-dependent state spaces $\mathcal{Y}(t)$*

In some situations it is necessary to introduce time-dependent state spaces which arise from time-dependent boundary conditions. In the most general situation we have a big state space $\mathcal{Y}$ on which the energy functional $\mathcal{E} : [0, T] \times \mathcal{Y} \to \mathbb{R}_\infty$ and the dissipation distance $\mathcal{D} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_\infty$ are defined. Then, the functional $\mathcal{E}(t, \cdot)$ may be $+\infty$ outside a set $\mathcal{Y}(t) \subset \mathcal{Y}$, which may be defined via time-dependent Dirichlet conditions. The problem is that in such situations it is not possible to satisfy condition (A2) concerning the time derivative $\partial_t \mathcal{E}$.

In continuum mechanics we often have $y = (\varphi, z) \in \mathcal{F} \times \mathcal{Z} = \mathcal{Y}$ and the time-dependence comes into play only through a set $\mathcal{F}(t) \subset \mathcal{F}$. Then, one may introduce a transformation $\varphi(t) = \Phi_t(\tilde{\varphi}(t))$ such that Φ_t maps $\widetilde{\mathcal{F}}$ homeomorphically into $\mathcal{F}(t)$ (e.g., by subtracting the time-dependent boundary conditions). Then, one defines the transformed energy

$$\widetilde{\mathcal{E}}(t,\tilde{\varphi},z)=\mathcal{E}\big(t,\Phi_t(\tilde{\varphi}),z\big)\quad \text{for } \tilde{\varphi}\in\widetilde{\mathcal{F}},\, t\in[0,T] \text{ and } z\in\mathcal{Z},$$

and the problem is reduced to the time-independent case. We refer to [FM05] for a careful treatment of time-dependent Dirichlet boundary data in the case of small strains as well as in the case of finite-strain elasticity.

However, in some situations this decoupling does not work and we now present a way how this situation can be modeled via the energetic formulation. The stability condition is easily transfered to the time-dependent case, as it is a static condition involving only one time instant. However, for the energy balance we need a replacement of the power of the external forces, previously written as $\partial_t\mathcal{E}(t,y)$.

For this purpose, we assume that $\mathcal{Y}$ is a Banach space and there exist a fixed subset $\widetilde{\mathcal{Y}}\subset\mathcal{Y}$ and invertible transformations

$$\Phi_t:\mathcal{Y}\to\mathcal{Y}\quad \text{with } \Phi_t\big(\widetilde{\mathcal{Y}}\big)=\mathcal{Y}(t) \text{ for all } t\in[0,T].$$

We define the functionals $\widetilde{\mathcal{E}}:[0,T]\times\mathcal{Y}\to\mathbb{R}_\infty$ and $\widetilde{\mathcal{D}}_{s,t}:\widetilde{\mathcal{Y}}\times\widetilde{\mathcal{Y}}\to[0,\infty]$ via

$$\widetilde{\mathcal{E}}(t,\tilde{y})=\mathcal{E}\big(t,\Phi_t(\tilde{y})\big)\quad\text{and}\quad \widetilde{\mathcal{D}}_{s,t}(\tilde{y},\hat{y})=\mathcal{D}\big(\Phi_s(\tilde{y}),\Phi_t(\hat{y})\big).$$

Hence, we introduce a time-dependent dissipation on the time-independent state space $\widetilde{\mathcal{Y}}$. Note that the solutions to be constructed have to lie in $\widetilde{\mathcal{Y}}$, but the functional $\widetilde{\mathcal{E}}$ is defined on all of $\mathcal{Y}$.

For all $s,t\in[0,T]$, we also define the transfer operators

$$\widetilde{\Phi}_{s,t}:\mathcal{Y}\to\mathcal{Y},\qquad \tilde{y}\mapsto\Phi_t^{-1}\big(\Phi_s(\tilde{y})\big),$$

which satisfy the evolution property $\widetilde{\Phi}_{r,s}\circ\widetilde{\Phi}_{s,t}=\widetilde{\Phi}_{r,t}$ and, by the definitions, we find

$$\begin{aligned}&\widetilde{\mathcal{D}}_{s,t}\big(\tilde{y}_0,\widetilde{\Phi}_{r,t}(\tilde{y}_1)\big)=\mathcal{D}\big(\Phi_s(\tilde{y}_0),\Phi_r(\tilde{y}_1)\big)=\widetilde{\mathcal{D}}_{s,r}(\tilde{y}_0,\tilde{y}_1)\quad\text{and}\\ &\widetilde{\mathcal{D}}_{s,t}\big(\tilde{y},\widetilde{\Phi}_{s,t}(\tilde{y})\big)=0.\end{aligned}\tag{5.15}$$

If $\mathcal{D}$ is generated from a dissipation metric $\Psi:\mathrm{T}\mathcal{Y}\to[0,\infty]$, then $\widetilde{\mathcal{D}}_{s,t}$ associates with the time-dependent dissipation metric $\widetilde{\Psi}$ given by

$$\widetilde{\Psi}(t,\tilde{y},\tilde{v})=\Psi\big(\Phi_t(\tilde{y}),\mathrm{D}\Phi_t(\tilde{y})v-\partial_t\Phi_t(\tilde{y})\big).$$

The main assumption on the model, replacing the former condition (A2), is now that for each $(t,\tilde{y})\in[0,T]\times\widetilde{\mathcal{Y}}$ with $\widetilde{\mathcal{E}}(t,\tilde{y})<\infty$ the function $s\mapsto\widetilde{\mathcal{E}}(s,\widetilde{\Phi}_{t,s}(\tilde{y}))$ is continuously differentiable and

$$\exists c_E^{(1)},c_E^{(0)}>0,\forall s\in[0,T],\qquad \left|\frac{\partial}{\partial s}\widetilde{\mathcal{E}}\big(s,\widetilde{\Phi}_{t,s}(\tilde{y})\big)\right|\leqslant c_E^{(1)}\big(\widetilde{\mathcal{E}}(t,\tilde{y})+c_E^{(0)}\big).\tag{5.16}$$

We now define the power of external forces via

$$\tilde{p}(t,\tilde{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{E}}\big(t,\Phi_t^{-1}(w)\big)\bigg|_{w=\Phi_t(\tilde{y})} = \frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{E}}\big(s,\widetilde{\Phi}_{t,s}(\tilde{y})\big)\bigg|_{s=t}.$$

EXAMPLE 5.14. Consider a smooth situation with $\mathcal{Y}=\mathbb{R}^N$, $\mathcal{E}(t,y)=\frac{1}{2}\langle Ay,y\rangle - \langle \ell(t),y\rangle$, $\Psi(y,v)=\widehat{\Psi}(v)$ and $\mathcal{Y}(t)=b(t)+V$, where V is an arbitrary, fixed subspace. Hence, the variational inequality reads

$$0\in\partial\widehat{\Psi}\big(\dot{y}(t)\big)+Ay(t)-\ell(t)+\partial I_{\mathcal{Y}(t)}\big(y(t)\big)\subset\mathbb{R}^N.$$

With $y=\Phi_t(\tilde{y})=b(t)+Q(t)\tilde{y}$, where $Q(t)\in\mathrm{Lin}(V,V)$, we obtain $\widetilde{\mathcal{E}}(t,\tilde{y})=\mathcal{E}(t,\Phi_t(\tilde{y}))=\frac{1}{2}\langle\widetilde{A}\tilde{y},\tilde{y}\rangle-\langle\tilde{\ell}(t),\tilde{y}\rangle+\tilde{e}(t)$ with $\widetilde{A}(t)=Q^{\mathsf{T}}AQ$ and $\tilde{\ell}=\ell-Ab$. Using $\dot{\tilde{A}}=\dot{Q}^{\mathsf{T}}AQ+Q^{\mathsf{T}}A\dot{Q}$ and $\dot{\tilde{\ell}}=\dot{\ell}-A\dot{b}$ we find

$$\begin{aligned}
&\partial_t\widetilde{\mathcal{E}}(t,\tilde{y})=\big\langle AQ(\tilde{y}+b)-\ell,\dot{Q}y\big\rangle-\big\langle\dot{\ell}-A\dot{b},Q\tilde{y}+b\big\rangle-\big\langle\ell,\dot{b}\big\rangle,\\
&\mathrm{D}\widetilde{\mathcal{E}}(t,\tilde{y})\big[Q^{-1}\big(\dot{Q}\tilde{y}+\dot{b}\big)\big]=\big\langle AQ(\tilde{y}+b)-\ell,\dot{Q}y+\dot{b}\big\rangle
\end{aligned}$$

and hence,

$$\begin{aligned}
\tilde{p}(t,\tilde{y})&=\partial_t\widetilde{\mathcal{E}}(t,\tilde{y})-\mathrm{D}\widetilde{\mathcal{E}}(t,\tilde{y})\big[Q^{-1}\big(\dot{Q}\tilde{y}+\dot{b}\big)\big]\\
&=-\big\langle\dot{\ell},Q\tilde{y}+b\big\rangle=\big\langle A\dot{b}-\dot{\tilde{\ell}},Q\tilde{y}+b\big\rangle.
\end{aligned}$$

Thus, we see the two contributions of the power of the changing boundary conditions via $\dot{b}$ and the power of the external forces via $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\ell}$. Moreover, the rate $\dot{Q}$ of the (unnecessary) transformation $Q(t)$ does not contribute to the power. With $a(t,\tilde{y})=\mathrm{D}\Phi_t(\tilde{y})\,\partial_t\Phi_t(\tilde{y})=Q^{-1}(\dot{Q}\tilde{y}+\dot{b})$ the transformed system in V takes the form

$$0\in\Big(\widehat{\Psi}\big(Q(t)\big[\dot{\tilde{y}}(t)-a\big(t,\tilde{y}(t)\big)\big]\big)+\widetilde{A}(t)\tilde{y}(t)-\tilde{\ell}(t)\Big)\cap V^*\subset V^*.$$

For fixed times, we define the set $\mathcal{S}(t)\subset\mathcal{Y}(t)$ of stable states via $\mathcal{S}(t)=\{y\in\mathcal{Y}(t)\mid \mathcal{E}(t,y)\leqslant\mathcal{E}(t,\hat{y})+\mathcal{D}(y,\hat{y})$ for all $\hat{y}\in\mathcal{Y}(t)\}$ as well as the transformed set

$$\widetilde{\mathcal{S}}(t)=\big\{\tilde{y}\in\widetilde{\mathcal{Y}}\;\big|\;\widetilde{\mathcal{E}}(t,\tilde{y})\leqslant\widetilde{\mathcal{E}}(t,\hat{y})+\widetilde{\mathcal{D}}_{t,t}(\tilde{y},\hat{y})\text{ for all }\hat{y}\in\widetilde{\mathcal{Y}}\big\}=\Phi_t^{-1}\big(\mathcal{S}(t)\big).$$

The dissipation of a curve $\tilde{y}:[0,T]\to\widetilde{\mathcal{Y}}$ on the interval $[r,s]\subset[0,T]$ is defined via

$$\begin{aligned}
&\mathrm{Diss}_{\widetilde{\mathcal{D}}}\big(\tilde{y},[r,s]\big)\\
&\quad=\sup\left\{\sum_{j=1}^{N}\widetilde{\mathcal{D}}_{\tau_{j-1},\tau_j}\big(\tilde{y}(\tau_{j-1}),\tilde{y}(\tau_j)\big)\;\middle|\;N\in\mathbb{N},r\leqslant\tau_0<\tau_1<\cdots<\tau_N\leqslant s\right\},
\end{aligned}$$

such that $y : t \mapsto \Phi_t(\tilde{y}(t)) \in \mathcal{Y}(t)$ satisfies $\mathrm{Diss}_{\mathcal{D}}(y; [r, s]) = \mathrm{Diss}_{\widetilde{\mathcal{D}}}(\tilde{y}, [r, s])$. We also define the power of the external forces in the original coordinates via

$$p(t, y) = \tilde{p}\big(t, \Phi_t^{-1}(y)\big) \quad \text{for } y \in \mathcal{Y}(t).$$

A simple application of the chain rule shows that, in the case that $\mathcal{Y}(t)$ is constant and $\mathcal{E}(t, y)$ is differentiable in t, we have $p(t, y) = \partial_t \mathcal{E}(t, y)$ as expected.

The following two energetic formulations (S) and (E) and ($\widetilde{\mathrm{S}}$) and ($\widetilde{\mathrm{E}}$) are equivalent via the transformation $y(t) = \Phi_t(\tilde{y}(t))$.

DEFINITION 5.15. A process $y : [0, T] \to \mathcal{Y}$ is called an *energetic solution* of the rate-independent problem for $(\mathcal{Y}(t))_{t \in [0,T]}$, $\mathcal{E}$ and $\mathcal{D}$, if (S) and (E) hold for all $t \in [0, T]$,

$$\text{(S)} \quad y(t) \in \mathcal{S}(t) \subset \mathcal{Y}(t);$$

$$\text{(E)} \quad \mathcal{E}\big(t, y(t)\big) + \mathrm{Diss}_{\mathcal{D}}\big(y, [0, t]\big) = \mathcal{E}\big(0, y(0)\big) + \int_0^t p\big(s, y(s)\big)\, \mathrm{d}s.$$

A process $\tilde{y} : [0, T] \to \widetilde{\mathcal{Y}}$ is called an *energetic solution* of the rate-independent problem for $\widetilde{\mathcal{Y}}$, $\widetilde{\mathcal{E}}$ and $(\widetilde{\mathcal{D}}_{s,t})_{0 \leqslant s \leqslant t \leqslant T}$, if ($\widetilde{\mathrm{S}}$) and ($\widetilde{\mathrm{E}}$) hold for all $t \in [0, T]$,

$$(\widetilde{\mathrm{S}}) \quad \tilde{y}(t) \in \widetilde{\mathcal{S}}(t) \subset \widetilde{\mathcal{Y}};$$

$$(\widetilde{\mathrm{E}}) \quad \widetilde{\mathcal{E}}\big(t, \tilde{y}(t)\big) + \mathrm{Diss}_{\widetilde{\mathcal{D}}}\big(\tilde{y}, [0, t]\big) = \widetilde{\mathcal{E}}\big(0, \tilde{y}(0)\big) + \int_0^t \tilde{p}\big(s, \tilde{y}(s)\big)\, \mathrm{d}s.$$

The important point is that both energetic formulations are strongly related to their associated time-incremental minimization problem (IP) and ($\widetilde{\mathrm{IP}}$), respectively. For a discretization $0 = t_0 < t_1 < \cdots < t_N = T$ and $y_0 \in \mathcal{Y}(0)$ we let $\tilde{y}_0 = \Phi_0^{-1}(y_0)$ and consider the two incremental problems

$$\text{(IP)} \qquad y_k \in \operatorname{Arg\,min}\big\{\mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \bigm| y \in \mathcal{Y}(t_k)\big\}. \tag{5.17}$$

$$(\widetilde{\mathrm{IP}}) \qquad \tilde{y}_k \in \operatorname{Arg\,min}\big\{\widetilde{\mathcal{E}}(t_k, \tilde{y}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}) \bigm| \tilde{y} \in \widetilde{\mathcal{Y}}\big\}. \tag{5.18}$$

Of course, these two incremental problems are equivalent via $y_k = \Phi_{t_k}(\tilde{y}_k)$. The following result shows that the basic a priori estimates for ($\widetilde{\mathrm{IP}}$) hold as in the case of a time-independent dissipation distance, cf. Theorem 3.2.

THEOREM 5.16. *If the above assumptions hold and* $\tilde{y}_0 \in \widetilde{\mathcal{S}}(0)$, *then every solution* $(\tilde{y}_k)_{k=1,\dots,N}$ *of* ($\widetilde{\mathrm{IP}}$) *satisfies the following properties.*

(i) *For* $k = 0, \dots, N$ *the state* $\tilde{y}_k$ *is stable at time* t_k, *i.e.,* $\tilde{y}_k \in \widetilde{\mathcal{S}}(t_k)$.

(ii) *For* $k = 1, \dots, N$ *we have*

$$\int_{t_{k-1}}^{t_k} \tilde{p}\big(s, \widetilde{\Phi}_{t_k,s}(\tilde{y}_k)\big)\, \mathrm{d}s \leqslant \tilde{e}_k - \tilde{e}_{k-1} + \tilde{\delta}_k \leqslant \int_{t_{k-1}}^{t_k} \tilde{p}\big(s, \widetilde{\Phi}_{t_{k-1},s}(\tilde{y}_{k-1})\big)\, \mathrm{d}s,$$

where $\tilde{e}_j = \widetilde{\mathcal{E}}(t_j, \tilde{y}_j)$ *and* $\tilde{\delta}_k = \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k)$.

(iii) *With* $E_0 = \widetilde{\mathcal{E}}(0, \tilde{y}_0) + c_E^{(0)}$ *we have*

$$\sum_{k=1}^{N} \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k) \leqslant E_0 \mathrm{e}^{c_E^{(1)} T}$$

and

$$\widetilde{\mathcal{E}}(t_k, \tilde{y}_k) \leqslant E_0\, \mathrm{e}^{c_E^{(1)} t_k} - c_E^{(0)} \quad \textit{for } k = 1, \ldots, N.$$

PROOF. For (i) use that $\tilde{y}_k$ minimizes and the triangle inequality. For $\tilde{y} \in \widetilde{\mathcal{Y}}$ we have

$$\begin{aligned}
\widetilde{\mathcal{E}}(t_k, \tilde{y}_k) &\leqslant \widetilde{\mathcal{E}}(t_k, \hat{y}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}) - \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k) \\
&\leqslant \widetilde{\mathcal{E}}(t_k, \tilde{y}) + \widetilde{\mathcal{D}}_{t_k,t_k}(\tilde{y}_k, \tilde{y}).
\end{aligned}$$

To obtain the upper estimate of (ii) we use $\hat{y}^* = \widetilde{\Phi}_{t_{k-1},t_k}(\tilde{y}_{k-1}) \in \widetilde{\mathcal{Y}}$ as a test function in ($\widetilde{\mathrm{IP}}$) at the kth step and employ (5.15):

$$\begin{aligned}
&\widetilde{\mathcal{E}}(t_k, \tilde{y}_k) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \tilde{y}_{k-1}) \\
&\quad \leqslant \widetilde{\mathcal{E}}\big(t_k, \hat{y}^*\big) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}\big(\tilde{y}_{k-1}, \hat{y}^*\big) - \widetilde{\mathcal{E}}(t_{k-1}, \tilde{y}_{k-1}) \\
&\quad = \widetilde{\mathcal{E}}\big(t_k, \widetilde{\Phi}_{t_{k-1},t_k}(\tilde{y}_{k-1})\big) - \widetilde{\mathcal{E}}(t_{k-1}, \tilde{y}_{k-1}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}\big(\tilde{y}_{k-1}, \widetilde{\Phi}_{t_{k-1},t_k}(\tilde{y}_{k-1})\big) \\
&\quad = \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\mathcal{E}}\big(s, \widetilde{\Phi}_{t_{k-1},s}(y_{k-1})\big)\, \mathrm{d}s + 0 = \int_{t_{k-1}}^{t_k} \tilde{p}\big(s, \widetilde{\Phi}_{t_{k-1},s}(y_{k-1})\big)\, \mathrm{d}s.
\end{aligned}$$

Similarly, we obtain the lower estimate in (ii) by using $\hat{y}_* = \widetilde{\Phi}_{t_k,t_{k-1}}(\tilde{y}_k)$ as a comparison function in the stability condition for $\tilde{y}_{k-1}$:

$$\begin{aligned}
&\widetilde{\mathcal{E}}(t_k, \tilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \tilde{y}_{k-1}) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k) \\
&\quad \geqslant \widetilde{\mathcal{E}}(t_k, \tilde{y}_k) - \widetilde{\mathcal{E}}(t_{k-1}, \hat{y}_*) - \widetilde{\mathcal{D}}_{t_{k-1},t_{k-1}}(\tilde{y}_{k-1}, \hat{y}_*) + \widetilde{\mathcal{D}}_{t_{k-1},t_k}(\tilde{y}_{k-1}, \tilde{y}_k) \\
&\quad = \widetilde{\mathcal{E}}(t_k, \tilde{y}_k) - \widetilde{\mathcal{E}}\big(t_{k-1}, \widetilde{\Phi}_{t_k,t_{k-1}}(\tilde{y}_k)\big) - \mathcal{D}\big(\Phi_{t_{k-1}}(\tilde{y}_{k-1}), \Phi_{t_k}(\tilde{y}_k)\big) \\
&\qquad + \mathcal{D}\big(\Phi_{t_{k-1}}(\tilde{y}_{k-1}), \widetilde{\Phi}_{t_k}(\tilde{y}_k)\big) \\
&\quad = \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\mathcal{E}}\big(s, \widetilde{\Phi}_{t_k,s}(\tilde{y}_k)\big)\, \mathrm{d}s + 0 = \int_{t_{k-1}}^{t_k} \tilde{p}\big(s, \widetilde{\Phi}_{t_k,s}(\tilde{y}_k)\big)\, \mathrm{d}s.
\end{aligned}$$

Estimate (iii) follows in the same way as shown in Section 3.3 by induction over k and using (5.16) and the upper estimate in (ii). □

Following the lines of Section 5.1 it should be possible to develop a suitable existence theory.

5.6. *Relaxation of rate-independent systems*

Rate-independent systems can also be used to study systems which develop microstructure. In mathematics, we say that a system develops microstructure if energy minimization for a functional $\mathcal{I}:\mathcal{Y}\to\mathbb{R}_\infty$ leads to infimizing sequences $(y^{(j)})$, whose weak limit y^∞ does not minimize $\mathcal{I}$. More precisely, we have

$$\mathcal{I}\big(y^{(j)}\big)\to\alpha=\inf\big\{\mathcal{I}(y)\mid y\in\mathcal{Y}\big\},\quad y^{(j)}\overset{\mathcal{Y}}{\rightharpoonup}y^\infty,\quad\text{and}\quad\mathcal{I}\big(y^\infty\big)>\alpha.\tag{5.19}$$

This means that the sublevels of $\mathcal{I}$ are not closed and the construction of minimizers via infimizing sequences does not work. In fact, the existence of minimizers may fail. In such a situation the functional $\mathcal{I}$ is usually relaxed to a new functional $\mathbf{I}:\mathbf{Y}\to\mathbb{R}_\infty$, which is lower semicontinuous and, hence, has a global minimizer $\mathbf{y}$ which is connected to the limit y^∞ from above and may also retain some information on the infimizing sequence $(y^{(j)})$.

Since rate-independent problems are strongly connected to energy minimization via the energetic formulation (S) and (E), a related philosophy may be applied to the associated incremental problems. This was first observed in [OrtR99,OrtRS00] where the occurrence of certain microstructures in plasticity was explained, see also [MiSL02,MiL03]. Independently this idea was used for the derivation of evolution equations for shape-memory alloys in [MieT99,MieTL02,The02,MieRou03]. The abstract framework presented here was developed in [Mie03b,Mie04a].

We return to the energetic formulation (S) and (E) via the functionals $\mathcal{E}:[0,T]\times\mathcal{Y}\to\mathbb{R}_\infty$ and $\mathcal{D}:\mathcal{Y}\times\mathcal{Y}\to\mathbb{R}_\infty$, where now $\mathcal{E}$ and $\mathcal{D}$ need no longer be lower semicontinuous. The motivation for the suggested relaxation relies on the incremental problem (IP), see (3.4), which is in general no longer solvable due to formation of microstructure, see (5.19). In this situation we suggest the following approximate incremental problem.

$$(\mathrm{AIP})_\varepsilon\qquad\begin{array}{l}\text{Given } \varepsilon>0 \text{ and } y_0\in\mathcal{Y}, \text{ find } y_k^\varepsilon\in\mathcal{Y} \text{ with}\\ \mathcal{E}(t_k,y_k^\varepsilon)+\mathcal{D}(y_{k-1}^\varepsilon,y_k^\varepsilon)\leqslant\varepsilon+\mathcal{E}(t_k,y)+\mathcal{D}(y_{k-1}^\varepsilon,y)\text{ for all } y\in\mathcal{Y}.\end{array}\tag{5.20}$$

Obviously, this problem has solutions for all $\varepsilon>0$. The difficult, remaining question is how the solutions y_k^ε behave for $\varepsilon\to0$. As we have seen in the above example, we cannot expect pointwise convergence but certain macroscopic quantities should have limits for $\varepsilon\to0$.

To define an abstract notion of relaxation we introduce a generalized convergence "$\overset{\mathbf{Y}}{\to}$" on an enlarged space $\mathbf{Y}$, whose elements are denoted by $\mathbf{y}$. This space is connected to $\mathcal{Y}$ via a continuous embedding $\mathcal{J}:\mathcal{Y}\mapsto\mathbf{Y}$. Moreover, generalized functionals $\mathbf{E}:[0,T]\times\mathbf{Y}\to\mathbb{R}$ and $\mathbf{D}:\mathbf{Y}\times\mathbf{Y}\to[0,\infty]$ replace the elastic functional $\mathcal{E}$ and the dissipation distance $\mathcal{D}$. The relaxation must be such that the associated *relaxed incremental problem* (RIP) for an initial datum $\mathbf{y}_0\in\mathbf{Y}$ and the time discretization $0=t_0<t_1<\cdots<t_N=T$ is solvable.

$$(\mathrm{RIP})\qquad\begin{array}{l}\text{For given } \mathbf{y}_0\in\mathbf{Y} \text{ find, for } k=1,\ldots,N,\\ \mathbf{y}_k\in\operatorname{Arg\,min}\{\mathbf{E}(t_k,\mathbf{y})+\mathbf{D}(\mathbf{y}_{k-1},\mathbf{y})\mid\mathbf{y}\in\mathbf{Y}\}.\end{array}\tag{5.21}$$

We do not ask for the conditions $\mathbf{D}(\mathcal{J}(0,z_0),\mathcal{J}(0,z_1))=\mathcal{D}(z_0,z_1)$ and $\mathbf{E}(t,\mathcal{J}(\boldsymbol{\varphi},z))=\mathcal{E}(t,\boldsymbol{\varphi},z)$. Hence, in general the relaxation will not be an extension.

DEFINITION 5.17. A four-tuple $(\mathbf{Y}, \mathcal{J}, \mathbf{E}, \mathbf{D})$ as defined above is called a *lower* (*or upper*) *incremental relaxation* of $(\mathcal{Y}, \mathcal{E}, \mathcal{D})$ if the following four conditions hold.

(R1) *Solvability*: For each $\mathbf{y}_0 \in \mathbf{Y}$ the relaxed incremental problem (RIP) has a solution.

(R2) *Approximation*: $\mathcal{J}(\mathcal{Y})$ is dense in $\mathbf{Y}$.

(R3) *Incremental consistency*: If $(y_k)_{k=1,\dots,N}$ solves (IP), then $\mathcal{J}(y_k)_{k=1,\dots,N}$ solves (RIP); and if $(\mathbf{y}_k)_{k=1,\dots,N}$ satisfies $\mathbf{y}_k = \mathcal{J}(y_k)$ and solves (RIP), then $(y_k)_{k=1,\dots,N}$ solves (IP).

$(\text{R4})_{\text{low}}$ *Lower incremental relaxation*: For each solution $(\mathbf{y}_k)_{k=1,\dots,N}$ of (RIP) there exist solutions $(y_k)_{k=1,\dots,N}$ of $(\text{AIP})_\varepsilon$ with $\mathcal{J}(y_k^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}_k$ for $\varepsilon \to 0$.

$(\text{R4})_{\text{upp}}$ *Upper incremental relaxation*: If $\mathcal{J}(y_k^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}_k$ and $(y_k^\varepsilon)_{k=1,\dots,N}$ solves $(\text{AIP})_\varepsilon$, then $(\mathbf{y}_k)_{k=1,\dots,N}$ solves (RIP).

Our definition implies that the relaxed problem has to be of the same energetic kind as the original one; we just give up the clear distinction between $\boldsymbol{\varphi} \in \mathcal{F}$ and $z \in \mathcal{Z}$. Condition (R1) forces us to consider only useful relaxations, namely those which have solutions. If the original problem is already solvable, then we can choose $\mathbf{Y} = \mathcal{F} \times \mathcal{Z}$, $\mathbf{E} = \mathcal{E}$ and $\mathbf{D} = \mathcal{D}$, since no relaxation is necessary. Condition (R2) says that the new state space $\mathbf{Y}$ should not be unnecessarily big in the sense that every $\mathbf{y} \in \mathbf{Y}$ can be approximated by a sequence $(\boldsymbol{\varphi}^\varepsilon, z^\varepsilon)_{\varepsilon>0}$ of classical elements in $\mathcal{F} \times \mathcal{Z}$, i.e., $\mathcal{J}(\boldsymbol{\varphi}^\varepsilon, z^\varepsilon) \xrightarrow{\mathbf{Y}} \mathbf{y}$ for $\varepsilon \to 0$. Condition (R3) is very important as it says that the relaxation must maintain classical solutions, if they exist for (IP) or if they are found by solving (RIP). Conditions $(\text{R4})_{\text{low}}$ and $(\text{R4})_{\text{upp}}$ link the rate-independent evolution of $(\mathcal{F} \times \mathcal{Z}, \mathcal{E}, \mathcal{D})$ to that of $(\mathbf{Y}, \mathbf{E}, \mathbf{D})$ via the approximate incremental problem $(\text{AIP})_\varepsilon$.

Moreover, the relaxed incremental problem (RIP) can be interpreted as the incremental problem associated to the following relaxed energetic formulation of a rate-independent time-continuous problem: A function $\mathbf{y} : [0, T] \mapsto \mathbf{Y}$ is a solution of the *relaxed energetic problem* associated with $(\mathbf{Y}, \mathbf{E}, \mathbf{D})$, if $(\mathbb{S})$ and $(\mathbb{E})$ hold for all $t \in [0, T]$:

$$(\mathbb{S}) \quad \forall \tilde{\mathbf{y}} \in \mathbf{Y}, \quad \mathbf{E}\big(t, \mathbf{y}(t)\big) \leqslant \mathbf{E}(t, \tilde{\mathbf{y}}) + \mathbf{D}\big(\mathbf{y}(t), \tilde{\mathbf{y}}\big);$$

$$(\mathbb{E}) \quad \mathbf{E}\big(t, \mathbf{y}(t)\big) + \mathbf{Diss}\big(\mathbf{y}; [0, t]\big) = \mathbf{E}\big(0, \mathbf{y}(0)\big) + \int_0^t \partial_s \mathbf{E}\big(s, \mathbf{y}(s)\big)\, \mathrm{d}s,$$

where the relaxed dissipation $\mathbf{Diss}$ is calculated via the relaxed dissipation distance $\mathbf{D}$.

A further desirable property for relaxations is the consistency for the time continuous problem.

(R5) *Time-continuous consistency*: If $(\boldsymbol{\varphi}, z) : [0, T] \mapsto \mathcal{Y}$ solves (S) and (E), then $\mathcal{J} \circ y : [0, T] \mapsto \mathbf{Y}$ solves $(\mathbb{S})$ and $(\mathbb{E})$; and if $\mathbf{y} : [0, T] \mapsto \mathbf{Y}$ satisfies $\mathbf{y}(t) = \mathcal{J}(y(t))$ and solves $(\mathbb{S})$ and $(\mathbb{E})$, then $y : [0, T] \mapsto \mathcal{Y}$ solves (S) and (E).

The major question is how suitable relaxations can be constructed. This problem is still unsolved. Following the ideas in [MieTL02] the abstract setting in [Mie03b,Mie04a] suggest to do a separate relaxation for $\mathcal{E}$ and $\mathcal{D}$ independently and to use for $\mathbf{Y}$ the set of associated Young measures generated by the convergence "$\xrightarrow{\mathcal{Y}}$" in $\mathcal{Y}$. It is then easy to show that conditions (R1)–(R3) hold. However, proving the validity of $(\text{R4})_{\text{low}}$ or $(\text{R4})_{\text{upp}}$ is very difficult.

Another way to define relaxations for rate independent problems of the type (S) and (E) is proposed in [The02]. This definition avoids totally the usage of incremental problems but needs instead a sequence of approximation operators $\mathcal{S}_n : \mathbf{Y} \mapsto \mathcal{Y}$ such that:

(R.i) For all $y \in \mathcal{Y}$, we have $\mathcal{S}_n(\mathcal{J}(y)) \xrightarrow{\mathbf{Y}} y$ for $n \to \infty$.
(R.ii) For all $t \in [0, T]$ and all $\mathbf{y} \in \mathbf{Y}$, we have $\mathcal{E}(t, \mathcal{S}_n(\mathbf{y})) \to \mathbf{E}(t, \mathbf{y})$ for $n \to \infty$.
(R.iii) For all $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{Y}$, we have $\mathcal{D}(\mathcal{S}_n(\mathbf{y}_0), \mathcal{S}_n(\mathbf{y}_1)) \to \mathbf{D}(\mathbf{y}_0, \mathbf{y}_1)$ for $n \to \infty$.

An application of this theory to phase transformations in elastic solids (see also Section 7.3) is given in [The02], where it is also shown that for the problem under consideration the relaxation axioms (R1)–(R3), (R5) and, most important, $(\text{R4})_{\text{low}}$ are satisfied. See also [ConT05] for a successful relaxation in a special situation in an elastoplastic problem at finite strains.

Formally, the same ideas were used in [MiSL02,MieTL02,HacH03,MiL03,KruO04, RouK04], however, the proofs of the important condition (R4) is missing.

6. Nonassociated dissipation laws

The above energetic formulations have the major advantage that the dissipational forces are derived from the dissipation potential Ψ. In the more nonlinear setting the energetic formulation could be reduced to the stability condition (S) and the energy balance (E). In several applications the dissipational forces are more general and we replace the subdifferential $\partial_v \Psi(y, \dot{y})$ by a more general set $\mathcal{R}(y, \dot{y})$ of dissipation forces. However, we will stay in the framework of rate-independent systems, which means that $\mathcal{R}(y, \cdot)$ is homogeneous of degree 0.

Typical applications in mechanics occur in plastic behavior of materials, in particular, for soils (cf. [vVd99,CeDTV02]), and in Coulomb friction for elastic bodies, where the dissipation is proportional to the product of the modulus of the sliding velocity and the normal pressure. Several new phenomena occur in such problems and so far the theory is much less developed than for associated flow rules. New types of instabilities and bifurcations occur [MarMG94,MarK99,MarP00,MarR02] as well as ill-posedness [VP96]. Some positive results in nonassociated plasticity are obtained in [Mró63,BrKR98], but they are restricted to the finite-dimensional case of point mechanics.

6.1. *General setup*

We consider a reflexive Banach manifold $\mathcal{Y}$ and assume that, for each $t \in [0, T]$ and each $(y, \dot{y}) \in \mathrm{T}\mathcal{Y}$, a closed set $\mathcal{R}(t, y, \dot{y}) \subset \mathrm{T}^*_y\mathcal{Y}$ for the dissipational forces is given. Rate independence is encoded into the problem by the assumption that $\mathcal{R}(t, y, \cdot)$ is homogeneous of degree 0, i.e.,

$$\forall \gamma > 0, \forall (t, y, v) \in [0, T] \times \mathrm{T}\mathcal{Y}: \quad \mathcal{R}(t, y, \gamma v) = \mathcal{R}(t, y, v).$$

In the framework of multivalued mappings $\mathcal{R}(t,y,\cdot):\mathrm{T}_y\mathcal{Y}\to\mathcal{P}(\mathrm{T}^*_y\mathcal{Y})$ we always find an inverse operator $\mathcal{V}(t,y,\cdot):\mathrm{T}^*_y\mathcal{Y}\to\mathcal{P}(\mathrm{T}_y\mathcal{Y})$ such that

$$\sigma\in\mathcal{R}(t,y,v)\quad\Longleftrightarrow\quad v\in\mathcal{V}(t,y,\sigma).$$

Rate independence now means that each $\mathcal{V}(t,y,\sigma)$ is a cone, i.e., $\gamma>0$ and $v\in\mathcal{V}(t,y,\sigma)$ imply $\gamma v\in\mathcal{V}(t,y,\sigma)$.

Moreover, the state $y\in\mathcal{Y}$ and the process time $t\in[0,T]$ determine the set of reaction forces $\Sigma(t,y)\subset\mathrm{T}^*_y\mathcal{Y}$, which may also be multivalued. In the above energetic setting we obviously have $\mathcal{R}(t,y,v)=\partial_v\Psi(y,v)$, $\mathcal{V}(t,y,\sigma)=\partial\mathcal{L}(\Psi(y,\cdot))(\sigma)$ and $\Sigma(t,y)=-\mathrm{D}\mathcal{E}(t,y)$. The problem to be solved is now the following differential inclusion:

$$\begin{aligned}&\text{For given } y_0\in\mathcal{Y} \text{ find } y\in\mathrm{W}^{1,1}([0,T],\mathcal{Y}) \text{ with } y(0)=y_0 \text{ and}\\ &0\in\mathcal{R}(t,y(t),\dot y(t))-\Sigma(t,y(t))\subset\mathrm{T}^*_{y(t)}\mathcal{Y} \text{ for a.a. } t\in[0,T].\end{aligned}\tag{6.1}$$

Very often the forces are assumed to have the form $\Sigma(t,y)=\Sigma_0(y)+\ell(t)$, then (6.1) takes the more familiar form

$$\ell(t)\in\mathcal{R}(t,y,\dot y)-\Sigma_0(y)\subset\mathrm{T}^*_{y(t)}\mathcal{Y}.\tag{6.2}$$

Using the inverse $\mathcal{V}$ we can also write (6.1) as

$$\dot y(t)\in\mathcal{V}\big(t,y(t),\Sigma\big(t,y(t)\big)\big)\subset\mathrm{T}_{y(t)}\mathcal{Y},\tag{6.3}$$

where the composition of the multivalued maps $\mathcal{V}$ and Σ is defined via $\mathcal{V}(t,y,\Sigma(t,y)):=\{v\in\mathrm{T}_y\mathcal{Y}\mid\exists\sigma\in\Sigma(t,y)\colon\ v\in\mathcal{V}(t,y,\sigma)\}$.

A general theory for equations of the type (6.1) is not to be expected, since only additional structures will enable us to develop a suitable existence and uniqueness theory, see [AuC84]. One such structure arises from thermodynamics. The forces in $\mathcal{R}$ are called *dissipative* (also called pre-monotone in [Alb98,Che03]), if for all $r\in\mathcal{R}(t,y,v)$ we have $\langle r,v\rangle\geqslant0$. By 0-homogeneity of $\mathcal{R}$ we may assume that there exists a function $\Psi_{\mathrm{low}}:[0,T]\times\mathrm{T}\mathcal{Y}\to[0,\infty]$ which is 1-homogeneous and satisfies

$$\forall(t,y,v)\in[0,T]\times\mathrm{T}\mathcal{Y},\forall r\in\mathcal{R}(t,y,v),\quad\langle r,v\rangle\geqslant\Psi_{\mathrm{low}}(v).\tag{6.4}$$

If additionally Σ is obtained as a (sub)differential of an energy functional $\mathcal{E}$, i.e., $\Sigma(t,y)=\mathrm{D}\mathcal{E}(t,y)$, then any solution of (6.1) satisfies the energy inequality

$$\mathcal{E}\big(t,y(t)\big)+\int_0^t\Psi_{\mathrm{low}}\big(s,y(s),\dot y(s)\big)\,\mathrm{d}s\leqslant\mathcal{E}(0,y_0)+\int_0^t\partial_s\mathcal{E}\big(s,y(s)\big)\,\mathrm{d}s.$$

6.2. *Existence theory*

So far, the main approach to an existence theory for such problems is via the theory of monotone operators in Hilbert spaces or accretive operators on general Banach spaces. In

Theorem 2.3 we already gave one such result. If $\mathcal{Y}$ is a Hilbert space, $\mathcal{R}(y,v)=\partial\Psi(v)$ with $\Psi:Y\to[0,\infty]$ being 1-homogeneous and weakly continuous, and $\Sigma_0:Y\to Y^*$ is Lipschitz continuous and strongly monotone, then (6.2) has a solution for suitable initial data.

Note that the theory of monotone operators applied to (6.3) does not give anything new. In fact, if $\mathcal{M}:Y\to\mathcal{P}(Y)$ is a maximal monotone operator such that all sets $\mathcal{M}(y)$ are closed convex cones, then maximality implies that the set $K=\{y\in Y \mid 0\in\mathcal{M}(y)\}$ is convex and closed and equals $D(\mathcal{M})$. Moreover, for each $y\in D(\mathcal{M})$ monotonicity implies $\mathcal{M}(y)\subset \mathrm{N}_K(y)=\{v\in Y \mid \langle v|y-\hat{y}\rangle\geqslant 0 \text{ for all } \hat{y}\in K\}$. Hence, maximality implies $\mathcal{M}=\partial I_K$. See [Alb98], Chapter 7, for rate-independent material models, which can be transformed into this setting.

Here we want to discuss a more general result which is based on [Gui00,Che03]. If $y=(\varphi,z)\in F\times Z=Y$ with an elastic, dissipationless part φ, then usually $\mathcal{R}$ takes the form $\mathcal{R}(\varphi,z,\dot\varphi,\dot z)=\{0\}\times\mathcal{R}_z(z,\dot z)\subset F^*\times Z^*$ and $\Sigma(t,\varphi,z)=\binom{\Sigma_\varphi(t,\varphi,z)}{\Sigma_z(t,\varphi,z)}$. Using the inverse $\mathcal{V}_z$ of $\mathcal{R}_z$, (6.1) may be written in the explicit form

$$0=\Sigma_\varphi(t,\varphi,z)\in F^*,\qquad 0\in\dot z+\mathcal{V}_z\big(z,\Sigma_z(t,\varphi,z)\big)\subset Z^*,$$

see [Che03], equation (CC). Assuming further that $\Sigma_\varphi(t,\varphi,z)=0$ can be solved uniquely for $\varphi=\phi(t,z)$, we may insert this into the second equation and we are left with a general differential inclusion

$$0\in\dot z(t)+B\big(t,z(t)\big)\subset Z^*,\quad z(0)=z_0. \tag{6.5}$$

This is a generalized form of (DI) (cf. (2.3)) which reads $0\in\dot y+\partial I_{-C_*}(Ay-\ell(t))$. Moreover, the equation also includes equations of the type $0\in\partial\Psi(\dot y)+\Sigma_0(y)-\ell(t)$, which were treated in Theorem 2.3. For this, just use the Legendre transform to obtain $0\in\dot y+\partial I_{-C_*}(\Sigma_0(y)-\ell(t))$. A closely related result was provided in [KuM97].

The following result is the abstract version of [Che03], Theorem 2.6.

THEOREM 6.1. *Let Z be a Hilbert space and C_* a closed convex subset of Z^*. Moreover, assume that $\Phi:Z^*\to Z^*$ is a $\mathrm{C}^{1,\mathrm{Lip}}$ diffeomorphism* (*i.e., Φ, Φ^{-1}, $\mathrm{D}\Phi$ and $\mathrm{D}\Phi^{-1}$ exist and are globally Lipschitz continuous*). *Moreover, let $A:Z\to Z^*$ be bounded, symmetric and positive definite* (*as in Section* 2). *Finally assume $\ell\in\mathrm{C}^{1,\mathrm{Lip}}([0,T],Z^*)$. If B in* (6.5) *has the form $B(t,z)=\mathcal{B}(Az-\ell(t))$ with the multivalued map $\mathcal{B}(\sigma)=\mathrm{D}\Phi(\sigma)^*\,\partial I_{-C_*}(\Phi(\sigma))$, then* (6.5) *has for each z_0 with $0\in\Phi(Az_0-\ell(0))+C_*$ a unique solution $z\in\mathrm{C}^{\mathrm{Lip}}([0,T],Z)$.*

Note that the equation has the form $0\in\dot z+\mathrm{D}\Phi(\sigma)^*\,\partial I_{-C_*}(\Phi(\sigma))$, where $\sigma=Az-\ell(t)$. Using $\Psi=\mathcal{L}I_{C_*}$ this can be rewritten by the Legendre transform as

$$-\Phi(\sigma)\in\partial\Psi\big(\mathrm{D}\Phi(\sigma)^{-*}\dot z\big)\quad\Longleftrightarrow\quad 0\in\mathcal{R}(\sigma,\dot z)+\sigma$$

with

$$\mathcal{R}(\sigma,v)=\Phi^{-1}\big(\partial\Psi\big(\mathrm{D}\Phi(\sigma)^{-*}v\big)\big)\quad\text{and}\quad \sigma=Az-\ell(t).$$

We see here, that $\mathcal{R}(\sigma, v)$ is obtained by applying Φ^{-1} to the convex set $\partial\Psi(\mathrm{D}\Phi(\sigma)^{-*}v)$, which means that $\mathcal{R}(\sigma, v)$ is not convex in general.

The *Skorokhod problem* forms another class of rate-independent systems with non-associated flow rules, see [KreV01,KreV03]. It is classically formulated in a Hilbert space using its scalar product, however, to stay consistent with the previous formulations we use our general notation where Y is a Hilbert space with dual Y^* and $A: Y \to Y^*$ is a positive definite isomorphism. As in (2.29) we start with a polyhedral convex set

$$C_* = \left\{\sigma \in Y^* \;\middle|\; \forall j = 1, \dots, K\colon \langle\sigma, n_j\rangle \leqslant \beta_j\right\} \subset Y^*,$$

with $\beta_j \geqslant 0$ and normal vectors $n_j \in Y\backslash\{0\}$.

In contrast to the classical subdifferential equation (SF) or the classical differential inclusion (DI), $\dot{y} \in \mathrm{N}_{C_*}(\ell(t) - Ay)$ which is treated in Theorem 2.8, we do not use the friction law $\partial(\mathcal{L}I_{C_*})$ but generalize it as follows. We define the multivalued operator $\mathcal{J}: Y^* \to \mathcal{P}(\{1, \dots, K\})$ of active indices via

$$\mathcal{J}(\sigma) = \left\{j \in \{1, \dots, K\} \;\middle|\; \langle\sigma, n_j\rangle = \beta_j\right\} \quad \text{for } \sigma \in C_* \quad \text{and}$$
$$\mathcal{J}(\sigma) = \emptyset \quad \text{for } \sigma \notin C_*.$$

Moreover, the reflection cone $\mathcal{V}: Y^* \to \mathcal{P}(Y)$ is given via vectors $m_1, \dots, m_K \in Y\backslash\{0\}$ as

$$\mathcal{V}(\sigma) = \left\{\sum_{j\in\mathcal{J}(\sigma)} \mu_j m_j \;\middle|\; \mu_j \geqslant 0\right\}.$$

Note that $m_j = n_j$ for all j implies $\mathcal{V}(\sigma) = \partial I_{C_*}(\sigma)$. The inverse $\mathcal{R}$ of $\mathcal{V}$ reads

$$\mathcal{R}(v) = \left\{\sigma \in C_* \;\middle|\; \exists\mu_j \geqslant 0\colon\ v = \sum_{j\in\mathcal{J}(\sigma)} \mu_j m_j\right\}.$$

The Skorokhod problem can now be written in the following two equivalent and dual forms:

$$0 \in \mathcal{R}\big(\dot{y}(t)\big) + Ay(t) - \ell(t) \subset Y^* \quad \text{or} \quad \dot{y}(t) \in \mathcal{V}\big(\ell(t) - Ay(t)\big) \subset Y. \tag{6.6}$$

We give an example of such a system in the next subsection.

Without loss of generality it is possible to assume further on that Y is finite-dimensional and equal to $\mathrm{span}\{m_1, \dots, m_K, n_1, \dots, n_k\}$, since the A-orthogonal complement can be decoupled like at the end of Section 2.4. The crucial property which has to be satisfied by the vectors $\{m_1, \dots, m_K, n_1, \dots, n_k\}$ is

$$\langle Am_j, n_j\rangle > 0 \quad \text{for } j = 1, \dots, K, \tag{6.7}$$

and ℓ-paracontractivity. The set $\{Q_j \mid j = 1, \dots, K\}$ containing the projections

$$Q_j: Y \to Y, \qquad y \mapsto \frac{\langle Ay, n_j\rangle}{\langle Am_j, n_j\rangle} m_j,$$

is called ℓ-paracontracting, if there exist a norm $\|\cdot\|$ on Y and a constant $\gamma > 0$ such that

$$\forall y \in Y, \forall j = 1, \ldots, K: \quad \|Q_j y\| + \gamma \|Q_j y - y\| \leqslant \|y\|. \tag{6.8}$$

The following results are established in [KreV01], Theorems 3.1 and 5.8. Further results can be found in [KreV03], where the case of time-dependent β_j is considered.

THEOREM 6.2. *Let Y and $m_1, \ldots, m_K, n_1, \ldots, n_k$ be given as above and such that* (6.7) *and* (6.8) *hold. Then, for each $\ell \in \mathrm{W}^{1,1}([0,T], Y^*)$ and each $\sigma_0 \in C_*$, problem* (6.6) *has a solution y with $y(0) = A^{-1}(\ell(0) - \sigma_0)$ and $y \in \mathrm{W}^{1,1}([0,T], Y)$.*

Under the additional transversality condition

$$\forall J' \subset \{1, \ldots, K\}, \quad \dim \operatorname{span}\{n_j \mid j \in J'\} = \dim \operatorname{span}\{m_j \mid j \in J'\}$$

the solution is unique and $(\sigma_0, \ell) \mapsto y$ is Lipschitz continuous from $C_* \times \mathrm{W}^{1,1}([0,T], Y^*)$ into $\mathrm{W}^{1,1}([0,T], Y)$ as well as from $C_* \times \mathrm{C}^0([0,T], Y^*)$ into $\mathrm{C}^0([0,T], Y)$.

EXAMPLE 6.3. This simple example from queuing theory is taken from [KreV01], Section 8, and it has the form of a Skorokhod problem.

With w_{ord} and w_{priv} we denote the number of ordinary and privileged customers waiting at a service point whose set of possible states is

$$W = \{w = (w_{\text{ord}}, w_{\text{priv}}) \in [0,1]^2 \mid w_{\text{ord}} - w_{\text{priv}} \leqslant c_{\text{tot}}\},$$

where $c_{\text{tot}} > 0$ is the total capacity of the waiting room. The customers which arrived in the time interval $[0,t]$ is the input $\tilde{\ell}(t) = (\tilde{\ell}_{\text{ord}}(t), \tilde{\ell}_{\text{priv}}(t))$ and the number of customers which left the service point during $[0,t]$ is $\tilde{y}(t)$, served or refused because of missing capacity. Thus, we have $\tilde{y} + w = \tilde{\ell}$. There are the two counters: O for ordinary customers and P for privileged customers. The following service rules apply.

(i) All customers are served at their respective counters, which work with their maximal capacities c_{ord} and c_{priv}, respectively, as long as there are customers.

(ii) If there is unused capacity at counter O, then it can be used by privileged customers.

(iii) If the waiting room is full, then for each refused privileged customer, there must be at least ρ refused ordinary customers, where $\rho > 0$ is fixed.

Define $c = (c_{\text{ord}}, c_{\text{priv}})$ and the final variables $y(t) = \tilde{y}(t) - tc$ and $\ell(t) = \tilde{\ell}(t) - tc$, such that $w(t) = \ell(t) - y(t)$. The evolution of y can be formulated as

$$\dot{y}(t) \in B\big(\partial I_W[\ell(t) - y(t)]\big) \quad \text{for a.a. } t \in [0,T]; \; y(0) = y_0,$$

where B maps the cones $\{\alpha n_j \mid \alpha \geqslant 0\}$ into the cones $\{\beta m_j \mid \beta \geqslant 0\}$, where $n_1 = (-1, 0)$, $n_2 = (0, -1)$ and $n_3 = (1, 1)$ are the normal vectors at the edges of W and $m_1 = (-1, 0)$, $m_2 = (1, -1)$ and $m_3 = (\rho, 1)$ are the reflection vectors.

The above theorem can be applied to show that this problem has a unique solution for each $\ell \in \mathrm{C}^{\text{Lip}}([0,T], \mathbb{R}^2)$ and each $y_0 \in \ell(0) - W$.

6.3. *Dry friction on surfaces*

The most important problem with nonassociated flow law is that of dry friction of elastic bodies on surfaces. There are two mostly disjoint areas. The first case is the finite-dimensional one which involves a structure composed of rigid bodies which are connected with elastic interactions and may slide along given surfaces. The second case concerns an elastic body which touches a surface along a part of its boundary and which is assumed to have only small deformations such that linearized elasticity theory and linearized contact laws can be used. However, in both cases the difficulty arises that the tangential frictional force is proportional to the normal pressure. See [MarK99,AndK01,MarR02] for surveys in this area.

In the first case the state of the structure is given by an element y of a smooth, finite-dimensional manifold $\widetilde{\mathcal{Y}}$. The contact surfaces are modeled via smooth constraints $c_j : \widetilde{\mathcal{Y}} \to \mathbb{R}$, $j = 1, \ldots, p$, such that the state space is given by

$$\mathcal{Y} = \big\{ y \in \widetilde{\mathcal{Y}} \bigm| c_j(y) \leqslant 0 \text{ for } j = 1, \ldots, p \big\}.$$

We assume that the derivatives $\mathrm{D}c_j(y)$ do not vanish on the boundary pieces $\Gamma_j = \mathcal{Y} \cap \{c_j(y) = 0\}$. Hence, the (outward) unit normal vectors $n_j(y) \in \mathrm{T}^*_y\mathcal{Y}$ exist on Γ_j. Several of the sets Γ_j may intersect in a lower-dimensional manifold, which just means that several bodies of the structure are in contact.

The elastic interactions between the bodies are given through a smooth, time-dependent energy functional $\mathcal{E} : [0, T] \times \mathcal{Y} \to \mathbb{R}$. For simplicity, we assume that there are no frictional forces other than the one arising if $y(t)$ touches the boundary $\partial\mathcal{Y} = \Gamma = \bigcup_{j=1}^{p} \Gamma_j$. For $(y, v) \in \mathrm{T}_y\Gamma$ we denote by $\mathcal{R}(y, v)$ the set of possible reaction forces of the boundary at the given velocity v. For y in the interior $\mathrm{int}(\mathcal{Y}) = \mathcal{Y} \setminus \Gamma$ of $\mathcal{Y}$, we simply set $\mathcal{R}(y, v) = \{0\}$. Then, the rate-independent friction problem takes the form

$$0 \in \mathcal{R}\big(y(t), \dot{y}(t)\big) + \mathrm{D}\mathcal{E}\big(t, y(t)\big) \subset \mathrm{T}^*_{y(t)}\widetilde{\mathcal{Y}}, \tag{6.9}$$

and the friction law is implemented through specifying $\mathcal{R}$.

For each contact point $y \in \Gamma_j$ we specify a static friction cone $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}}$ which is closed, convex and contains $n_j(y)$. If a single body $y^j \in \mathbb{R}^d$ is in contact, this is usually done by decomposing the reaction forces $r^j \in \mathrm{T}^*_{y^j}\mathbb{R}^d$ into a tangential part r^j_{t} and a normal part $r^j_{\mathrm{n}} = \alpha n_j(y)$ and by setting

$$\mathcal{R}^j(y) = \big\{ r^j = r^j_{\mathrm{t}} + r^j_{\mathrm{n}} \bigm| \big|r^j_{\mathrm{t}}\big| \leqslant \mu^j(y) r^j_{\mathrm{n}} \big\} \subset \mathrm{T}^*_{y^j}\mathbb{R}^d,$$

where $\mu^j(y) \geqslant 0$ is the coefficient of (isotropic) friction for the jth body. To obtain now $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}}$ we simply fill in 0 for all reaction forces of the other bodies.

For situations in which $y \in \Gamma$ has several contacts, we make the assumptions that the different contacts do not influence each other. To describe this mathematically, we extend the vectors $n_j : \Gamma_j \to \mathrm{T}^*\mathcal{Y}$ and the cones $\mathcal{R}_j(y) \subset \mathrm{T}^*_y\mathcal{Y}$ to all of $\mathcal{Y}$ by 0 and $\{0\}$, respectively.

The tangential directions $\mathcal{T}(y)$ and the outward normal cones $\mathcal{N}(y)$ are

$$\mathcal{T}(y)=\left\{v\in \mathrm{T}_y\widetilde{\mathcal{Y}}\;\middle|\;\langle n_j(y),v\rangle=0 \text{ for } j=1,\dots,p\right\},$$
$$\mathcal{N}^*(y)=\left\{\sum_{j=1}^p \alpha_j n_j(y)\;\middle|\;\alpha_j\geqslant 0\right\}.$$

Additionally, we prescribe at each $y\in\Gamma$ a projection $P(y)$ which maps $\mathrm{T}_y\widetilde{\mathcal{Y}}$ onto $\mathcal{T}(y)$. The adjoint projector $P(y)^*$ has the kernel span$(\mathcal{N}(y))$ and it decomposes reaction forces $r\in\mathrm{T}_y^*\widetilde{\mathcal{Y}}$ into its tangential part $r_\mathrm{t}=P(y)^*r$ and its normal part $r_\mathrm{n}=r-r_\mathrm{t}\in\mathcal{N}(y)$. With this, we define the total static friction cone as the sum

$$\mathcal{R}^*(y)=\sum_{j=1}^p\mathcal{R}_j(y)=\left\{\sum_{j=1}^p r_j\;\middle|\;r_j\in\mathcal{R}_j(y)\right\}$$

of the cones $\mathcal{R}_j(y)$, which gives again a closed, convex cone with $\mathcal{N}^*(y)\subset\mathcal{R}^*(y)$, and the velocity-dependent friction cone via

$$\mathcal{R}(y,v)=\begin{cases}\left\{r\in\mathcal{R}^*(y)\;\middle|\;v\in P(y)\mathrm{N}_{\mathcal{R}^*(y)}(r)\right\} & \text{if } v\in\mathcal{T}(y),\\ \{0\} & \text{if } v\notin\mathcal{T}(y).\end{cases}$$

In particular, we have $\mathcal{R}(y,0)=\mathcal{R}^*(y)$ for the sticking particle. However, sliding can only occur in that direction where the critical tangential force (relative to the normal force) is reached.

The easiest example is $\widetilde{\mathcal{Y}}=\mathbb{R}^3$, $c_1(y)=y_3$ and $\mathcal{R}_1((y_1,y_2,0))=\{r\mid(r_1^2-r_2^2)^{1/2}\leqslant\mu r_3\}$ and gives the time-dependent friction cone

$$\mathcal{R}\big((y_1,y_2,0),v\big)=\begin{cases}\{0\} & \text{for } v_3\neq 0,\\ \mathcal{R}_1(y_1,y_2,0) & \text{for } v=0,\\ \big\{\alpha\big(-\mu v_1,-\mu v_2,|v|\big)\mid\alpha\geqslant 0\big\} & \text{for } v=(v_1,v_2,0) \text{ with } |v|>0.\end{cases}\tag{6.10}$$

Thus, it can be easily seen that there exists no $\Psi:\mathbb{R}^3\to[0,\infty]$ such that $\mathcal{R}(0,v)=\partial\Psi(v)$.

There is a substantial body of work for this type of finite-dimensional friction problems, however, in most cases the inertia terms are used to regularize the problem, i.e., an equation like $0\in M(t,y)\ddot{y}+\mathcal{R}(t,y,\dot{y})+\mathrm{D}\mathcal{E}(t,y)$ is considered. We refer to [MarK99, MarR02] for surveys and to [AndK97,MarPS02,PM03,MarR02,MarMP05] for some relevant mathematical work. In [GaMM98] it was shown that in quasistatic problems even in simple linear systems we have to expect jumps in the solution.

There is a way to reformulate the problem such that it almost looks like a rate-independent system with a dissipation potential. Using the decomposition $\mathrm{T}_y^*\widetilde{\mathcal{Y}}=\mathrm{span}(\mathcal{N}(y))\oplus\mathcal{T}^*(y)$ with $\mathcal{T}^*(y)=P(y)^*\mathrm{T}_y^*\widetilde{\mathcal{Y}}$ we define, for $y\in\mathcal{Y}$ and $r_n\in\mathcal{N}^*(y)$, the set of possible tangential forces via

$$\begin{aligned}C_*(y, r_\mathrm{n}) &= \big\{r_\mathrm{t} \in \mathcal{T}^*(y) \,\big|\, r_\mathrm{n} + r_\mathrm{t} \in \mathcal{R}^* y\big\}\\ &= P(y)^*\big(\mathcal{R}^*(y) \cap \big\{r \,\big|\, \big(\mathbf{1} - P(y)\big)^* r = r_\mathrm{n}\big\}\big).\end{aligned}$$

Using this set we use the Legendre transform on $\mathcal{T}(y)$ to define the dissipation functional $\Psi(y, r_\mathrm{n}, \cdot) : \mathrm{T}_y\widetilde{\mathcal{Y}} \to [0, \infty]$ via

$$\Psi(y, r_\mathrm{n}, v) = [\mathcal{L}I_{C_*(y,r_\mathrm{n})}]\big(P(y)v\big) = \sup\big\{\big\langle r_\mathrm{t}, P(y)v\big\rangle \,\big|\, r_\mathrm{t} \in C_*(y, r_\mathrm{n})\big\}.$$

For $r_\mathrm{n} \notin \mathcal{N}(y)$ we have $C_*(y, r_\mathrm{n}) = \emptyset$ and hence $\Psi(y, r_\mathrm{n}, \cdot) \equiv \infty$. Some elementary calculations show that if $v = P(y)v$, then $r \in \mathcal{R}(y, v)$ is equivalent to $r_\mathrm{t} \in \partial_v\Psi(y, r_\mathrm{n}, y)$. Thus, the friction laws are reduced to an associated flow law (a principle of maximal dissipation) in the tangential direction, if the normal forces are considered to be given.

Note that Ψ is defined for all velocities, but only the tangential part $v_\mathrm{t} = P(y)v$ contributes. Thus, $\partial\Psi$ always includes the whole space $\mathrm{span}(\mathcal{N}(y))$, which is useful in the following equivalent rewriting of (6.9)

$$\begin{aligned}&0 \in \partial\Psi\big(y(t), \sigma_\mathrm{n}(t), \dot{y}(t)\big) - \sigma(t) \subset \mathrm{T}^*_y\widetilde{\mathcal{Y}},\\ &\quad \text{where } \sigma(t) = -\mathrm{D}\mathcal{E}\big(t, y(t)\big) \text{ and } \sigma_\mathrm{n} = \big(\mathbf{1} - P(y)\big)^*\sigma.\end{aligned}$$

Thus, the structure is somewhat similar to the case of general dissipation metrics. However, the main difficulty coming into play here is that the function Ψ which is built using the normal vectors n and the projections P is not continuous. Whenever a new contact arises or a contact disappears, then there are jump discontinuities.

EXAMPLE 6.4. In [Mon93], Section 5.3, the following friction problem is solved. Let $\widetilde{\mathcal{Y}} = \mathbb{R}^3$ and $c_1(y) = y_3$ which gives $\mathcal{Y} = \{y \in \mathbb{R}^3 \mid y_3 \leqslant 0\}$. As an energy functional we choose $\mathcal{E}(t, y) = \frac{\alpha}{2}|y - \ell(t)|^2$, where $|\cdot|$ denotes the Euclidean norm. The friction law on $\Gamma = \partial\mathcal{Y} = \{y_3 = 0\}$ is defined via (6.10) and the friction coefficient function $\mu : \Gamma \to (0, \mu_{\max}]$ with $|\mu(y) - \mu(\tilde{y})| \leqslant \beta|y - \tilde{y}|$ for $y, \tilde{y} \in \Gamma$.

It is proved that the corresponding friction problem has for each loading $\ell \in \mathrm{W}^{1,1}([0, T]; \mathbb{R}^3)$ and each equilibrated (i.e., stable) state y^0 a solution y with $y(0) = y^0$ and $y \in \mathrm{W}^{1,1}([0, T]; \mathbb{R}^3)$, if additionally the smallness condition $\beta\|\ell_3(\cdot)\|_\infty < 1$ holds. Here a state y^0 is called equilibrated with $\ell(0)$, if for $\ell_3(0) \leqslant 0$ we have $y^0 = \ell(0)$ and for $\ell_3(0) > 0$ we have $y^0_3 = 0$ and $|y^0 - (\ell_1(0), \ell_2(0), 0)| \leqslant \ell_3(0)\mu(y^0)$.

Counterexamples to uniqueness and existence of solutions are given in [Kla90,Bal99], and [GaMM98,And95] provides examples in which the solutions are in general not continuous. However, general systems of the type described above need much further study.

The second class of friction problems involves a linearly elastic body, which may touch a surface with parts of its boundary. Throughout we assume small displacements since the general case seems out of reach at the present stage of research. The first major steps in this field were done in [DuL76], where the static problem was solved and simplified evolution variational inequalities were considered. The time-dependent problem was first studied including inertia terms and sometimes viscoelastic damping, which keep the solution from making undesirable jumps, see [MarO87,Kut97,Eck02,EckJ03]. Here we restrict ourselves to the rate-independent case, which is usually called the quasistatic case in contrast to the dynamic case.

The system consists of the elastic bulk energy $\frac{1}{2}\langle Au, u\rangle - \langle \ell(t), u\rangle$, where $A : Y \to Y^*$ is the usual symmetric elastic operator with $Y = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) = \{u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \mid u|_{\Gamma_D} = 0\}$, where $\Omega \subset \mathbb{R}^d$ is a domain with Lipschitz boundary and $\Gamma_\mathrm{D} \subset \partial\Omega$, and $\langle Au, u\rangle = \int_\Omega C_{ijkl}\,\partial_i u_j\,\partial_k u_l\,\mathrm{d}x \geqslant c\|u\|^2_{\mathrm{H}^1}$ for some $c > 0$. At a contact part $\Gamma_\mathrm{c} \subset \partial\Omega$, which has positive distance from Γ_D, the body may touch a given obstacle which is prescribed by the function $g : \Gamma_\mathrm{c} \to \mathbb{R}$. Let ν be the normal vector on $\partial\Omega$, then there is no contact, if the normal component $u_\mathrm{n} = u\cdot\nu$ satisfies $u_\mathrm{n} < g$. Contact means that the penetration depth $u_\mathrm{n} - g$ is nonnegative. Note that the tangential displacement $u_\mathrm{t} = u - u_\mathrm{n}\nu$ is not involved in the contact condition, since we are in a situation of small displacements.

The normal stress vector $\sigma \in \mathbb{R}^d$ and its normal and tangential components at a point $x \in \Gamma_\mathrm{c}$ are defined via

$$\sigma[u] = \Big(\sum_{ijk} C_{ijkl}\,\partial_i u_j\,\nu_k\Big)_{l=1,\dots,d},$$

$$\sigma[u]_\mathrm{n} = \sigma[u]\cdot\nu \quad \text{and} \quad \sigma[u]_\mathrm{t} = \sigma[u] - \sigma[u]_\mathrm{n}\nu.$$

In the case of normal compliance, one assumes that the penetration depth can become positive due to some elastic behavior of the obstacle. This induces a normal stress according to a compliance law

$$-\sigma[u]_\mathrm{n} = H(x, u_\mathrm{n} - g), \quad \text{where } H(x,\delta) = 0 \text{ for } \delta \leqslant 0.$$

Usually one chooses $H(x,\delta) = \lambda(x)\max\{0,\delta\}^m$ for suitable parameters $\lambda, m > 0$. Hard Coulomb friction is modeled via $H(\delta) = +\infty$ for $\delta > 0$. Associated with this elasticity law is the functional

$$\mathcal{H}(u) = \int_{\Gamma_\mathrm{c}} h\big(x, u(x) - g(x)\big)\,\mathrm{d}a(x), \quad \text{where } h(x,u) = \int_0^u H(x,\delta)\,\mathrm{d}\delta.$$

The total stored energy now defines the energy functional

$$\mathcal{E}_\mathcal{H}(t,u) = \frac{1}{2}\langle Au, u\rangle - \big\langle \ell(t), u\big\rangle + \mathcal{H}(u).$$

As in the rigid-body case, the friction law is now specified best by a local dissipation function ψ in the form

$$\psi(x,\sigma_\mathrm{n}, v) = \begin{cases} \infty & \text{for } \sigma_\mathrm{n} > 0, \\ -\sigma_\mathrm{n}\,\mu(x)|v_\mathrm{t}| & \text{for } \sigma_\mathrm{n} \leqslant 0. \end{cases}$$

The friction law now asks the stress vector σ and the velocity v to satisfy $\sigma_{\mathrm{t}} \in \partial\Psi(x, \sigma_{\mathrm{n}}, v)$.

Thus, the whole problem can be written as a variational inequality using the stress-dependent dissipation functional

$$\Psi(\sigma_{\mathrm{n}}; v) = \int_{\Gamma_{\mathrm{c}}} \psi\big(x, \sigma_{\mathrm{n}}(x), v(x)\big)\, \mathrm{d}a(x)$$

in the following way,

$$\begin{aligned} &\big\langle \mathrm{D}\mathcal{E}_{\mathcal{H}}(t,u), v - \dot{u}\big\rangle + \Psi\big(\sigma[u]_{\mathrm{n}}; v\big) - \Psi\big(\sigma[u]_{\mathrm{n}}; \dot{u}\big) \geqslant 0 \\ &\qquad \text{for all } v \in Y = \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega). \end{aligned} \tag{6.11}$$

This model with $H(\delta) = \lambda \max\{0, \delta\}^m$ and $m \in [1, \frac{d}{d-2})$ was treated in a series of papers [And91,And95,And99]. Under the assumption of small friction ($\|\mu\|_{\mathrm{L}^\infty} \ll 1$) existence of solutions is shown. The approach follows exactly the one explained in Section 3.6 for general state-dependent dissipation metrics. The smallness of the friction coefficient corresponds to the smallness of ψ_* in (3.16), which controls the deviation from the convex part obtained from the energy. Roughly spoken, the result is the one which one expects, namely that for each loading $\ell = (f_{\mathrm{vol}}, f_{\mathrm{surf}}) \in \mathrm{W}^{1,1}([0,T]; \mathrm{L}^2(\Omega; \mathbb{R}^d) \times \mathrm{H}^{-1/2}(\partial\Omega; \mathbb{R}^d))$ with $\ell(0) = 0$ there exists a function $u \in \mathrm{W}^{1,1}([0,T]; Y)$ with $u(0) = 0$ such that (6.11) holds a.e. on $[0, T]$.

The case of a real hard unilateral constraint with $h(\delta) = \infty$ for $\delta > 0$ is handled in [And00], again using the smallness of the friction coefficient μ. Studying the solutions of the compliance problem (6.11) for the compliance parameter λ tending to ∞, it is shown that the Coulomb friction problem has also a solution. Defining

$$K_g = \big\{u \in Y \mid u_{\mathrm{n}}|_{\Gamma_{\mathrm{c}}} \leqslant g\big\} \quad \text{and} \quad \mathcal{E}_0(t,u) = \frac{1}{2}\langle Au, u\rangle - \big\langle \ell(t), u\big\rangle,$$

the variational inequality now reads with $Y = \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$,

$$\begin{aligned} \forall v \in Y, \quad &\big\langle \mathrm{D}\mathcal{E}_0(t,u), v - \dot{u}\big\rangle - \big\langle \sigma_{\mathrm{n}}[u], v_{\mathrm{n}} - \dot{u}_{\mathrm{n}}\big\rangle \\ &- \Psi\big(\sigma[u]_{\mathrm{n}}; v\big) - \Psi\big(\sigma[u]_{\mathrm{n}}; \dot{u}\big) \geqslant 0, \\ \forall w \in K_g, \quad &\big\langle \sigma_{\mathrm{n}}[u], w_{\mathrm{n}} - u_{\mathrm{n}}\big\rangle \geqslant 0. \end{aligned} \tag{6.12}$$

In fact, both inequalities can be put into one equation by introducing the potential $\mathcal{E}_\infty(t,y) = \mathcal{E}_0(t,u) + I_{K_g}(u)$,

$$0 \in \partial\Psi\big(\sigma[u]_{\mathrm{n}}; \dot{u}\big) + \partial\mathcal{E}_\infty(t,u).$$

After doing some slight modifications and specifying the assumptions fully, it is shown in [And00] that (6.12) has for each $\ell = (f_{\mathrm{vol}}, f_{\mathrm{surf}})$ and for each suitable initial data a solution $u \in \mathrm{W}^{1,1}([0,T], Y)$ with $u(t) \in K_g$ for all $t \in [0, T]$.

The case of large friction coefficient μ and normal compliance is handled in [And95]. For this the solution concept needs to be modified, since solutions will no longer be continuous and the variational inequality (6.11) has to be replaced by a more energetic formulation.

7. Applications to continuum mechanics

To unify the presentation of the applications in continuum models we refrain from the full generality and restrict ourselves to standard situations like, for instance, simple (dead) loadings and time-independent Dirichlet boundary conditions. For time-dependent boundary conditions we refer to [FM05], where they are treated with similar ideas as explained in Section 5.5.

Throughout we will consider a body $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$, which is open, bounded and has a Lipschitz boundary such that integration by parts and Sobolev embeddings are available. The deformation is $\varphi : \Omega \to \mathbb{R}^d$ and we will use $u : \Omega \to \mathbb{R}^d; x \mapsto \varphi(x) - x$ to denote the displacement in the case of linearized elasticity. For the general situation we use $y : \Omega \to \mathbb{R}^d$ to denote φ or u. In addition, there will be an internal variable $z : \Omega \to Z \subset \mathbb{R}^m$. The two constitutive functions are the stored-energy density (stress potential) $W : \Omega \times \mathbb{R}^{d\times d} \times Z \to \mathbb{R}_\infty$ and the dissipation potential $\psi : \Omega \times \mathrm{T}Z \to [0, \infty]$. The latter generates the dissipation distance $D : \Omega \times Z \times Z \to [0, \infty]$ such that the functionals have the form

$$
\begin{aligned}
\mathcal{E}(t, y, z) &= \int_\Omega W\big(x, \mathrm{D}y(x), z(x)\big) + \frac{\kappa}{r}\big|\mathrm{D}z(x)\big|^r \,\mathrm{d}x - \big\langle \ell(t), y\big\rangle, \\
\mathcal{D}(z_0, z_1) &= \int_\Omega D\big(x, z_0(x), z_1(x)\big)\,\mathrm{d}x,
\end{aligned}
\tag{7.1}
$$

where $t \mapsto \ell(t)$ denotes the loading which is considered as input data. We add the regularizing term $\frac{\kappa}{r}|\mathrm{D}z|^r$ with suitable $r > 1$ which is also called “nonlocal” in mechanics terminology. For $\kappa > 0$ it provides helpful compactness properties.

Throughout the following subsections we will assume that the deformations or displacements are taken from a space $F = \mathrm{W}^{1,p}_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ (denoted $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ for $p = 2$), where $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ is such that in the case of linearized elasticity Korn’s inequality holds in F, i.e., there exists a constant $c > 0$ such that

$$
\forall u \in F, \quad \big\|\varepsilon(u)\big\|_{\mathrm{L}^p} \geqslant c\|u\|_{\mathrm{W}^{1,p}}, \quad \text{where } \varepsilon(u) = \frac{1}{2}\big(\mathrm{D}u + (\mathrm{D}u)^{\mathsf{T}}\big).
$$

Here, $\varepsilon(u) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is called the linearized strain tensor. In the case of finite elasticity, we only impose Poincaré’s inequality

$$
\forall \varphi \in F, \quad \|\mathrm{D}\varphi\|_{\mathrm{L}^p} \geqslant c\|\varphi\|_{\mathrm{W}^{1,p}}.
$$

The different applications below differ in the form of the variable z and in the nonlinearities or nonconvexities in the functions W and D. For instance, in elastoplasticity z will contain a plastic tensor in $\mathbb{R}^{d\times d}$ as well as hardening variables (Sections 7.1 and 7.2), in shape-memory materials $z \in \{z \in [0,1]^m \mid \sum_1^m z^{(j)} = 1\}$ contains the portions of the phases (Section 7.3), in ferromagnetic materials $z \in \mathbb{S}^{d-1}$ is the magnetization (Section 7.4), and in damage problems $z \in [0,1]$ denotes the proportion of intact material [MieRou05]. There are also applications, where z is not defined on all of Ω but only on hypersurfaces like in delamination (Section 7.5) and in crack propagation (Section 7.6).

7.1. *Linearized elastoplasticity*

The theory of linearized elastoplasticity has been the major driving force for the theory of rate-independent systems since the mid-1970s. So we give the main structure of the theory, but refer to the huge list of references for further details, see, e.g., [HanR99] for a recent monograph.

A bounded body $\Omega \subset \mathbb{R}^d$ is subject to small deformations which are described by the displacement $u : \Omega \to \mathbb{R}^d$. As internal variables we have $z = (\varepsilon_{\text{pl}}, q)$, where $\varepsilon_{\text{pl}} \in \mathbb{R}_0^{d\times d} = \{A \in \mathbb{R}^{d\times d} \mid \operatorname{tr} A = 0,\ A = A^{\mathsf{T}}\}$ is the plastic strain and $q \in \mathbb{R}^n$ denotes hardening variables. The stored-energy functional has the form

$$\mathcal{E}(t,u,\varepsilon_{\text{pl}},q) = \int_\Omega \frac{1}{2}\mathbb{A}\big(\varepsilon(u)-\varepsilon_{\text{pl}}\big):\big(\varepsilon(u)-\varepsilon_{\text{pl}}\big) + Q(\varepsilon_{\text{pl}},q)\,\mathrm{d}x - \big\langle \ell(t),u\big\rangle,$$

where $\mathbb{A}$ is the (fourth-order) elastic tensor and $Q : \mathbb{R}_0^{d\times d} \times \mathbb{R}^n \to [0,\infty)$ is a quadratic form which describes hardening effects.

The dissipation functional Ψ takes the form $\Psi(\dot\varepsilon_{\text{pl}}, \dot q) = \int_\Omega \psi(\dot\varepsilon_{\text{pl}}(x), \dot q(x))\,\mathrm{d}x$, where $\psi : \mathbb{R}_0^{d\times d} \times \mathbb{R}^n \to [0,\infty)$ is convex, 1-homogeneous and coercive.

Under the additional assumption that there is enough hardening, i.e., the quadratic form Q is coercive, it is quite standard to apply the existence results of Section 2.3. For this we choose

$$Y = \mathrm{H}^1_{\Gamma_{\text{Dir}}}\big(\Omega;\mathbb{R}^d\big) \times \mathrm{L}^2\big(\Omega;\mathbb{R}_0^{d\times d}\big) \times \mathrm{L}^2\big(\Omega;\mathbb{R}^n\big).$$

However, in most plasticity models the hardening is weaker. In particular, we have $Q(\varepsilon_{\text{pl}},q) = Q(0,q)$, such that the energy is not coercive in the variables $\varepsilon(u)$ and ε_{pl}, but only in their difference $\varepsilon(u)-\varepsilon_{\text{pl}}$. In this situation, the dissipation can be used to control ε_{pl} as well, but more careful bookkeeping is necessary, see [Joh76,Suq81,HanR95,AlbeC00]. To explain the general approach we restrict ourselves to the case of von Mises plasticity, where q is a scalar hardening variable and the dissipation potential takes the form

$$\psi(\dot\varepsilon_{\text{pl}},\dot q) = s_2|\dot\varepsilon_{\text{pl}}| \quad \text{for } \dot q \geqslant s_1|\dot\varepsilon_{\text{pl}}| \quad \text{and} \quad \psi(\dot\varepsilon_{\text{pl}},\dot q) = \infty \quad \text{else}.$$

The elastic domain is $C_* = \partial\psi(0) = \{(\sigma,r) \in \mathbb{R}_0^{d\times d} \times \mathbb{R} \mid r \leqslant 0,\ |\sigma| + s_1 r \leqslant s_2\}$. The energy density is given as above with $Q(\varepsilon_{\text{pl}},q) = \frac{s_3}{2}q^2$. Here all the constants s_j are strictly positive. The dissipation distance $\mathcal{D}$ is given via $\mathcal{D}((\varepsilon^0_{\text{pl}},q^0),(\varepsilon^1_{\text{pl}},q^1)) = \Psi((\varepsilon^1_{\text{pl}},q^1) - (\varepsilon^0_{\text{pl}},q^0))$.

The arising difficulty is that neither the dissipation functional Ψ nor the stored energy density are coercive in the sense assumed in the abstract Section 3. However, the sum of stored and dissipated energies is coercive, namely for each $(\varepsilon_{\mathrm{pl}}^0, q^0)$ the mapping

$$\mathcal{K}_{(\varepsilon_{\mathrm{pl}}^0,q^0)} : (u, \varepsilon_{\mathrm{pl}}, q) \mapsto \mathcal{E}(t, u, \varepsilon_{\mathrm{pl}}, q) + \Psi\big((\varepsilon_{\mathrm{pl}}, q) - \big(\varepsilon_{\mathrm{pl}}^0, q^0\big)\big)$$

satisfies either $\mathcal{K}_{(\varepsilon_{\mathrm{pl}}^0,q^0)}(u, \varepsilon_{\mathrm{pl}}, q) = \infty$ or

$$\begin{aligned}&\mathcal{K}_{(\varepsilon_{\mathrm{pl}}^0,q^0)}(u, \varepsilon_{\mathrm{pl}}, q)\\ &\quad \geqslant \frac{a}{2}\big\|\varepsilon(u) - \varepsilon_{\mathrm{pl}}\big\|_{\mathrm{L}^2}^2 - \big\|\ell(t)\big\|_{\mathrm{H}^{-1}}\|u\|_{\mathrm{H}^1} + \frac{s_3}{2}\|q\|_{\mathrm{L}^2}^2 + s_2\big\|\varepsilon_{\mathrm{pl}} - \varepsilon_{\mathrm{pl}}^0\big\|_{\mathrm{L}^1}\end{aligned}$$

if $s_1|\varepsilon_{\mathrm{pl}}(x) - \varepsilon_{\mathrm{pl}}^0(x)| \leqslant |q(x) - q^0(x)|$ for a.e. $x \in \Omega$. Using this pointwise constraint and Korn's inequality on $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$, it is then easy to find constants $c, C^0 > 0$ such that

$$\mathcal{K}_{(\varepsilon_{\mathrm{pl}}^0,q^0)}(u, \varepsilon_{\mathrm{pl}}, q) \leqslant c\big(\|u\|_{\mathrm{H}^1}^2 + \|\varepsilon_{\mathrm{pl}}\|_{\mathrm{L}^2}^2 + \|q\|_{\mathrm{L}^2}^2\big) - C^0.$$

Here C^0 depends only on $(\varepsilon_{\mathrm{pl}}^0, q^0)$ and $\ell(t)$.

Thus, we choose the underlying space $\mathcal{Y} = Y = \mathrm{H}^1(\Omega; \mathbb{R}^d) \times \mathrm{L}^2(\Omega; \mathbb{R}_0^{d\times d}) \times \mathrm{L}^2(\Omega)$, which makes $\mathcal{E}$ and Ψ weakly lower semicontinuous due to convexity. Fixing an initial datum $(u^0, \varepsilon_{\mathrm{pl}}^0, q^0) \in Y$, the abstract condition (A2) can be replaced by

$$\begin{aligned}\exists c_E^{(0)}, c_E^{(1)} > 0: \quad &\mathcal{E}(t, \varepsilon_{\mathrm{pl}}, q) < \infty \quad \text{and} \quad \Psi\big((\varepsilon_{\mathrm{pl}}, q) - \big(\varepsilon_{\mathrm{pl}}^0, q^0\big)\big) < \infty\\ &\quad \Longrightarrow \quad \big|\partial_t\mathcal{E}(t, u, \varepsilon_{\mathrm{pl}}, q)\big| \leqslant c_E^{(1)}\big(c_E^{(0)} + \mathcal{E}(t, u, \varepsilon_{\mathrm{pl}}, q)\big).\end{aligned}$$

Moreover, since $\partial_t\mathcal{E}(t, u, \varepsilon_{\mathrm{pl}}, q) = -\langle\dot{\ell}(t), u\rangle$ is linear in u, it is weakly continuous. Finally, we can use the quadratic nature of $\mathcal{E}$ to show that the stable sets are convex and strongly closed, and hence weakly closed. Thus, existence can be deduced from Theorem 5.2 using assumption (A7).

The case without any hardening, i.e., $s_3 = 0$, is also called perfect plasticity. In this case no a priori bounds in L^2-spaces are possible, but the dissipation provides L^1-bounds. Since this space is not weakly closed, one is led to consider $\varepsilon_{\mathrm{pl}}$ as a bounded measure and u in the set of *bounded deformations*, where $\varepsilon(u)$ is a bounded measure, see [TeS80,Suq81,EbR04, DalDM04].

For the further rich literature in linearized elastoplasticity we refer to [Alb98,HanR99] and the references therein.

7.2. *Finite-strain elastoplasticity*

While the theory of linearized elastoplasticity is well developed in terms of existence and uniqueness results and also provides reliable and efficient finite-element discretizations, there is a big lack of theory for the case of finite-strain elastoplasticity.

The reason is that the linearized theory is based on the additive decomposition

$$\varepsilon = \frac{1}{2}\big(\mathrm{D}u + \mathrm{D}u^{\mathsf{T}}\big) = \varepsilon_{\mathrm{elast}} + \varepsilon_{\mathrm{plast}},$$

which is well suited for methods in linear functional analysis, whereas the finite-strain theory is based on the *multiplicative decomposition*

$$F = \mathrm{D}y = F_{\mathrm{elast}} P \quad \text{with } P = F_{\mathrm{plast}}. \tag{7.2}$$

The main feature here is that the nonlinearities arise from the multiplicative group of invertible matrices. (In finite-strain elasticity this is also called geometrically nonlinear elasticity.) The main open question is to understand the interaction of functional analytical tools, mainly based on linear function spaces, and these algebraic nonlinearities.

At the present stage there are only very little results in this direction. In [Nef03] a local existence result for a viscously regularized director model is obtained. For the rate-independent setting there is a series of negative result, in the sense that it is shown that the incremental problem (IP) studied in Section 3 does not have solutions in general, see [OrtR99,OrtRS00,CaHM02,HacH03,Mie03a]. This mathematical difficulty is also observed in experiments where the dislocations accumulate on interfaces which have microstructure. Here we want to address some results in the positive direction, namely where it is possible to establish existence of solutions for (IP) for an arbitrary number of steps, see [Mie02,Mie03a]. However, the limit for the time step tending to 0 is not yet understood.

To be more specific, let $y : \Omega \to \mathbb{R}^d$ be the deformation of the body $\Omega \subset \mathbb{R}^d$. The energy $\mathcal{E}$ stored in a deformed body depends only on the elastic strain $F_{\mathrm{elast}} = \mathrm{D}y P^{-1}$ of the deformation tensor and on suitable hardening variables $p \in \mathbb{R}^n$. But it is not allowed to depend on the plastic strain $P = F_{\mathrm{plast}}$, which is contained in $\mathrm{SL}(\mathbb{R}^d) = \{P \mid \det P = 1\}$ or another Lie group $\mathfrak{G}$ contained in $\mathrm{GL}_+(\mathbb{R}^d) = \{G \in \mathrm{Lin} \mid \det G > 0\}$. The energy functional takes the form

$$\mathcal{E}\big(t, y, (P, p)\big) = \int_\Omega W\big(x, \mathrm{D}y(x) P(x)^{-1}, p(x)\big)\, \mathrm{d}x - \big\langle \ell(t), y \big\rangle$$

with external loading $\langle \ell(t), y\rangle = \int_\Omega f_{\mathrm{ext}}(t,x) \cdot y(x)\, \mathrm{d}x + \int_\Gamma g_{\mathrm{ext}}(t,x) \cdot y(x)\, \mathrm{d}a$.

To model the plastic effects one prescribes either a plastic flow law or, equivalently, a dissipation potential $\psi : \Omega \times \mathrm{T}(\mathfrak{G} \times \mathbb{R}^m) \to [0, \infty]$, which generates the global dissipation distance $D(x, \cdot, \cdot)$ on $\mathfrak{G} \times \mathbb{R}^m$. Thus, the second ingredient of the material model is the dissipation distance between two internal states $z_j = (P_j, p_j) : \Omega \to \mathrm{SL}(d) \times \mathbb{R}^m$

$$\mathcal{D}(z_1, z_2) = \int_\Omega D\big(x, \big(P_1(x), p_1(x)\big), \big(P_2(x), p_2(x)\big)\big)\, \mathrm{d}x.$$

The main assumption in plasticity theory is that the actual plastic tensor P does not appear in the constitutive laws. Changes of P can only influence the forces through the hardening variable. Thus, the dissipation potential ψ and the dissipation distance D have to satisfy *plastic invariance*, namely

$$\psi\big(x,(P,p),(\dot P,\dot p)\big)=\psi\big(x,(\mathbf{1},p),(\dot P\,P^{-1},\dot p)\big),$$
$$D\big(x,(P_1,p_1),(P_2,p_2)\big)=D\big(x,(\mathbf{1},p_1),(P_2P_1^{-1},p_2)\big).$$

This symmetry leads naturally to a logarithmic behavior of D which contradicts any convexity properties.

Allowing for finite strains we are forced to abolish convexity of the stored-energy density W. Instead it has to be polyconvex or quasiconvex and frame indifferent, see [Bal77]. These notions work well together with the philosophy that $F=\mathrm{D}y$ is an element of $\mathrm{GL}_+(\mathbb{R}^d)$, i.e., we set $W(F)=\infty$ for $\det F\leqslant 0$.

The associated incremental problem (IP) has the form

$$\begin{aligned}&(y_k,P_k,p_k)\\&\quad\in \operatorname{Arg\,min}\big\{\mathcal{E}(t_k,y,P,p)+\mathcal{D}\big((P_{k-1},p_{k-1}),(P,p)\big)\;\big|\;y\in\mathcal{F},(P,p)\in\mathcal{Z}\big\},\end{aligned}$$

where the spaces $\mathcal{F}$ and $\mathcal{Z}$ still need to be specified. Typical choices are $\mathcal{F}=y_{\mathrm{Dir}}+\mathrm{W}^{1,q_y}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d)$ and $\mathcal{Z}$ is a subset of $\mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d})\times\mathrm{L}^{q_p}(\Omega;\mathbb{R}^m)$. Under suitable assumptions on W and D it is then possible to prove coercivity, but there is no hope to proof weak lower semicontinuity in the variable P.

The crucial observation in [CaHM02,Mie03a,Mie04b] is that weak lower semicontinuity is by far not needed to prove existence of minimizers for (IP). The point is that $z=(P,p)$ appears only pointwise under the integral. Thus, the minimization can be done pointwise in $x\in\Omega$. This leads to the *condensed energy density* W^{cond} as defined in Section 3.4,

$$\begin{aligned}&W^{\mathrm{cond}}(P_{\mathrm{old}},p_{\mathrm{old}};x,F)\\&\quad=\min\big\{W\big(x,FP^{-1},p\big)+D\big(x,(P_{\mathrm{old}},p_{\mathrm{old}}),(P,p)\big)\;\big|\;(P,p)\in\mathfrak{G}\times\mathbb{R}^m\big\}.\end{aligned}$$

Plastic invariance is inherited in the form

$$W^{\mathrm{cond}}(P_{\mathrm{old}},p_{\mathrm{old}};x,F)=W^{\mathrm{cond}}\big(\mathbf{1},p_{\mathrm{old}};x,FP_{\mathrm{old}}^{-1}\big).$$

In [Mie04b] it is shown that under the usual technical assumptions on W and D the solvability of (IP) can be shown, if additionally the following conditions hold:

$$\begin{aligned}&\text{(a)}\quad W^{\mathrm{cond}}\big((\mathbf{1},p_*);x,\cdot\big):\mathbb{R}^{d\times d}\to\mathbb{R}_\infty\quad\text{is polyconvex},\\&\text{(b)}\quad W^{\mathrm{cond}}\big((\mathbf{1},p_*);F\big)\geqslant c|F|^{q_F}-C\quad\text{and}\\&\qquad\; D\big((\mathbf{1},p_*),(P,p)\big)\geqslant c|P|^{q_P}-C,\\&\text{(c)}\quad \frac{1}{q_F}+\frac{1}{q_P}\leqslant\frac{1}{q_y}<\frac{1}{d}.\end{aligned}\tag{7.3}$$

The major problem with these assumptions is that in practice only W and ψ are given and W^{cond} and D are defined implicitly. In particular, condition (7.3)(a) is difficult to check. By now, only one nontrivial model is known, which is only two-dimensional, see [Mie04b], Section 4. On the one-dimensional setting with $\mathfrak{G} = \mathrm{GL}_+(\mathbb{R}^1) = (0, \infty)$ the solvability of (IP) is rather straightforward, since polyconvexity equals convexity. Even the convergence of the solutions of (IP) to solutions of (S) and (E) can be shown, see [Mie04b], Sections 5 and 6.

In microscopical models for finite-strain elastoplasticity it is often desirable to introduce regularizing (also called nonlocal) terms, which generate a small length scale which stops formation of microstructure. One typical term of this kind is $(\mathrm{curl} P)P^{\mathsf{T}}$, which is also called the dislocation-density tensor. In [MieM05] it is shown that the functional

$$(y, P) \mapsto \int_\Omega W\big(\mathrm{D}y P^{-1}\big) + D(P) + \frac{\kappa}{r}\big|(\mathrm{curl} P)P^{\mathsf{T}}\big|^r \,\mathrm{d}x$$

is weakly lower semicontinuous on $\mathrm{W}^{1,q_y}(\Omega; \mathbb{R}^d) \times \mathrm{L}^{q_P}(\Omega; \mathrm{SL}(\mathbb{R}^d))$, if both, W and D, are polyconvex. Based on this result, existence for (IP) is established.

7.3. *Phase transformations in shape-memory alloys*

Over the last decade the shape-memory effect became important in many mechanical and medical applications. Thus, the need of good models for simulation and optimization of such materials arises. The energetic formulation (S) and (E) was in fact developed for models of phase transformations induced by stress or strain, rather than by temperature changes, cf. [MieT99,MieTL02]. For such isothermal cases the energetic formulation is a tool which can model the static behavior quite well, but the dynamical effects are modeled only crudely. The dissipation potential is fitted phenomenologically to obtain the desired hysteresis loops.

We assume that in each microscopic point $x \in \Omega$ the shape-memory material is free to choose one of m crystallographic phases, denoted by $\{e_1, \ldots, e_m\} \in \mathbb{R}^m$, and that the elastic energy density W is then given by $W(\mathrm{D}\varphi, e_j)$. If the model is made on the mesoscopic level, then the internal variables are phase portions $z^{(j)} \in [0, 1]$ for the jth phase. We set

$$Z = \left\{ z \in [0,1]^p \subset \mathbb{R}^m \;\middle|\; \sum_1^m z^{(j)} = 1 \right\} \quad \text{and} \quad \mathcal{Z} = \mathrm{L}^1(\Omega; Z) \subset \mathrm{L}^1\big(\Omega; \mathbb{R}^m\big).$$

The material properties are given via a *mixture function* (also called cross-quasiconvexification) $W : \mathbb{R}^{d\times d} \times Z \to [0, \infty]$, see [MieTL02,GoMH02]. The dissipation can be shown to have the form $D(z_0, z_1) = \psi(z_1 - z_0)$ with $\psi(v) = \max\{\sigma_m \cdot v \mid m = 1, \ldots, M\} \geqslant C_\psi |v|$, where $\sigma_m \in \mathbb{R}^p$ are thermodynamically conjugated threshold values. The derivation of this model is, in fact, a special case of the relaxation described in Section 5.6.

In the case of no regularization term, i.e., $\kappa = 0$ in (7.1), we are unable to prove existence results for this model in its full generality. The mixture function W is constructed as a relaxation, which means that the associated energy functional is weakly lower semicontinuous on $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^d) \times \mathrm{L}^1(\Omega, Z)$. Clearly $\mathcal{D}$ is convex and weakly lower semicontinuous.

Thus, our abstract conditions (A1)–(A5) can be satisfied easily. However, condition (A6) (weak continuity of $\mathcal{D}$) does not hold. The weak continuity of $\partial_t\mathcal{E}$ in condition (A7) also holds, but the weak closedness of the stable sets $\mathcal{S}(t)$ seems to be wrong in general. Hence, in general we are able to prove existence of solutions for the incremental problem (IP), but the convergence of the piecewise constant interpolants to solutions of the energetic formulation (S) and (E) is still open. However, the case with only two phases ($m = 2$) has been treated in [MieTL02] under the additional assumption that the elastic behavior is linear and both phases have the same elastic tensor. The missing closedness of the stable sets is shown via an explicit representation of the set in terms of a pseudo-differential operator of order 0 and a finite number of quadratic inequalities which have to hold pointwise. Then, a careful analysis using H-measures (cf. [Tar90]) shows that the nonconvex sets $\mathcal{S}(t)$ are weakly closed.

The situation is much better if a regularizing term $\frac{\kappa}{r}|\mathrm{D}z|^r$, with $r \geqslant 1$ and $\kappa > 0$, is added to the stored energy. In this case the underlying space can be chosen as $\mathcal{F} \times \mathcal{Z}$, with $\mathcal{F} = \mathrm{W}^{1,p}_{\Gamma_{\mathrm{Dir}}}(\Omega, \mathbb{R}^d)$ and $\mathcal{Z} = \mathrm{W}^{1,r}(\Omega, \mathbb{R}^m) \cap \mathrm{L}^1(\Omega, Z)$ equipped with the weak topologies in both cases. Conditions (A1)–(A5) remain valid but now (A6) also holds since $\mathrm{W}^{1,r}(\Omega)$ is compactly embedded into $\mathrm{L}^1(\Omega)$ and $\mathcal{D}$ is strongly continuous on $\mathrm{L}^1(\Omega, Z)$. Such regularizations are used in [Rou02,ArGR03,MieRou03,FM05].

In [Mai05] a microscopic model without phase mixtures is considered, i.e., we assume $z \in Z_{\mathrm{p}} := \{e_1, \dots, e_m\} \subset \mathbb{R}^m$, where e_j is the jth unit vector. The subscript "$_{\mathrm{p}}$" stands for "pure" phases, and the functions $z \in \mathcal{Z}_{\mathrm{p}} = \{z \in \mathrm{BV}(\Omega, \mathbb{R}^m) \mid z(x) \in Z_{\mathrm{p}} \text{ a.e. in } \Omega\}$ are like characteristic functions which indicate exactly one phase at each material point. The dissipation is assumed as above, but now the elastic energy contains an additional term measuring the surface area of the interfaces between the different regions

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(\mathrm{D}\varphi, z)\,\mathrm{d}x + \kappa \int_\Omega |\mathrm{D}z| - \big\langle \ell_{\mathrm{ext}}(t), \varphi \big\rangle,$$

where κ is a positive constant and the total variation of z over Ω is $\int_\Omega |\mathrm{D}z|$, which equals $\sqrt{2}$ times the area of all interfaces. Thus, the underlying space $\mathcal{Y} = \mathcal{F} \times \mathcal{Z}_{\mathrm{p}}$ is defined with $\mathcal{F}$ as above and $\mathcal{Z}_{\mathrm{p}}$ is equipped with the weak* topology, i.e., the measure $\mathrm{D}z$ is tested with functions in $\mathrm{C}^0(\overline{\Omega})$. Again we have a compact embedding of $\mathrm{BV}(\Omega)$ into $\mathrm{L}^1(\Omega)$, which shows that $\mathcal{Z}_{\mathrm{p}}$ is a weakly* closed and $\mathcal{D}$ is weakly* continuous on $\mathcal{Z}_{\mathrm{p}}$. We refer to [Mai05] for details.

A totally different approach to shape-memory materials is given in [AurP02]. This model is isotropic and uses the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathsf{T}})$. The internal variable is the mesoscopically averaged transformation strain, namely

$$z \in Z = \mathbb{R}^{d\times d}_0 = \big\{A \in \mathbb{R}^{d\times d} \mid \operatorname{tr} A = 0\big\}.$$

For a given fixed temperature, the stored-energy density W takes the form

$$\begin{aligned} &W(\varepsilon, z) = \frac{1}{2}\mathbb{A}(\varepsilon - z) : (\varepsilon - z) + H(z) \\ &\quad \text{with } H(r) = c_1|z| + c_2|z|^2 + I_{\{|z| \leqslant c_3\}}(z). \end{aligned}$$

The dissipation potential is given simply by $\psi(\dot{z}) = c_4|\dot{z}|$. Thus, the functionals

$$\mathcal{E}(t, u, z) = \int_\Omega W\big(\varepsilon(u), z\big)\,\mathrm{d}x - \big\langle \ell(t), u\big\rangle$$

and

$$\mathcal{D}(z_0, z_1) = \int_\Omega c_4\big|z_1(x) - z_0(x)\big|\,\mathrm{d}x$$

are convex and weakly lower semicontinuous on the Hilbert space $\mathcal{Y} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d) \times \mathrm{L}^2(\Omega; Z)$. (Throughout the constants c_j are positive.) Moreover, $\mathcal{E}(t, \cdot)$ is uniformly convex, which implies that the incremental problem (IP) has unique solutions and that these solutions satisfy an a priori Lipschitz bound in time, see Section 3.5. However, our theory does not provide existence of solutions. For the existence proof in the convex case we lack the necessary smoothness (see Theorem 4.3) whereas for the nonsmooth case we lack the suitable compactness (see (A6) or (A7) in Theorem 5.2).

However, a slight modification and a regularization make the problem accessible for our general theory, see [AurMS05]. We replace the special function H by a smooth version H_ρ which satisfies

$$\begin{aligned}&H \in \mathrm{C}^3(Z, R),\ H \text{ uniformly convex and}\\ &\exists c, C > 0, \forall z \in Z\colon\ c|z|^2 \leqslant H(z) \leqslant C|z|^2.\end{aligned}$$

Moreover, we require that for z fixed $H_\rho(z) \to H(z)$ if $\rho \searrow 0$. As an example we may choose

$$H_\rho(z) = c_1\big(\sqrt{\rho^2 + |z|^2} - \rho\big) + \left(c_2 + \frac{1}{\rho}h\left(\frac{1}{\rho}\big(|z|^2 - c_3^2\big)\right)\right)|z|^2,$$

where

$$h(s) = \frac{\mathrm{e}^s}{1 + \mathrm{e}^s}.$$

Moreover, we add a spatial regularization and obtain the energy functional

$$\mathcal{E}_\rho(t, u, z) = \int_\Omega \frac{1}{2}\mathbb{A}\big(\varepsilon(u) - z\big) : \big(\varepsilon(u) - z\big) + H_\rho(z) + \frac{\rho c_5}{2}|\mathrm{D}z|^2\,\mathrm{d}x.$$

Now the suitable function space is $\mathcal{Y} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d) \times \mathrm{H}^1(\Omega; Z)$ which, on the one hand, makes $\mathcal{D}$ weakly continuous (cf. (A6)) and, on the other hand, makes $\mathcal{E}_\rho$ uniformly convex and C^3. Thus, now both abstract existence results are applicable. In particular, one obtains a unique solution for the associated energetic formulation (S) and (E) and the solutions of the incremental problem (IP) converge strongly like $\sqrt{\text{stepsize}}$.

Note that the smoothness of H_ρ is not enough to guarantee smoothness if $c_5 = 0$. Then, the suitable space for uniform convexity is again $\mathrm{H}^1 \times \mathrm{L}^2$ as above. However, a functional

$$\mathcal{H} : \mathrm{L}^2(\Omega, Z) \to \mathbb{R}, \qquad z \mapsto \int_\Omega h_*\big(z(x)\big)\,\mathrm{d}x$$

is C^3 if and only if $h_* : Z \to \mathbb{R}$ is a quadratic functional.

7.4. *Models in ferromagnetism*

Hysteretic effects in ferromagnetism were one of the driving forces in the development of hysteresis operators like the Preisach operator as a superposition of many relay operators, see [Vis94], I.4 and IV. These models are mainly used for mean field models, which replace the continuum by a system with a finite number of degrees of freedom. Here we want to propose a continuum model with infinitely many degrees of freedom. Thus, we use simpler hysteresis operators which will also generate complicated hysteretic behavior because of spatial variations of the internal variable.

In ferromagnetism we are interested in the interplay between elastic effects and magnetic effects, sometimes also called magnetostriction since magnetic fields may deform a body. The models described here and studied in [EfM05] lie in between the parabolic and hyperbolic models considered in [Vis94,Vis05] and the purely static models in [DeS93,DeSJ02]. Thus, our models describe the statics as good as the latter works but our dynamics are not as good as in the former ones.

For simplicity, we assume small strains and use the linearized strain tensor $\varepsilon(u)$. The internal variable is the magnetization $z \in Z = \mathbb{S}^{d-1} = \{z \in \mathbb{R}^d \mid |z| = 1\}$. More standard notations for the magnetization are the symbols m and M, but we stay with z to remain consistent with the other parts of this chapter. The functional of the stored energy takes the form

$$\mathcal{E}(t, u, z) = \int_\Omega W\big(\varepsilon(u), z\big) + \frac{\kappa^2}{2}|\mathrm{D}z|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} \frac{\mu_0}{2}|\nabla\phi_z|^2\,\mathrm{d}x \\ - \big\langle \ell_{\mathrm{mech}}(t), u\big\rangle - \big\langle \mu_0 H_{\mathrm{ext}}(t), z\big\rangle,$$

where $\ell_{\mathrm{mech}}(t)$ denotes the mechanical loading, $H_{\mathrm{ext}}(t)$ is the external magnetic field satisfying $\operatorname{div} H_{\mathrm{ext}} = 0$. The stored energy density W contains information on the interaction between the elastic behavior and the magnetic directions and $\kappa > 0$ is the exchange length which gives the thickness of the domain walls.

The potential ϕ_z describes the field induced by the magnetization inside the body, i.e., the magnetic flux is $B = \mu_0(H + \mathbb{E}_\Omega z)$ with $H = H_{\mathrm{ext}} - \nabla\phi_z$. Here $\mathbb{E}_\Omega$ denotes the operator which extends a function on Ω by 0 to all of $\mathbb{R}^d$. Thus, $\operatorname{div} B = 0$ yields the definition of ϕ_z as a solution of

$$\operatorname{div}(-\nabla\phi_z + \mathbb{E}_\Omega z) = 0 \quad \text{on } \mathbb{R}^d.$$

Of course, ϕ_z is defined only up to a constant, but $\widehat{\mathcal{G}}: \mathrm{L}^2(\Omega, \mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d, \mathbb{R}^d)$, $z \mapsto \nabla\phi_z$, is a bounded linear operator. Moreover, $\mathcal{G}: z \mapsto (\widehat{\mathcal{G}}z)|_\Omega$ is an orthogonal projection on $\mathrm{L}^2(\Omega, \mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} |\nabla\phi_z|^2 \,\mathrm{d}x = \int_\Omega z \cdot (\mathcal{G}z) \,\mathrm{d}x$, see [DeS93].

In addition to the stored energy we define a dissipation distance via a metric on $Z = \mathbb{S}^{d-1}$. The simplest distance which respects the geometry is

$$D(z_0, z_1) = \frac{\delta}{\pi} \arccos(z_0 \cdot z_1) \quad \text{giving} \quad \mathcal{D}(z_0, z_1) = \int_\Omega D\big(z_0(x), z_1(x)\big) \,\mathrm{d}x.$$

Using these functionals with a suitable W and the space

$$\mathcal{Y} = \mathrm{H}^1\big(\Omega, \mathbb{R}^d\big) \times \big\{ z \in \mathrm{H}^1\big(\Omega, \mathbb{R}^d\big) \;\big|\; \big|z(x)\big| = 1 \text{ a.e.} \big\},$$

it is shown in [EfM05] that for $\kappa > 0$ the energetic formulation (S) and (E) has a solution. Again the crucial compactness condition (A6) is satisfied.

In [Vis05] the bulk energy W and the exchange energy are neglected (i.e., $\kappa = 0$). Moreover, the models are considered to be mesoscopic and z is considered to be a mesoscopic average satisfying $z \in Z = \{z \in \mathbb{R}^d \mid |z| \leqslant 1\}$. Thus, the energy reduces to

$$\mathcal{E}_0(t, z) = \int_{\mathbb{R}^d} \frac{\mu_0}{2} |\nabla\phi_z|^2 \,\mathrm{d}x + \int_\Omega I_Z\big(z(x)\big) \,\mathrm{d}x - \langle \mu_0 H_{\mathrm{ext}}, z \rangle.$$

The dissipation is taken to be in the form $\psi(z, \dot z) = \hat\psi(\dot z)$, since Z is a closed convex subset of $\mathbb{R}^d$. With these assumptions the problem is convex and using the formula $\mathrm{D}\mathcal{E}_0(t, z)[\tilde z] = \mu_0(\mathcal{G}z - H_{\mathrm{ext}}) = -\mu_0 H|_\Omega$ we find the subdifferential formulation which is equivalent to (S) and (E), namely

$$0 \in \partial\hat\psi\big(\dot z(t, x)\big) + \partial I_Z(z) - \mu_0(H_{\mathrm{ext}} - \nabla\phi_z) \quad \text{for a.a. } (t, x) \in [0, T] \times \Omega.$$

In [Vis05] a “scalar relay” is used which corresponds to the choice $\hat\psi(\dot z) = \delta_0 \dot z \cdot \theta + (\delta_0 + \delta_1)|\dot z \cdot \theta|$ with $\delta_0, \delta_1 > 0$ and a given vector $\theta \in \mathbb{S}^{d-1}$. Existence and uniqueness results for parabolic versions (not rate-independent) of this problem are then established.

7.5. *A delamination problem*

In this section we provide a simple model for rate-independent delamination and refer to [KoMR05] for a better model and the detailed analysis. Thus, we remove all unnecessary distractions and focus the attention on the interplay of the different continuity properties in the suitable topologies.

The body $\Omega \subset \mathbb{R}^d$ consists of several pieces which are glued together at certain interfaces. The model is based on the assumption that sufficiently strong forces can destroy the glue. To be precise, we assume that $\mathrm{int}(\mathrm{cl}(\Omega))$ differs from Ω by a finite set of sufficiently smooth hypersurfaces Γ_j, $j = 1, \ldots, n$, along which parts of the body are glued together. This means that with $\Gamma := \bigcup_{j=1}^n \Gamma_j$ we have $\mathrm{int}(\mathrm{cl}(\Omega)) = \Omega \cup \Gamma$ and $\Omega \cap \Gamma = \emptyset$. The two sides of the body are glued together along these surfaces with a glue that is softer than the material itself. Upon loading, some parts of the glue may break and thus lose their effectiveness. The remaining fraction of the glue which is still effective is denoted by the internal state function $z:\Gamma\to[0,1]$.

We let $\mathcal{Z}=\{z:\Gamma\to[0,1]\mid z \text{ measurable}\}\subset \mathrm{L}^1(\Gamma)$. The dissipation distance $\widetilde{\mathcal{D}}(z_0,z_1)$ is proportional to the amount of glue that is broken from state z_0 to state z_1,

$$\widetilde{\mathcal{D}}(z_0,z_1)=\int_\Gamma \psi_{\mathrm{delam}}\big(z_1(y)-z_0(y)\big)\,\mathrm{d}a(y)$$
$$\text{with } \psi_{\mathrm{delam}}(v)=-\kappa v \quad \text{for } v\leqslant 0 \text{ and } +\infty \text{ else.}$$

Here we explicitly forbid the healing of the glue by setting $\psi_{\mathrm{delam}}(v)$ equal to ∞ for $v>0$.

The energy is given by the elastic energy in the body, the elastic energy in the glue and the potential of the external loadings

$$\mathcal{E}(t,\varphi,z)=\int_\Omega W(\mathrm{D}\varphi)\,\mathrm{d}x+\int_\Gamma z(y)Q\big(y,[\![\varphi]\!]_\Gamma(y)\big)\,\mathrm{d}a(y)-\big\langle \ell_{\mathrm{ext}}(t),\varphi\big\rangle,$$

where for $y\in\Gamma$ the vector $[\![\varphi]\!]_\Gamma(y)$ denotes the jump of the deformation φ across the interface Γ and $Q(y,\cdot)$ is the potential defining the elastic properties of the glue.

For simplicity we assume further that W is coercive and provides linearized elasticity and that Q is quadratic as well. Then there is a unique minimizer $\varphi=\Phi(t,z)\in\mathcal{F}:=\{\phi\in\mathrm{H}^1(\Omega,\mathbb{R}^d)\mid \phi|_{\Gamma_{\mathrm{Dir}}}=\varphi_{\mathrm{Dir}}\}$ of $\mathcal{E}(t,\cdot,z)$. We let $\mathcal{Y}=\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega,\mathbb{R}^d)\times\mathcal{Z}$ be equipped with the weak topology of $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^1(\Gamma)$.

Note that $\mathcal{E}$ is not convex since the integral over Γ is trilinear. Nevertheless $\mathcal{E}$ is weakly lower semicontinuous since the compactness of the embedding $\mathrm{H}^{1/2}(\Gamma)\subset\mathrm{L}^2(\Gamma)$ implies $Q([\![\varphi_k]\!]_\Gamma)\to Q([\![\varphi^*]\!]_\Gamma)$ strongly in $\mathrm{L}^1(\Gamma)$ if $\varphi\rightharpoonup\varphi^*$ in $\mathrm{H}^1(\Omega)$. Moreover, $z_k\rightharpoonup z^*$ in $\mathrm{L}^1(\Gamma)$ and $\|z_k\|_\infty\leqslant 1$ implies $z_k\overset{*}{\rightharpoonup}z^*$ in $\mathrm{L}^\infty(\Gamma)$. Thus, we conclude $\int_\Gamma z_kQ([\![\varphi_k]\!]_\Gamma)\,\mathrm{d}a\to\int_\Gamma z^*Q([\![\varphi^*]\!]_\Gamma)\,\mathrm{d}a$, as it is desired for lower semicontinuity.

Thus, it is not difficult to satisfy assumptions (A1)–(A5). Since (A6) (weak continuity of $\mathcal{D}$) does not hold, we have to show that the stable sets $\mathcal{S}(t)$ are weakly closed. Note that each element (φ,z) in $\mathcal{S}(t)$ satisfies $\varphi=\Phi(t,z)$ since the elastic problem for fixed z is strictly convex. Moreover, it can be shown that the mapping $\Phi(t,\cdot)$ is compact, in the sense that $z_k\rightharpoonup z^*$ in $\mathcal{Z}$ implies $\Phi(t,z_k)\to\Phi(t,z)$ in $\mathrm{H}^1(\Omega)$. We refer to [KoMR05], Lemma 2.1, for the proof of this delicate continuity result. Finally, an application of Proposition 5.11 and the usage of the special form of ψ_{delam} establish the weak closedness of $\mathcal{S}(t)$. Thus, assumption (A7) holds and existence of solutions follows according to Theorem 5.2, namely, for each stable initial state z_0 and each loading $\ell_{\mathrm{ext}}\in\mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{H}^{-1}(\Omega))$, the energetic formulation (S) and (E) of the delamination problem has a solution (φ,z) with $\varphi\in\mathrm{L}^\infty([0,T],\mathrm{H}^1(\Omega,\mathbb{R}^d))$ and $z\in\mathrm{BV}_{\widetilde{\mathcal{D}}}([0,T],\mathcal{Z})$. Recall that this condition on z is equivalent to the monotonicity $z(s)\geqslant z(t)$ on Γ for $s<t$ and that then $\mathrm{Diss}_{\widetilde{\mathcal{D}}}(z;[s,t])=\widetilde{\mathcal{D}}(z(s),z(t))$.

In [KoMR05] also numerical simulations are given which display the different contributions in the energy balance (E).

7.6. *Crack growth in brittle materials*

In a series of papers starting with [FM93,FM98,Bul98,Bul00,BoFM00] and culminating with [DalT02,Cha03,FL03,DalFT05] the following fracture model is developed and analyzed. We follow the notation of the latter paper and will show that the approach taken there is exactly that of the energetic formulation (S) and (E), if the notions are reinterpreted correctly. This problem is technically very difficult and needs certain adjustments to the abstract theory which we will not discuss here. Nevertheless the main strategy of the proof is exactly as it was described in Sections 3 and 5, namely by using the incremental problem in the form of a minimization problem and then passing to the limit as the stepsize of the discretization goes to 0. We will present here only a simplified version which omits certain finer details. But this enables us to compare these result with our abstract theory without too much effort. For full details we refer to [DalFT05].

The model deals with nonlinear elasticity such that the energy density W is given as a quasiconvex function of the displacement gradient $A = \mathrm{D}u$ with suitable growth restrictions, namely

$$\exists p > 1, \exists c, C > 0, \forall A \in \mathbb{R}^{m\times d}, \quad c|A|^p - C \leqslant W(A) \leqslant C|A|^p + C.$$

The time-dependent exterior forces are assumed to have a potential F which is nonlinear and coercive. This is needed because pieces which are broken off by cracks all around could "fall to infinity". The assumption is

$$\exists c > 0, \exists f \in \mathrm{L}^1(\Omega), \forall u \in \mathbb{R}^m, \quad -F(t,x,u) \geqslant c|u|^q - f(x)$$

for a suitable q related to p and d.

The internal variables are the cracks themselves. A crack is considered to be a subset $\Gamma \subset \overline{\Omega}$ which satisfies $\mathcal{H}^{d-1}(\Gamma) < \infty$, where $\mathcal{H}^{d-1}$ denotes the surface measure or the $(d-1)$-dimensional Hausdorff measure. (More precisely, a crack Γ is the equivalence class of all $\widetilde{\Gamma}$ which satisfy $\mathcal{H}^{d-1}(\Gamma\backslash\widetilde{\Gamma}) + \mathcal{H}^{d-1}(\widetilde{\Gamma}\backslash\Gamma) = 0$. All inclusions $\subset$ are also meant to be up to sets N with $\mathcal{H}^{d-1}(N) = 0$.) The state space $\mathcal{Y}$ is then given as

$$\begin{aligned}\mathcal{Y} = \big\{(u,\Gamma) \bigm| {}& \Gamma \subset \overline{\Omega}, \mathcal{H}^{d-1}(\Gamma) < \infty, \Gamma \text{ rectifiable},\\ & u \in \mathrm{GSBV}\big(\Omega,\mathbb{R}^m\big), u|_{\Gamma_{\mathrm{Dir}}} = u_{\mathrm{Dir}}, J(u) \subset \Gamma\big\},\end{aligned}$$

where $\mathrm{GSBV}(\Omega,\mathbb{R}^m)$ is the set of generalized special functions of bounded variations and $J(u)$ denotes the jump set of such functions. This space is equipped with the following weak convergence:

$$\begin{aligned}&(u_k,\Gamma_k) \xrightarrow{\mathcal{Y}} (u,\Gamma)\\ \iff \quad &\begin{cases} u_k \to u \quad \text{a.e. in } \Omega, & \mathrm{D}u_k \rightharpoonup \mathrm{D}u \quad \text{in } \mathrm{L}^p\big(\Omega,\mathbb{R}^{m\times d}\big),\\ \sup\big\{\mathcal{H}^{d-1}\big(J(u_k)\big) \mid k \in \mathbb{N}\big\} < \infty, & \Gamma_k \xrightarrow{\sigma_p} \Gamma.\end{cases}\end{aligned}$$

See [DalFT05], Definition 4.1, for the exact definition of σ_p convergence of sets.

The functional for the stored energy (in our convention) is given as

$$\mathcal{E}(t,u,\Gamma)=\int_\Omega W\big(\mathrm{D}u(x)\big)-F\big(t,x,u(x)\big)\,\mathrm{d}x,$$

which does not depend directly on Γ, since Γ has Lebesgue measure 0. The dissipation is associated with the crack propagation. For simplicity we take

$$\mathcal{D}(\Gamma_0,\Gamma_1)=\begin{cases}\kappa\mathcal{H}^{d-1}(\Gamma_1\backslash\Gamma_0) & \text{for } \Gamma_0\subset\Gamma_1,\\ \infty & \text{else,}\end{cases}$$

but more general x-dependent and anisotropic surface measures can be used.

In [DalFT05] the notation are somewhat different. They use total energy $\mathbb{E}(t)(u,\Gamma)=\mathcal{E}(t,u,\Gamma)+\mathcal{D}(\emptyset,\Gamma)$ to write the energetic formulation in the following way:

(a) *global stability*: for every $t\in[0,T]$ the pair $u(t),\Gamma(t)$ is a minimum energy configuration, i.e., $\mathbb{E}(t)(u(t),\Gamma(t))\leqslant\mathbb{E}(t)(\tilde u,\widetilde\Gamma)$ for all $(\tilde u,\widetilde\Gamma)\in\mathcal{Y}$ with $\Gamma(t)\subset\widetilde\Gamma$;

(b) *irreversibility*: $\Gamma(s)$ is contained in $\Gamma(t)$ for $0\leqslant s\leqslant t\leqslant T$;

(c) *energy balance*: the increment in stored energy plus the energy spent in crack increase equals the work of the external forces, i.e.,

$$\mathbb{E}(t)\big(u(t),\Gamma(t)\big)=\mathbb{E}(s)\big(u(s),\Gamma(s)\big)-\int_s^t\int_\Omega\partial_\tau F\big(\tau,x,u(\tau,x)\big)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Using the above definition of $\mathcal{D}$ it is easy to see that (a)–(c) are equivalent to our energetic formulation (S) and (E). Thus, the existence results there are also existence results for (S) and (E).

Like in the delamination case, it is possible to show that the functionals $\mathcal{E}$ and $\mathcal{D}$ are lower semicontinuous with compact sublevels, since $\mathcal{E}$ is a volume integral and $\mathcal{D}$ is a surface integral. However, the analysis is much deeper, since here the crack Γ is not prescribed a priori. Thus, already the lower semicontinuity property is nontrivial. The most difficult part is the proof of the closedness of the stable set which relies on the so-called "jump transfer in GSBV" ([DalFT05], Theorem 5.3), which supplies assumption (5.9) of our abstract Proposition 5.11.

It should be noted that the analysis in [DalFT05] provides several new tools. In fact, they form a significant part of the basis for the abstract existence result in Theorem 5.2 (see also [FM05]). In particular, the t-dependent choice of subsequences for the u-component (see Step 2), the approximation of Lebesgue integrals by suitable Riemann sums in the proof of Proposition 5.7 and, most importantly, the following concrete version of Proposition 5.6

$$\left.\begin{array}{l}(u_k,\Gamma_k)\overset{\mathcal{Y}}{\to}(u,\Gamma)\\ \mathcal{E}(t,u_k,\Gamma_k)\to\mathcal{E}(t,u,\Gamma)\end{array}\right\}$$
$$\Longrightarrow\quad \mathrm{D}_u\mathcal{E}(t,u_k,\Gamma_k)\rightharpoonup\mathrm{D}_u\mathcal{E}(t,u,\Gamma)\quad\text{in }\big(\mathrm{W}^{1,p}\big(\Omega,\mathbb{R}^m\big)\big)^*.$$

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CHAPTER 7

On the Global Weak Solutions to a Variational Wave Equation

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Abstract

A nonlinear wave equation arises in a simplified liquid crystal model through the variational principle. The wave speed of the wave equation is a given function of the wave amplitude. In the earlier study to this equation, Hunter and Saxton have derived a simple asymptotic equation for weakly nonlinear unidirectional waves of the equation. Previous work has established the existence of weak solutions to the initial value problem for the asymptotic equation for data in the space of bounded variations. We improve the previous work to the natural space of square integrable functions, and we establish the uniqueness of weak solutions for both the dissipative and conservative types.

We also have results on the full nonlinear wave equation. It has been known from joint work of the second author with Glassey and Hunter for the equation that smooth initial data may develop singularities in finite time, a sequence of weak solutions may develop concentrations, while oscillations may persist. For monotone wave speed functions in the equation, we find an invariant region in the phase space in which we discover: (a) smooth data evolve smoothly forever; (b) the smooth solutions obtained through data mollification and step (a) for not-as-smooth initial data yield weak solutions to the Cauchy problem of the nonlinear variational wave equation with initial data in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Furthermore, for initial data outside the invariant region, we can also prove the global existence of weak solution with initial Riemann invariant in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. The main tool for the weak solution is the Young measure theory and related techniques.

More specifically, we will present the following results.

1. On the asymptotic equation, we have existence and uniqueness of multiple weak solutions in the weak norm L^2 of the derivative u_x.
2. For the nonlinear wave equation, with monotone wave speed, we found some invariant regions and some global smooth solutions.
3. For the nonlinear wave equation, with monotone wave speed, we prove the global existence of weak solution with initial Riemann invariant in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$.

1. Introduction

In this chapter we study the Cauchy problem for the nonlinear wave equation

$$\begin{cases} u_{tt} - c(u)\big[c(u)u_x\big]_x = 0, & t > 0, x \in \mathbb{R}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1)(x). \end{cases} \tag{0.0.1}$$

1.1. *Origin of the variational wave equation*

The equation in (0.0.1) is the Euler–Lagrange equation of the least action principle

$$\frac{\delta}{\delta u} \iint \big\{(\partial_t u)^2 - c^2(u)(\partial_x u)^2\big\}\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{1.1.1}$$

The equation in (0.0.1) may be regarded as a generalization of equations for harmonic wave maps and arises in a number of different physical contexts, including nematic liquid crystals [54], long waves on a dipole chain in the continuum limit [30,31,77] and in classical field theories and general relativity [30]. Let us take, for example, nematic liquid crystals. We know that the mean orientation of the long molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n} \in \mathbb{S}^2$, the unit sphere. Associated with the director field $\mathbf{n}$, there is the well-known Oseen–Franck potential energy density W given by

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha\big|\mathbf{n} \times (\nabla \times \mathbf{n})\big|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2. \tag{1.1.2}$$

The positive constants α, β and γ are elastic constants of the liquid crystal. For the special case $\alpha = \beta = \gamma$, the potential energy density reduces to

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha|\nabla\mathbf{n}|^2,$$

which is the potential energy density used in harmonic maps into the sphere $\mathbb{S}^2$. There are many studies on the constrained elliptic system of equations for $\mathbf{n}$ derived through variational principles from the potential (1.1.2), and on the parabolic flow associated with it, see [5,15,22,33,41,64] and references therein. In the regime in which inertia effects dominate viscosity, however, the propagation of the orientation waves in the director field may then be modeled by the least action principle [54],

$$\frac{\delta}{\delta u} \int \big\{\partial_t\mathbf{n} \cdot \partial_t\mathbf{n} - W(\mathbf{n}, \nabla\mathbf{n})\big\}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1. \tag{1.1.3}$$

In the special case $\alpha = \beta = \gamma$, this variational principle (1.1.3) yields the equation for harmonic wave maps from $(1+3)$-dimensional Minkowski space into the two sphere, see, for example, [9,57,58]. For planar deformations depending on a single space variable x, the director field has the special form

$$\mathbf{n} = \cos u(x,t)\mathbf{e}_x + \sin u(x,t)\mathbf{e}_y,$$

where the dependent variable $u \in \mathbb{R}^1$ measures the angle of the director field to the x-direction, and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the coordinate vectors in the x and y directions, respectively. In this case, the variational principle (1.1.3) reduces to (1.1.1) with the wave speed c given specifically by

$$c^2(u) = \alpha \cos^2 u + \beta \sin^2 u. \tag{1.1.4}$$

The general problem of global existence and uniqueness of solutions to the Cauchy problem of the nonlinear variational wave equation (0.0.1) is open. It has been demonstrated in [29] that (0.0.1) is rich in structural phenomena associated with weak solutions. Writing the highest derivatives of (0.0.1) in conservative form

$$\partial_t^2 u - \partial_x \left(c^2 \, \partial_x u\right) = -cc'(\partial_x u)^2,$$

we see that the strong precompactness in L^2 of the derivatives $\{\partial_x u\}$ of a sequence of approximate solutions is essential in establishing the existence of a global weak solution. However, the equation has the phenomenon of persistence of oscillation [20] and annihilation in which a sequence of exact solutions with bounded energy can oscillate forever so that the sequence $\{\partial_x u\}$ is not precompact in L^2, but the weak limit of the sequence is still a weak solution. Secondly, the equation has short-time smooth solutions that blow up in finite time. Thirdly, from the study of its asymptotic equation (see further), it is clear that a positive amount of energy of a solution can concentrate in a set of measure zero, and there are multiple choices for the continuation of the solution beyond blow-up time. To put the equation and its particulars into context of nonlinear wave equations under current research, we compare this equation with the equation

$$\partial_t^2 u - \partial_x \left[p(\partial_x u)\right] = 0, \tag{1.1.5}$$

where $p(\cdot)$ is a given function, considered by Lax [44]. The derivative $\partial_x u$ remains bounded for (1.1.5), but we find that it is merely in L^2 for (0.0.1). We note interestingly that solutions of (1.1.5) – with a "stronger" $\partial_x u$-dependent nonlinearity – are more regular than solutions of (0.0.1) – with an apparently "weaker" u-dependent nonlinearity. (This kind of behavior is well known for nonlinear parabolic partial differential equations.) In each case it appears that singularities develop to the maximum extent permitted by the existence of global weak solutions. For further research into (1.1.5) and its generalizations, we refer the reader to [43] and [50]. Another related equation is

$$\partial_t^2 u - c^2(u) \Delta u = 0 \tag{1.1.6}$$

considered by Lindblad [47], who established the global existence of smooth solutions of (1.1.6) with smooth, small, and spherically symmetric initial data in $\mathbb{R}^3$, where the large-time decay of solutions in high space dimensions is crucial. The multidimensional generalization of (0.0.1),

$$\partial_t^2 u - c(u) \nabla \cdot \left(c(u) \nabla u\right) = 0, \tag{1.1.7}$$

contains a lower-order term proportional to $cc'|\nabla u|^2$, which (1.1.6) lacks. This lower-order term is responsible for the blow-up in the derivatives of u. Finally, we note that (0.0.1) also looks related to the perturbed wave equation

$$\partial_t^2 u - \Delta u + f(u, \nabla u, \nabla\nabla u) = 0, \tag{1.1.8}$$

where $f(u, \nabla u, \nabla\nabla u)$ satisfies an appropriate convexity condition (for example, $f = u^p$ or $f = a(\partial_t u)^2 + b|\nabla u|^2$) or some nullity condition. Blow-up for (1.1.8) with a convexity condition has been studied extensively, see [2,28,32,38,40,46,55,60,61] and [62] for more reference. Global existence and uniqueness of solutions to (1.1.8) with a nullity condition depend on the nullity structure and large time decay of solutions of the linear wave equation in higher dimensions (see [42] and references therein). Therefore (0.0.1) with the dependence of $c(u)$ on u and the possibility of sign changes in $c'(u)$ is familiar yet truly different.

1.2. *Asymptotic equations*

Despite its simplicity, (0.0.1) is not easy. A study of its geometric optical solutions is helpful and interesting. Look for solutions of the form

$$\psi(t, x) = u_0 + \varepsilon u(\varepsilon t, x - c_0 t) + \mathrm{O}\big(\varepsilon^2\big),$$

where u_0 is a constant state and $c_0 = c(u_0) > 0$ is its speed, Hunter and Saxton [35] found that $u(\cdot, \cdot)$ satisfies

$$\boxed{(u_t + uu_x)_x = \frac{1}{2}u_x^2} \tag{1.2.9}$$

up to a scaling factor, assuming that $c'(u_0) \neq 0$.

If u_0 is such that $c'(u_0) = 0$, $c''(u_0) \neq 0$, then the equation is

$$\big(u_t + u^2 u_x\big)_x = uu_x^2.$$

In general, if u_0 is such that $c^{(k)}(u_0) = 0$, $k = 1, 2, \ldots, n-1$, but $c^{(n)}(u_0) \neq 0$, the equation is

$$\big(u_t + u^n u_x\big)_x = \frac{1}{2}nu^{n-1}u_x^2.$$

Another form of the first asymptotic equation is

$$\boxed{\partial_t v + \partial_x(uv) = \frac{1}{2}v^2, \qquad u_x = v.} \tag{1.2.10}$$

We shall not study the second and higher asymptotic equations in this short talk.

1.3. *Young measure*

We shall prove the compactness of the approximate solution sequence by applying Young measure theory [63] and the ideas used by Lions [49] in the proof of the global existence of weak solutions to multidimensional compressible Navier–Stokes equation, and by Joly, Métivier and Rauch [39] in the rigorous justification of weakly nonlinear geometric optics for a semilinear wave equation (see also [69]). For the convenience of the reader, we quote the following lemma from [39] (see also [20,23]) that we use in this chapter.

LEMMA 1.3.1. *Let U be an open subset of $\mathbb{R}^n$, whose boundary has zero Lebesgue measure. Given a bounded family $\{u^\varepsilon(y)\} \subset L^s(U)$, $s > 1$, of $\mathbb{R}^N$-valued functions, then there exist a subsequence $\{\varepsilon_j\}$ and a measurable family of probability measures on $\mathbb{R}^N$, $\{\mu_y(\cdot), y \in U\}$, such that for all continuous functions $F(y,\lambda)$ with $F(y,\lambda) = \mathrm{O}(|\lambda|^q)$ as $|\lambda| \to \infty$, and $q < s$, there holds*

$$\lim_{\varepsilon_j \to 0} \int_U \varphi(y) F\big(y, u^{\varepsilon_j}(y)\big)\, \mathrm{d}y = \int_U \int_{\mathbb{R}^N} \varphi(y) F(y,\lambda)\, \mathrm{d}\mu_y(\lambda)\, \mathrm{d}y \tag{1.3.11}$$

for all $\varphi(y) \in L^r(U)$ with compact support in the closure of U, where $1/r + q/s = 1$. Moreover,

$$\iint |\lambda|^s\, \mathrm{d}\mu_y(\lambda)\, \mathrm{d}y \leqslant \underline{\lim}_{\varepsilon_j \to 0} \big\| u^{\varepsilon_j}(y) \big\|_{L^s}^s. \tag{1.3.12}$$

1.4. *Relation to Camassa–Holm equation*

Physically by approximating directly the Hamiltonian for Euler equations in the shallow water regime, Camassa and Holm [7] derived the following equation

$$\partial_t u - \partial_x^2 \partial_t u + 3u\, \partial_x u = 2\, \partial_x u\, \partial_x^2 u + u\, \partial_x^3 u, \quad t > 0,\ x \in \mathbb{R}. \tag{1.4.13}$$

Mathematically, (1.4.13) is obtained and proved to be formally integrable by Fuchssteiner and Fokas [25] as a bi-Hamiltonian generalization of Korteweg–de Vries (KDV) equation. Equation (1.4.13) has several important features that distinguish it from the well-known KDV equation. First, Camassa and Holm discovered that (1.4.13) possesses peaked solutions with a corner at their crest, which is in sharp contrast to the solitary waves for KDV. Second, physical water waves often break down, which can not be predicted by the solutions to the KDV equation.

Formally (1.4.13) is equivalent to

$$\begin{cases} \partial_t u + u\, \partial_x u + \partial_x P = 0, & t > 0, x \in \mathbb{R}, \\ P(t,x) = \int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|} \big(u^2 + \frac{1}{2}(u_y)^2\big)(t,y)\, \mathrm{d}y. \end{cases} \tag{1.4.14}$$

In [11,52], the authors proved the finite time break down of smooth solution to (1.4.14). In particular, McKean gives a necessary and sufficient criterion on the initial data for the finite-time formulation of singularities in a smooth solution to (1.4.14). Furthermore, McKean describes the blow-up process by showing the formation of cusps instead of shocks for compressible fluids. Motivated by the Young measure approach in Section 2 of this chapter (see [72]), Xin and the first author of this chapter (see [67]) proved the global existence of weak solution to (1.4.14) with initial data in $H^1(\mathbb{R})$.

Set $v = \partial_x u$, by taking ∂_x to the first equation of (1.4.14), we get

$$\partial_t v + u\,\partial_x v = -\frac{1}{2}v^2 - P + u^2. \tag{1.4.15}$$

Compared this equation with (1.2.10), we find that (1.2.10) can be considered as the first-order approximation to (1.4.14).

1.5. *Open problems*

Up to now, we already proved the global existence and uniqueness of both dissipative and energy conservative solution to (1.2.10) through Young measure approach and explicit construction of the approximate solutions (see Section 2 of this chapter or [72]). Notice the relations between (1.2.10) and (1.4.14), we can step-by-step follow the method of [67] to prove the global existence of dissipative solution by vanishing viscosity.

OPEN PROBLEM 1. The global existence of energy conservative weak solution to (1.2.10) via the vanishing dispersion limit to (1.2.10).

As for the original variational wave equation (0.0.1), up to my recent result [75,76], we only proved the global existence of weak solution to (0.0.1) with monotone wave speed.

OPEN PROBLEM 2. The uniqueness of weak solution is open.

And finally we have the third problem.

OPEN PROBLEM 3. The global existence and uniqueness of weak solution to (0.0.1) is completely open.

Finally, let us outline the main contents of this chapter. In Section 2 we present some global existence and uniqueness results to the asymptotic equation. In Section 3 we prove some results on invariant region with monotone wave speed and in Section 4 we present one general global existence result to this equation. For some recent progress on this subject see [76].

2. The first asymptotic equation

2.1. *Introduction*

In this section we establish the global existence and uniqueness of admissible weak solutions to the initial–boundary value problem

$$\begin{cases} \partial_t v + u\,\partial_x v = -\frac{1}{2}v^2, & x > 0, t > 0, \\ \partial_x u = v(t,x), \\ u(t,x)|_{x=0} = 0, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.1.1}$$

where $v_0(x) \in L^2(\mathbb{R}^+)$. We use the notation $\mathbb{R}^+ := (0,\infty)$. We recall that Hunter and Zheng [37] established the global existence of both dissipative and conservative weak solutions to (2.1.1) with initial data $v_0(x) \in BV(\mathbb{R}^+)$. We established in [70] the global existence of dissipative weak solutions to (2.1.1) with nonnegative L^p initial data for any $p > 2$ by applying the theory of Young measures. In [71] we established the global existence and uniqueness of dissipative weak solutions to (2.1.1) with nonnegative $L^2(\mathbb{R}^+)$ initial data.

Among the many solutions that (2.1.1) may have for general data, the interesting ones are the dissipative and conservative weak solutions. The difference between the dissipative and conservative weak solutions is in the continuation of the solution beyond its blow-up time. The conservative solution is the one which preserves the energy, while the dissipative solution loses all the energy at the blow-up time. The conservative solution preserves the complete integrability of the system [36], while the dissipative solution is the limit of vanishing artificial viscosity [37].

In this section we first establish the existence of a global dissipative weak solution to (2.1.1) for any given $v_0 \in L^2$. The space L^2 is the natural space from the perspective of the energy estimate. This generalizes maximally earlier existence results in the spaces BV, L^p $(p > 2)$ or L^2 of nonnegative functions. Our main result of this section is the uniqueness of both the dissipative and conservative weak solutions. The key condition for uniqueness of the dissipative weak solutions is an entropy condition similar to Oleĭnik's entropy condition for conservation laws. The key condition for uniqueness of conservative weak solutions is the conservation of energy between characteristics of the solutions. In comparison with the entropy rate criterion of Dafermos [16] for conservation laws, we comment that the energy of a dissipative weak solution decays at the fastest possible rate, while that of a conservative weak solution decays the slowest (i.e., no decay at all). We mention in passing that there are many weak solutions that are in between these two extreme solutions. Finally, the existence of a global conservative weak solution is established together with its uniqueness.

We now begin to present precisely our results. Let us first give the definition of admissible dissipative weak solutions, which is in consistency with earlier work [37]. Let $Q_\infty := [0,\infty) \times [0,\infty)$.

DEFINITION 2.1.1 (Admissible dissipative weak solutions). We call $(v(t,x), u(t,x))$ an admissible dissipative weak solution of (2.1.1) if (d1) the functions have the regularity

$$(v,u)(t,x) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+, L^2\big(\mathbb{R}^+\big)\big) \otimes C(Q_\infty);$$

(d2) the functions satisfy in the sense of distributions the equations

$$\partial_t v + \partial_x(uv) = \frac{1}{2}v^2, \qquad \partial_x u = v;$$

(d3) the energy $\int_{\mathbb{R}^+} v^2(t,x)\,\mathrm{d}x$ is nonincreasing in $t \in [0,\infty)$;

(d4) the function $u(t,x)$ is equal to zero at $x=0$ as a continuous function and the function $v(t,x)$ takes on the initial value $v_0(x)$ in the sense $C([0,\infty), L^1(\mathbb{R}^+))$;

(d5) the entropy condition holds

$$v(t,x) \leqslant \frac{2}{t} \quad \text{a.e. } (t,x) \in Q_\infty. \tag{2.1.2}$$

We remark that the entropy condition (2.1.2) is of Oleǐnik type $\partial_x u \leqslant C/t$, a one-sided inequality. Also, for both the existence and uniqueness purposes, we can replace condition (d3) by either the right continuity of $v(t,\cdot)$ at $t=0+$ in L^2, or the energy inequality $\lim_{t\to 0+}\int_{\mathbb{R}^+} v^2(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^+} v_0^2(x)\,\mathrm{d}x$; the decreasing of energy for $t>0$ is then a consequence. Without condition (d3), however, the solution is not unique and an example is the zero initial datum which would have, besides the zero solution, the nontrivial solution

$$v(t,x) = \frac{2}{t}\mathbb{1}_{\{0<x<t^2/4\}},$$

where $\mathbb{1}_{\{0<x<t^2/4\}}$ denotes the characteristic function of the set $\{0<x<t^2/4\}$.

We give the definition of admissible conservative weak solutions, strengthened from the earlier papers [36,37].

DEFINITION 2.1.2 (Admissible conservative weak solutions). We call $(v(t,x), u(t,x), \Phi_t(x))$ an admissible conservative weak solution of (2.1.1) if

(c1) the functions have the regularity

$$\big(v(t,x), u(t,x), \Phi_t(x)\big) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+, L^2\big(\mathbb{R}^+\big)\big) \otimes C(Q_\infty) \otimes C(Q_\infty),$$
$$\partial_t \Phi_t(x) \in C(Q_\infty);$$

(c2) the functions (v,u) satisfy the equations

$$\partial_t v + \partial_x(uv) = \frac{1}{2}v^2, \qquad \partial_x u = v, \qquad \partial_t\big(v^2\big) + \partial_x\big(uv^2\big) = 0$$

in the sense of distributions, while $\Phi_t(x)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = u\big(t, \Phi_t(x)\big), \quad \Phi_0(x) = x,\ x \geqslant 0; \tag{2.1.3}$$

(c3) the local energy conservation holds

$$\int_{\Phi_t(x_1)}^{\Phi_t(x_2)} v^2(t,y)\,\mathrm{d}y = \int_{x_1}^{x_2} v_0^2(x)\,\mathrm{d}x \quad \text{a.e. } t \in \mathbb{R}^+,\ \forall x_1 < x_2;$$

(c4) the function $u(t,x)$ is equal to zero at $x = 0$ as a continuous function and the function $v(t,x)$ takes on the initial value $v_0(x)$ in $C([0,\infty), L^1(\mathbb{R}^+))$.

Now we state the main results of this section.

THEOREM 2.1.3 (Dissipative solutions). *Let $v_0(x) \in L^2(\mathbb{R}^+)$ have compact support. Then* (2.1.1) *has a unique global admissible dissipative weak solution (v,u) in the sense of Definition* 2.2.6. *In addition, the solution satisfies $v \in L^p_{\mathrm{loc}}(Q_\infty)$, $u \in W^{1,p}_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$, $v(t,x) \in C([0,\infty), L^q(\mathbb{R}^+))$ for all $q < 2$ and $v(t,x) \in C_+([0,\infty), L^2(\mathbb{R}^+))$* (*the right continuity*). *Moreover,*

$$\Phi_t(x) = \int_0^x \left(1 + \frac{1}{2}v_0(y)t\right)^2 \mathbb{1}_{\{1+(1/2)v_0(y)t \geqslant 0\}}\,\mathrm{d}y$$

is the unique solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = u\big(t, \Phi_t(x)\big), \quad \Phi_0(x) = x, \tag{2.1.4}$$

and for almost all $x \in \mathbb{R}^+$ there hold the formula

$$v\big(t, \Phi_t(x)\big) = \frac{2v_0(x)}{2 + v_0(x)t}\mathbb{1}_{\{2+v_0(x)t \geqslant 0\}}$$

and the energy inequality

$$\int_{\Phi_t(x_1)}^{\Phi_t(x_2)} v^2(t,y)\,\mathrm{d}y \leqslant \int_{x_1}^{x_2} v_0^2(y)\,\mathrm{d}y \quad \forall t \geqslant 0,\ 0 \leqslant x_1 \leqslant x_2. \tag{2.1.5}$$

We comment that the solution $(v(t,x), u(t,x))$ also satisfies

$$\partial_t f(v) + u\,\partial_x f(v) + \frac{1}{2}v^2 f'(v) \leqslant 0$$

in the sense of distributions for all functions f of the form $f(\xi) = \int_0^\xi\int_0^\zeta \psi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\zeta$, where $\psi(\sigma) \in C_c^\infty(\mathbb{R})$ and $\psi \geqslant 0$. This will follow from the proof of the existence part of Theorem 2.1.3. We remark further that in [70] and [71], we have proved that any dissipative solution $v(t,x)$ belongs to $C([0,\infty), L^q(\mathbb{R}^+))$ for $q \geqslant 2$ if the initial data are $v_0(x) \geqslant 0$ and $v_0(x) \in L^q(\mathbb{R}^+)$. But here we do not have $v(t,x) \in C([0,\infty), L^2(\mathbb{R}^+))$ for a dissipative solution due to the sign change of the initial data and dissipation. An explicit example of piecewise smooth dissipative solution can be easily constructed to show that $v(t,x) \notin C([0,\infty), L^2(\mathbb{R}^+))$ for $v_0(x) \in L^2(\mathbb{R}^+)$.

For those interested in generalizations of the uniqueness of flow maps of the type studied by DiPerna and Lions [19], we mention with great interest that we establish in Propositions A and B in Section 2.5, two uniqueness results for flow maps of u, where only the conditions $\partial_x u = v$, (d1) and (d3)–(d5) (or their generalizations), but not (d2), are assumed.

THEOREM 2.1.4 (Conservative solutions). *Let $v_0 \in L^2(\mathbb{R}^+)$ have compact support. Then* (2.1.1) *has a unique global admissible conservative weak solution $(v(t,x), u(t,x), \Phi_t(x))$ in the sense of Definition* 2.1.2. *In addition, the solution satisfies $v \in L^p_{\mathrm{loc}}(Q_\infty)$, $u \in W^{1,p}_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$, $v(t,x) \in C([0,\infty), L^q(\mathbb{R}^+))$ for all $q < 2$, and $v(t,x) \in C([0,\infty), L^2(\mathbb{R}^+))$ for almost all t. Moreover, there hold the formulas*

$$\Phi_t(x) = \int_0^x \left(1 + \frac{1}{2}v_0(y)t\right)^2 \mathrm{d}y,$$

$$u\big(t, \Phi_t(x)\big) = \int_0^x \left(1 + \frac{1}{2}v_0(y)t\right)v_0(y)\,\mathrm{d}y,$$

$$v\big(t, \Phi_t(x)\big) = \frac{2v_0(x)}{2 + v_0(x)t}$$

for almost all $(t,x) \in Q_\infty$.

We remark that Theorem 2.1.3 may have in general multiple flow maps for a given u with the regularity available in Theorem 2.1.4. The uniqueness of the flow is established in Theorem 2.1.4 because it is required to preserve local energy. We find that this uniqueness is interesting if we contrast it with the uniqueness of DiPerna and Lions [19,48] and [49].

The methods we use for both theorems are Young measures and mollifiers. The existence approach in Theorem 2.1.3 is a refined version of the Young measure method used in [70]. The proof utilizes the Young measures and a technical splitting of the approximate solution v^n into its positive and negative parts, so that the partial variances of the Young measures corresponding to the positive and negative parts decouple weakly.

We have attempted other conceivable methods. It seems to us that, due to the swaying of the characteristics of the approximate solutions, the Cauchy property of the sequence of approximate solutions is not easy to obtain. The removal of the set of singular points, or the shielding technique of the test functions as used in [20], is not easy for implementation either, due to the conceivable denseness of the singular set.

Our main tool for the uniqueness is the mollification technique, used in [71] earlier, but here we also need to estimate the size of the set of blow-up points. For conservative solutions, we utilize additionally that the characteristics are integral parts of the solution. This way, as a by-product, the existence proof of the conservative solutions is more direct than that for the dissipative solutions, as the former involves mainly the characteristics method and simple weak convergence techniques. This is fortunate, since the Young measure approach for conservative solutions fails in our attempt due to the generation of positive parts from the negative parts of the solutions.

2.2. *Existence of dissipative solutions*

2.2.1. *Approximate solutions.* We construct approximate solutions to (2.1.1). As in [37], we solve Definition 2.1.1 with simple functions as initial data. Without loss of generality, we assume that $\operatorname{supp} v_0 \subset [0,1)$. We approximate $v_0(x)$ by step functions $\{v_0^n(x)\}$ defined by

$$v_0^n(x) = v_i^n := n\int_{(i-1)/n}^{i/n} v_0(y)\,\mathrm{d}y, \quad x \in \left[\frac{i-1}{n}, \frac{i}{n}\right), \ i = 1, 2, \ldots, n. \tag{2.2.6}$$

Without loss of generality, we assume that every point of $[0,1]$ is a Lebesgue point of $v_0(x)$, thus

$$\lim_{n\to\infty} \left\| v_0^n - v_0 \right\|_{L^2([0,1])} = 0 \quad \text{and} \quad \lim_{n\to\infty} v_0^n(x) = v_0(x) \quad \forall x \in [0,1]. \tag{2.2.7}$$

We construct solutions of (2.1.1) with initial data $v_0(x)$ replaced by $v_0^n(x)$. From [37] or by directly applying the characteristic method, we obtain the admissible dissipative weak solutions

$$v^n(t,x) = \frac{2v_i^n}{2 + v_i^n t}, \quad x_{i-1}^n(t) \leqslant x < x_i^n(t), \tag{2.2.8}$$

where $x_0^n(t) := 0$ and

$$x_i^n(t) := \frac{1}{n}\sum_{j=1}^{i}\left(1 + \frac{1}{2}v_j^n t\right)^2 \mathbb{1}_{\{2+v_j^n t \geqslant 0\}}. \tag{2.2.9}$$

The associated u^n follows from integration of v^n.

2.2.2. *Primitive estimates and precompactness*

LEMMA 2.2.1 (Primitive estimates). *For all $p \in [2,3)$, $T > 0$ and $R > 0$, the approximate solution sequence $\{v^n, u^n\}$ constructed above satisfies the estimates*

$$\text{(a)} \quad v^n(t,x) \leqslant \frac{2}{t}, \tag{2.2.10}$$

$$\text{(b)} \quad \left\| v^n(t_2,\cdot) \right\|_{L^2(\mathbb{R}^+)} \leqslant \left\| v^n(t_1,\cdot) \right\|_{L^2(\mathbb{R}^+)} \leqslant \left\| v_0^n \right\|_{L^2([0,1])}, \quad 0 < t_1 < t_2,$$

$$\text{(c)} \quad \left\| v^n \right\|^p_{L^p([0,T]\times\mathbb{R}^+)} \leqslant C_{T,p} \left\| v_0^n \right\|^2_{L^2([0,1])}.$$

Moreover, $\{u^n(t,x)\}$ are uniformly bounded in $W^{1,p}_{\mathrm{loc}}(Q_\infty)$.

PROOF. The proof can be found on p. 329 of [37]. Inequality (c) can also be deduced from Theorem 3 of [70]. □

LEMMA 2.2.2 (Basic precompactness). *There exist some $u \in W^{1,p}_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$ and a subsequence of $\{u^n\}$ which we still denote by $\{u^n\}$ such that*

$$u^n(t,x) \to u(t,x) \tag{2.2.11}$$

uniformly on any compact subset of Q_∞. Moreover,

$$v^n(t,x) = \partial_x u^n(t,x) \rightharpoonup \partial_x u(t,x) =: v(t,x) \tag{2.2.12}$$

weakly in $L^p_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$. Further, there hold

$$\left\|u(t,\cdot)\right\|_{L^\infty} \leqslant \int_0^1 \left|v_0(x)\right| \mathrm{d}x + \frac{t}{2}\int_0^1 |v_0|^2 \,\mathrm{d}x \tag{2.2.13}$$

and

$$\operatorname{supp} v^n(t,\cdot) \subset \big[0, K(t)\big) \tag{2.2.14}$$

for some $K(t) < \infty$.

PROOF. The convergence in (2.2.11) and (2.2.12) follows from Lemma 2.2.1 directly. We only prove (2.2.13) and (2.2.14). In fact, by the construction of $\{v^n(t,x)\}$, we find

$$\begin{aligned}
\left\|u^n(t,\cdot)\right\|_{L^\infty} &\leqslant \int_0^\infty \left|v^n(t,x)\right| \mathrm{d}x \\
&= \frac{1}{n}\sum_{i=1}^n \frac{2|v_i^n|}{|2+v_i^n t|}\left(1+\frac{1}{2}v_i^n t\right)^2 \mathbb{1}_{\{2+v_i^n t \geqslant 0\}} \\
&\leqslant \int_0^1 \left|v_0^n(x)\right| \mathrm{d}x + \frac{t}{2}\int_0^1 \left|v_0^n(x)\right|^2 \mathrm{d}x \\
&\leqslant C(t) \quad \forall n,
\end{aligned} \tag{2.2.15}$$

for some locally bounded function $C(t)$. Hence, by the convergence in (2.2.11) and (2.2.12), we obtain (2.2.13). Moreover, by (2.2.15), we obtain (2.2.14) for $K(t) = C(t)t + 1$. $\square$

In the sequel, we shall use the notation $Q_T := [0,T] \times [0, K(T)]$. In particular, $Q_\infty = [0,\infty) \times [0,\infty)$ is consistent with our earlier notation.

We need the strong precompactness of $\{v^n(t,x)\}$ which we establish in late.

2.2.3. *Strong precompactness.* We prove the precompactness of the solution sequence $\{v^n(t,x)\}$ in $L^p([0,\ T] \times \mathbb{R}^+)$ for all $T < \infty$, $p < 3$, by applying the Young measure theory [20,23,63,68], the ideas used by Lions in the proof of the global existence of weak solutions to multidimensional Navier–Stokes equations [49], and the ideas used by Joly, Métivier and Rauch [39] in the rigorous justification of the weakly nonlinear geometric optics for semilinear wave equations.

LEMMA 2.2.3 (Time-distinguished Young measures). *There exist a subsequence of the solution sequence* $\{v^n(t,x)\}$, *for convenience we still denote it by* $\{v^n(t,x)\}$, *and a family of Young measures* $\mu(t,x,\mathrm{d}\lambda)$ *such that for all continuous functions* $f(t,x,\lambda) = \mathrm{o}(|\lambda|^q)$, $\partial_\lambda f(t,x,\lambda) = \mathrm{o}(|\lambda|^{q-1})$ *as* $\lambda \to \infty$ *for* $q < 2$, *and for all* $\psi(x) \in L^r_c([0,\infty))$ *with* $1/r + q/2 = 1$, *there holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}^+} f\big(t,x,v^n(t,x)\big)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^+} \overline{f(t,x,v)}\psi(x)\,\mathrm{d}x \tag{2.2.16}$$

uniformly in every compact subset of $[0,\infty)$, *where*

$$\overline{f(t,x,v)} := \int_{\lambda\in\mathbb{R}} f(t,x,\lambda)\mu(t,x,\mathrm{d}\lambda) \in C\big([0,\infty), L^{q'/q}\big(\mathbb{R}^+\big)\big) \tag{2.2.17}$$

for all $q' \in (q,2)$. *Moreover, for all* $T > 0$, *there hold*

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T\int_{\mathbb{R}^+} g\big(t,x,v^n\big)\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^+} \overline{g(t,x,v)}\varphi\,\mathrm{d}x\,\mathrm{d}t \quad \textit{and}\\
&\lambda \in L^p_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R},\ \mathrm{d}t \otimes \mathrm{d}x \otimes \mu(t,x,\mathrm{d}\lambda)\big) \quad \textit{for all}\ \ p < 3,
\end{aligned} \tag{2.2.18}$$

where the continuous function $g(t,x,\lambda) = \mathrm{o}(|\lambda|^p)$ *as* $\lambda \to \infty$ *for some* $p < 3$, *and* $\varphi(t,x) \in L^m(Q_T)$ *with* $1/p + 1/m = 1$.

The proof is the same as that of Lemma 3 in [70], so we omit it here.

REMARK 2.2.4. From Proposition 3.1.3 of [39] and the above lemma, we find that

$$\lambda \in L^\infty\big(\mathbb{R}^+, L^2\big(\mathbb{R}^+ \times \mathbb{R}, \mathrm{d}x \otimes \mu(t,x,\mathrm{d}\lambda)\big)\big). \tag{2.2.19}$$

Also, comparing the notation in (2.2.12) with that of (2.2.17), we have $\bar{v} \equiv v$.

LEMMA 2.2.5. *For almost all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$, *there holds* $\mu(t,x,\mathrm{d}\lambda) = \delta_{\bar{v}(t,x)}(\lambda)$.

PROOF.
Step 1. We derive an evolution equation for the variance of the Young measures.

Step 1.1. From the construction of $\{v^n\}$, we have

$$\partial_t v^n + u^n\,\partial_x v^n = -\frac{1}{2}\big(v^n\big)^2 \tag{2.2.20}$$

in the weak sense. We wish to multiply (2.2.20) with v^n and send $n \to \infty$, but the equation holds only in the weak sense and a cubic term like $(v^n)^3$ is not bounded in L^1. We use mollifiers to regularize the equation. So, by Lemma II.1 of [19], we have

$$\partial_t v^{n,\varepsilon} + u^n\, \partial_x v^{n,\varepsilon} = -\frac{1}{2}\big(v^{n,\varepsilon}\big)^2 + r_n^{\varepsilon}, \tag{2.2.21}$$

where $v^{n,\varepsilon}(t,x) := \int_{\mathbb{R}} v^n(t,y) j_\varepsilon(x-y)\,\mathrm{d}y$, $r_n^\varepsilon(t,x) := -(u^n\,\partial_x v^n) * j_\varepsilon + u^n\,\partial_x v^{n,\varepsilon} + \frac{1}{2}((v^{n,\varepsilon})^2 - (v^n)^2 * j_\varepsilon)$, with $j_\varepsilon(x)$ the standard Friedrichs' mollifier, $r_n^\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^+)) \cap L^{p/2}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$ for all $p < 3$. We remark that across the boundary $x = 0$ we extend the functions v^n and u^n with zero and (2.2.20) holds weakly. We then use a cut-off function to chop off v^n. So we let, as in [48],

$$T_R^+(\xi) = \begin{cases} 0, & \xi < 0, \\ \xi, & 0 \leqslant \xi \leqslant R, \\ R, & \xi \geqslant R, \end{cases} \qquad S_R^+(\xi) = \begin{cases} 0, & \xi < 0, \\ \frac{1}{2}\xi^2, & 0 \leqslant \xi \leqslant R, \\ R\xi - \frac{1}{2}R^2, & \xi \geqslant R. \end{cases} \tag{2.2.22}$$

That is, $T_R^+(\xi)$ is a cut-off function of ξ and $S_R^+(\xi)$ is an antiderivative of $T_R^+(\xi)$. Now we multiply $T_R^+(v^{n,\varepsilon})$ to both sides of (2.2.21) to yield

$$\begin{aligned} &\partial_t S_R^+\big(v^{n,\varepsilon}\big) + \partial_x\big(u^n S_R^+\big(v^{n,\varepsilon}\big)\big) \\ &\quad = v^n S_R^+\big(v^{n,\varepsilon}\big) - \frac{1}{2} T_R^+\big(v^{n,\varepsilon}\big)\big(v^{n,\varepsilon}\big)^2 + T_R^+\big(v^{n,\varepsilon}\big) r_n^\varepsilon. \end{aligned} \tag{2.2.23}$$

Trivially, $T_R^+(v^{n,\varepsilon}) r_n^\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^+))$ as $\varepsilon \to 0$ for each fixed R since $|T_R^+| \leqslant R$. Thus by taking $\varepsilon \to 0$ in (2.2.23), we find

$$\partial_t S_R^+\big(v^n\big) + \partial_x\big(u^n S_R^+\big(v^n\big)\big) = v^n S_R^+\big(v^n\big) - \frac{1}{2} T_R^+\big(v^n\big)\big(v^n\big)^2. \tag{2.2.24}$$

Hence, by Lemmas 2.2.2 and 2.2.3, we obtain

$$\partial_t \overline{S_R^+(v)} + \partial_x\big(u \overline{S_R^+(v)}\big) = \overline{v S_R^+(v) - \frac{1}{2} T_R^+(v) v^2} =: F^+. \tag{2.2.25}$$

Step 1.2. On the other hand, by directly applying Lemmas 2.2.2 and 2.2.3 on the equation

$$\partial_t v^n + \partial_x\big(u^n v^n\big) = \frac{1}{2}\big(v^n\big)^2$$

in the limit $n \to \infty$, we find

$$\partial_t \bar{v} + \partial_x(u\bar{v}) = \frac{1}{2}\overline{v^2},$$

which is

$$\partial_t \bar{v} + u\, \partial_x \bar{v} = \left(\frac{1}{2}\overline{v^2} - \bar{v}^2\right). \tag{2.2.26}$$

Thus, by a similar argument as that from (2.2.20)–(2.2.24), we find

$$\partial_t S_R^+(\bar{v}) + \partial_x\big(u S_R^+(\bar{v})\big) = T_R^+(\bar{v})\left(\frac{1}{2}\overline{v^2} - \bar{v}^2\right) + \bar{v} S_R^+(\bar{v}) =: G^+. \tag{2.2.27}$$

Step 1.3. Subtracting (2.2.27) from (2.2.25), we obtain

$$\partial_t\big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big) + \partial_x\big(u\big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big)\big) = F^+ - G^+. \tag{2.2.28}$$

But

$$\begin{aligned} F^+ &= \int_{\mathbb{R}} \left(\lambda S_R^+(\lambda) - \frac{1}{2} T_R^+(\lambda)\lambda^2\right)\mu(t,x,\mathrm{d}\lambda) \\ &= \int_{\mathbb{R}} \left(\frac{1}{2} R\lambda(\lambda - R)\mathbb{1}_{\lambda \geqslant R}\right)\mu(t,x,\mathrm{d}\lambda), \\ G^+ &= \frac{1}{2} T_R^+(\bar{v})\big(\overline{v^2} - \bar{v}^2\big) + \frac{1}{2} R\bar{v}(\bar{v} - R)\mathbb{1}_{\bar{v} \geqslant R}. \end{aligned} \tag{2.2.29}$$

And by the construction of $\{v^n\}$, we find that both $\bar{v}(t,x)$ and $v^n(t,x)$ are less than or equal to $2/t$. Thus we have $\operatorname{supp} \mu(t,x,\cdot) \subset (-\infty, \frac{2}{t}]$ and

$$F^+ = 0, \qquad \frac{1}{2} R\bar{v}(\bar{v} - R)\mathbb{1}_{\bar{v} \geqslant R} = 0 \quad \text{for } t > \frac{2}{R}. \tag{2.2.30}$$

Summing up (2.2.28)–(2.2.30), we find

$$\partial_t\big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big) + \partial_x\big(u\big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big)\big) \leqslant 0, \quad t > \frac{2}{R}. \tag{2.2.31}$$

Step 1.4. Similar to the proof of (2.2.28), we can also prove that

$$\partial_t\big(\overline{S_R^-(v)} - S_R^-(\bar{v})\big) + \partial_x\big(u\big(\overline{S_R^-(v)} - S_R^-(\bar{v})\big)\big) = F^- - G^-, \tag{2.2.32}$$

where

$$T_R^-(\xi) = \begin{cases} 0, & \xi > 0, \\ \xi, & -R \leqslant \xi \leqslant 0, \\ -R, & \xi \leqslant -R, \end{cases} \qquad S_R^-(\xi) = \begin{cases} 0, & \xi > 0, \\ \frac{1}{2}\xi^2, & -R \leqslant \xi \leqslant 0, \\ -R\xi - \frac{1}{2}R^2, & \xi \leqslant -R, \end{cases} \tag{2.2.33}$$

and

$$\begin{aligned} F^- &:= \overline{vS_R^-(v) - \frac{1}{2}T_R^-(v)v^2} \\ &= \int_{\mathbb{R}} \left(-\frac{1}{2}R\lambda(\lambda+R)\mathbb{1}_{\lambda\leqslant -R}\right)\mu(t,x,\mathrm{d}\lambda), \\ G^- &:= \bar{v}S_R^-(\bar{v}) + T_R^-(\bar{v})\left(\frac{1}{2}\overline{v^2} - \bar{v}^2\right) \\ &= \frac{1}{2}T_R^-(\bar{v})\big(\overline{v^2} - \bar{v}^2\big) - \frac{1}{2}R\bar{v}(\bar{v}+R)\mathbb{1}_{\bar{v}\leqslant -R}. \end{aligned} \tag{2.2.34}$$

Therefore,

$$F^- - G^- = -\frac{1}{2}RJ_R - \frac{1}{2}T_R^-(\bar{v})\big(\overline{v^2} - \bar{v}^2\big), \tag{2.2.35}$$

where

$$J_R := \int_{\mathbb{R}} \lambda(\lambda+R)\mathbb{1}_{\lambda\leqslant -R}\mu(t,x,\mathrm{d}\lambda) - \bar{v}(\bar{v}+R)\mathbb{1}_{\bar{v}\leqslant -R}.$$

To handle $(\overline{v^2} - \bar{v}^2)$, we use the splitting

$$\overline{v^2} = \overline{(v_+)^2} + \overline{(v_-)^2}, \qquad \bar{v}^2 = \big((\bar{v})_+\big)^2 + \big((\bar{v})_-\big)^2, \tag{2.2.36}$$

and the identity

$$\frac{1}{2}\Big(\overline{(v_-)^2} - \big((\bar{v})_-\big)^2\Big) = \overline{S_R^-(v)} - S_R^-(\bar{v}) + \frac{1}{2}J_R + \frac{1}{2}RH_R, \tag{2.2.37}$$

where we have introduced the notation

$$\begin{aligned} &w_\pm = \pm\max\{0, \pm w\}, \qquad w = v, \lambda, \text{ or } \bar{v}; \\ &\overline{(v_\pm)^2} = \int_{\mathbb{R}} (\lambda_\pm)^2\mu(t,x,\mathrm{d}\lambda); \\ &H_R := \int (\lambda+R)\mathbb{1}_{\lambda\leqslant -R}\mu(t,x,\mathrm{d}\lambda) - (\bar{v}+R)\mathbb{1}_{\bar{v}\leqslant -R}. \end{aligned}$$

We then have from (2.2.35)–(2.2.37)

$$\begin{aligned} &F^- - G^- \\ &\quad = -\frac{1}{2}RJ_R - \frac{1}{2}T_R^-(\bar{v})(J_R + RH_R) \\ &\qquad - T_R^-(\bar{v})\big(\overline{S_R^-(v)} - S_R^-(\bar{v})\big) - \frac{1}{2}T_R^-(\bar{v})\Big(\overline{(v_+)^2} - \big((\bar{v})_+\big)^2\Big). \end{aligned} \tag{2.2.38}$$

And since $R + T_R^-(\bar{v}) \geqslant 0$, $-T_R^-(\bar{v}) \geqslant 0$ and $\lambda(\lambda + R)\mathbb{1}_{\lambda \leqslant -R}$ is a convex function, while $(\lambda + R)\mathbb{1}_{\lambda \leqslant -R}$ is a concave function, we obtain

$$-\frac{1}{2}\big(R + T_R^-(\bar{v})\big) J_R \leqslant 0, \qquad -\frac{1}{2} T_R^-(\bar{v}) H_R \leqslant 0. \tag{2.2.39}$$

Thus, by summing up (2.2.32), (2.2.38) and (2.2.39), we find

$$\begin{aligned} &\partial_t\big(\overline{S_R^-(v)} - S_R^-(\bar{v})\big) + \partial_x\Big(u\big(\overline{S_R^-(v)} - S_R^-(\bar{v})\big)\Big) \\ &\quad \leqslant R\bigg\{\Big[\overline{S_R^-(v)} - S_R^-(\bar{v})\Big] + \frac{1}{2}\Big[\overline{(v_+)^2} - \big((\bar{v})_+\big)^2\Big]\bigg\}. \end{aligned} \tag{2.2.40}$$

Step 2. We show that the family of Young measures are Dirac masses.

Step 2.1. Let us first claim the right continuity

$$\lim_{t \to 0+} \int \bar{v}^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} v_0^2(x)\,\mathrm{d}x. \tag{2.2.41}$$

In fact, by (2.2.26) satisfied by $\bar{v}(t,x)$, we can derive sufficient continuity so that

$$\bar{v}(t,x) \rightharpoonup v_0(x) \quad \text{in } L^2(\mathbb{R}) \text{ as } t \to 0. \tag{2.2.42}$$

Hence, by Theorem 1 on p. 4 of [23], we have

$$\int v_0^2(x)\,\mathrm{d}x \leqslant \varliminf_{t \to 0} \int \bar{v}^2(t,x)\,\mathrm{d}x. \tag{2.2.43}$$

But by Lemma 2.2.1, we trivially have

$$\int_{\mathbb{R}} \bar{v}^2(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} v_0^2(x)\,\mathrm{d}x \quad \forall t > 0. \tag{2.2.44}$$

By summing up (2.2.43) and (2.2.44), we obtain (2.2.41).

Step 2.2. We then claim that

$$\int_{\mathbb{R}} \big(\overline{(v_+)^2} - (\bar{v}_+)^2\big)(t,x)\,\mathrm{d}x = 0 \quad \forall t \in \mathbb{R}^+. \tag{2.2.45}$$

In fact, by Lemma 2.2.3 and (2.2.31), we have for $t > 2/R$

$$\int_{\mathbb{R}} \big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big)(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \big(\overline{S_R^+(v)} - S_R^+(\bar{v})\big)\Big(\frac{2}{R}, x\Big)\,\mathrm{d}x. \tag{2.2.46}$$

By (2.2.19) and Lebesgue dominated convergence theorem, we have

$$\lim_{R\to\infty}\int_{\mathbb{R}}\big(\overline{S_R^+(v)}-S_R^+(\bar v)\big)(t,x)\,\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}}\big(\overline{(v_+)^2}-(\bar v_+)^2\big)(t,x)\,\mathrm{d}x. \tag{2.2.47}$$

Further, by the definition of $S_R^+(\xi)$ and similar to (2.2.37), we find

$$\frac{1}{2}\big(\overline{(v_+)^2}-\big((\bar v)_+\big)^2\big)=\overline{S_R^+(v)}-S_R^+(\bar v)+\frac{1}{2}I_R,$$

where

$$I_R:=\int(\lambda-R)^2\mathbb{1}_{\lambda\geqslant R}\,\mu(t,x,\mathrm{d}\lambda)-(\bar v-R)^2\mathbb{1}_{\bar v\geqslant R}\geqslant 0$$

due to convexity, which implies

$$\int_{\mathbb{R}}\big(\overline{S_R^+(v)}-S_R^+(\bar v)\big)\left(\frac{2}{R},x\right)\mathrm{d}x\leqslant\frac{1}{2}\int_{\mathbb{R}}\big(\overline{v^2}-\bar v^2\big)\left(\frac{2}{R},x\right)\mathrm{d}x. \tag{2.2.48}$$

By Lemma 2.2.1 and (2.2.41), we find

$$\lim_{R\to\infty}\int_{\mathbb{R}}\big(\overline{v^2}-\bar v^2\big)\left(\frac{2}{R},x\right)\mathrm{d}x\leqslant\int_{\mathbb{R}}v_0^2(x)\,\mathrm{d}x-\lim_{R\to\infty}\int_{\mathbb{R}}\bar v^2\left(\frac{2}{R},x\right)\mathrm{d}x=0. \tag{2.2.49}$$

By summing up (2.2.46)–(2.2.49), we obtain (2.2.45).

Step 2.3. Integrating (2.2.40) with respect to x and using (2.2.45), we have

$$\partial_t\int_{\mathbb{R}}\big(\overline{S_R^-(v)}-S_R^-(\bar v)\big)\,\mathrm{d}x\leqslant R\int_{\mathbb{R}}\big(\overline{S_R^-(v)}-S_R^-(\bar v)\big)\,\mathrm{d}x. \tag{2.2.50}$$

Thus, by Lemma 2.2.3 and Gronwall's inequality, we immediately obtain

$$\int_{\mathbb{R}}\big(\overline{S_R^-(v)}-S_R^-(\bar v)\big)(t,x)\,\mathrm{d}x=0\quad\forall t\in\mathbb{R}^+. \tag{2.2.51}$$

Then again by (2.2.19) and Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}}\big(\overline{(v_-)^2}-(\bar v_-)^2\big)\,\mathrm{d}x=\lim_{R\to\infty}\int_{\mathbb{R}}\big(\overline{S_R^-(v)}-S_R^-(\bar v)\big)(t,x)\,\mathrm{d}x=0. \tag{2.2.52}$$

Thus, by summing up (2.2.45) and (2.2.52), we in fact obtain

$$\int_{\mathbb{R}}\big(\overline{v^2}-\bar v^2\big)\,\mathrm{d}x=\iint_{\mathbb{R}^2}\big|\lambda-\bar v(t,x)\big|^2\mu(t,x,\mathrm{d}\lambda)\,\mathrm{d}x=0. \tag{2.2.53}$$

This shows that $\mu(t,x,\mathrm{d}\lambda)=\delta_{\bar v(t,x)}(\lambda)$ for almost all $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. This completes the proof of Lemma 2.2.5. □

We remark that the difficulty in establishing the strong precompactness of $\{v^n\}$ lies in the possible swaying of characteristics at a positive time t. The Young measure tool handles this difficulty successfully. Other methods, including removal of a small singularity set, the Cauchy sequence property, or Sobolev embedding, are not successful in our attempts.

2.2.4. *The existence proof.* We now prove the existence and some regularity parts of Theorem 2.1.3.

By Lemmas 2.2.1 and 2.2.5, we have that $v^n(t,y)\to v(t,y)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+)$ for all $p<3$, and $v(t,y)\leqslant 2/t$. Thus by Lemma 2.2.2, we can take the limit $n\to\infty$ in the equation

$$\partial_t v^n+\partial_x\big(u^n v^n\big)=\frac{1}{2}\big(v^n\big)^2.$$

This proves that $v(t,x)$ of (2.2.12) is indeed a global weak solution of (2.1.1). (The explicit formula of v will be established in the proof of uniqueness.) Moreover, by Lemma 2.2.1, we find that $v(t,x)\in L^p_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+)$ for all $p<3$. The continuity $v\in C([0,\infty),L^q(\mathbb{R}^+))$ for all $q<2$ is standard. The right continuity $v(t,x)\in C_+([0,\infty),L^2(\mathbb{R}^+))$ follows from the claim (2.2.41) applied at any time $t_0\geqslant 0$. The estimate on energy in a strip (2.1.5) is shown in the next subsection.

2.2.5. *The flow map.* Let $\{\Phi^n_t(x)\}$ be the flow of $u^n(t,x)$; i.e., $\Phi^n_t(x)$ satisfies

$$\begin{cases}\frac{\mathrm{d}\Phi^n_t(x)}{\mathrm{d}t}=u^n\big(t,\Phi^n_t(x)\big)=\int_0^{\Phi^n_t(x)}v^n(t,y)\,\mathrm{d}y,\\ \Phi^n_t(x)\big|_{t=0}=x.\end{cases}\tag{2.2.54}$$

We study the flow of $u(t,x)$, namely in the following lemma.

LEMMA 2.2.6 (Existence of flow). *There exists a subsequence of $\{u^n\}$ which we still denote by $\{u^n\}$ such that the corresponding flow sequence $\{\Phi^n_t(x)\}$ is uniformly convergent on every compact subset of Q_∞ to some $\Phi_t(x)\in C(Q_\infty)$ with $\partial_t\Phi_t(x)\in C(Q_\infty)$. Moreover, it satisfies*

$$\begin{cases}\frac{\mathrm{d}\Phi_t(x)}{\mathrm{d}t}=u\big(t,\Phi_t(x)\big)=\int_0^{\Phi_t(x)}v(t,y)\,\mathrm{d}y,\\ \Phi_t(x)|_{t=0}=x\end{cases}\tag{2.2.55}$$

in the classical sense, where (v,u) are from (2.2.11) *and* (2.2.12). *Further, there holds*

$$\Phi_t(x)=\int_0^x\left(1+\frac{1}{2}v_0(z)t\right)^2\mathbb{1}_{\{2+v_0(z)t\geqslant 0\}}\,\mathrm{d}z.\tag{2.2.56}$$

PROOF.

Step 1. We first establish

$$\Phi_t^n(x) = \int_0^x \left(1 + \frac{1}{2} v_0^n(z) t\right)^2 \mathbb{1}_{\{2+v_0^n(z)t \geqslant 0\}} \, \mathrm{d}z. \tag{2.2.57}$$

For any $x_1 \in [0, 1]$, say $x_1 \in [\frac{m-1}{n}, \frac{m}{n})$ for some m, we have from (2.2.54) that

$$\begin{aligned} \Phi_t^n(x_1) &= x_1 + \int_0^t \int_0^{\Phi_s^n(x_1)} v^n(s, y) \, \mathrm{d}y \, \mathrm{d}s \\ &= x_1 + \int_0^t \left(\sum_{k=1}^{m-1} \int_{x_{k-1}(s)}^{x_k(s)} \frac{2v_k^n}{2 + v_k^n s} \, \mathrm{d}y + \int_{x_{m-1}(s)}^{\Phi_s^n(x_1)} \frac{2v_m^n}{2 + v_m^n s} \, \mathrm{d}y \right) \mathrm{d}s. \end{aligned} \tag{2.2.58}$$

Using this formula we obtain the distance between two particular paths $\Phi_t^n(x_1)$ and $x_{m-1}(t)$ by subtraction

$$\begin{aligned} &\Phi_t^n(x_1) - x_{m-1}(t) \\ &\quad = x_1 - \frac{m-1}{n} + \int_0^t \int_{x_{m-1}(s)}^{\Phi_s^n(x_1)} \frac{2v_m^n}{2 + v_m^n s} \, \mathrm{d}y \, \mathrm{d}s \\ &\quad = x_1 - \frac{m-1}{n} + \int_0^t \frac{2v_m^n}{2 + v_m^n s} \big(\Phi_s^n(x_1) - x_{m-1}(s)\big) \, \mathrm{d}s \end{aligned} \tag{2.2.59}$$

since the solution is independent of y in a strip. Thus, if we let $Y^n(t) := \Phi_t^n(x_1) - x_{m-1}(t)$, we then find

$$\begin{cases} \frac{\mathrm{d}Y^n(t)}{\mathrm{d}t} = \frac{2v_m^n}{2+v_m^n t} Y^n(t), \\ Y^n(t)\big|_{t=0} = x_1 - \frac{m-1}{n}. \end{cases}$$

Integrating the ordinary differential equation before the possible blow-up time, we obtain

$$\Phi_t^n(x_1) - x_{m-1}(t) = \left(x_1 - \frac{m-1}{n}\right) \left(1 + \frac{1}{2} v_m^n t\right)^2 \mathbb{1}_{\{2+v_m^n t \geqslant 0\}}. \tag{2.2.60}$$

Similarly (or directly from (2.2.9)), we find

$$x_k(t) - x_{k-1}(t) = \frac{1}{n} \left(1 + \frac{1}{2} v_k^n t\right)^2 \mathbb{1}_{\{2+v_k^n t \geqslant 0\}}. \tag{2.2.61}$$

Summing up (2.2.58), (2.2.60) and (2.2.61), we obtain

$$\begin{aligned}
\Phi_t^n(x_1) = x_1 + \int_0^t \Bigg[\sum_{k=1}^{m-1} \frac{1}{2n} v_k^n \big(2 + v_k^n t\big) \mathbb{1}_{\{2+v_k^n t \geqslant 0\}} \\
+ \frac{1}{2} v_m^n \big(2 + v_m^n t\big) \mathbb{1}_{\{2+v_m^n t \geqslant 0\}} \left(x_1 - \frac{m-1}{n} \right) \Bigg] \mathrm{d}s \\
= \sum_{k=1}^{m-1} \left[\frac{1}{n} + \frac{1}{4n} v_k^n \big(4t + v_k^n t^2\big) \right] \mathbb{1}_{\{2+v_k^n t \geqslant 0\}} \\
+ \left[1 + \frac{1}{4} v_m^n \big(4t + v_m^n t^2\big) \right] \mathbb{1}_{\{2+v_m^n t \geqslant 0\}} \left(x_1 - \frac{m-1}{n} \right) \\
= \int_0^{x_1} \left(1 + v_0^n(z) t + \frac{1}{4} \big(v_0^n(z) t\big)^2 \right) \mathbb{1}_{\{2+v_0^n(z) t \geqslant 0\}} \, \mathrm{d}z
\end{aligned}$$

which yields (2.2.57).

Step 2. By (2.2.15), we have

$$\left\| \frac{\mathrm{d}\Phi_t^n(\cdot)}{\mathrm{d}t} \right\|_{L^\infty} = \big\| u^n\big(t, \Phi_t^n(\cdot)\big) \big\|_{L^\infty} \leqslant C(T) \tag{2.2.62}$$

for all $t \leqslant T$. By (2.2.7), (2.2.57) and Dunford–Pettis' theorem, we see that $\{\Phi_t^n(x)\}$ is equicontinuous with respect to x for all $(t,x) \in Q_T$. By Ascoli–Arzelà theorem and a diagonal process for the time T, we obtain the strong precompactness of $\{\Phi_t^n(x)\}$. That is, there exists a subsequence of $\{\Phi_t^n(x)\}$ such that $\{\Phi_t^n(x)\}$ uniformly converges to some $\Phi_t(x) \in C(Q_\infty)$ in Q_T for all $T > 0$.

By (2.2.54) we have

$$\Phi_t^n(x) = x + \int_0^t u^n\big(s, \Phi_s^n(x)\big) \, \mathrm{d}s.$$

By the above precompactness, we find that there holds

$$\Phi_t(x) = x + \int_0^t u\big(s, \Phi_s(x)\big) \, \mathrm{d}s.$$

Since $u(t,x) \in C(Q_\infty)$, we obtain that $\partial_t \Phi_t(x) \in C(Q_\infty)$ and $\Phi_t(x)$ satisfies (2.2.55) in the classical sense.

Furthermore by (2.2.7) and Dunford–Pettis' theorem, we easily deduce (2.2.56) from (2.2.57). This completes the proof of Lemma 2.2.6. $\square$

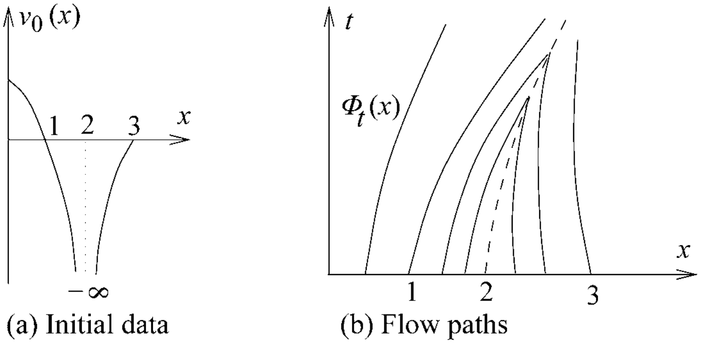

Fig. 1. An illustration of flow paths.

We remark that the flow map is unique, see Proposition A in Section 2.5. The set of singularity points of v has 2D Lebesgue measure zero, see Lemma 2.3.1. The flow map is invertible almost everywhere, see Lemma 2.3.3. See Figure 1 for an illustration of a flow map.

LEMMA 2.2.7 (Stripwise energy inequality). *There holds*

$$\int_{\Phi_t(x_1)}^{\Phi_t(x_2)} v^2(t,y)\,\mathrm{d}y \leqslant \int_{x_1}^{x_2} v_0^2(y)\,\mathrm{d}y \quad \forall t \in \mathbb{R}^+,\ 0 \leqslant x_1 < x_2 \leqslant 1, \tag{2.2.63}$$

for any weak solution v satisfying Definition 2.1.1 (*including the limit* (v,u) *obtained though* (2.2.11) *and* (2.2.12)), *where Φ is the associated flow.*

SKETCH OF THE PROOF. The idea is to mollify the first equation in (2.1.1) as in (2.2.21), use cut-off multipliers as in (2.2.22) and (2.2.33), then integrate the equation between two characteristics. The entropy condition (2.1.2) and the right continuity of $v(t,\cdot)$ in L^2 at $t=0$ are essential. The proof of Lemma 4 of [71] from its (3.9) to (3.27) contains completely the proof of (2.2.63); the new difficulty here that v may be negative does not cause any difficulty in its step (3.24). □

Summing up, we complete the proof of the existence and some regularity parts of Theorem 2.1.3. We establish the explicit formula and uniqueness of the solution in the next subsection.

2.3. *Uniqueness of dissipative solutions*

In this subsection, we establish the uniqueness and the explicit formula parts of Theorem 2.1.3 by utilizing the idea of the proof of the uniqueness of entropy solutions of (2.1.1) with nonnegative $L^2(\mathbb{R}^+)$ initial data in [71]. The approach is through the method of characteristics and regularization. The new difficulty for general $L^2(\mathbb{R}^+)$ initial data here is that characteristics may merge at finite times. We need to establish detailed structural properties of the solutions, as well as uniqueness, continuity, and differentiability properties, of the characteristics.

PROOF OF THE UNIQUENESS. We divide the proof into several steps.

Step 1. In this step we establish more details of the basic structure of the flow.

Let $(v(t,x), u(t,x))$ be a dissipative solution of (2.1.1) by Definition 2.1.1, and $\Phi_t(x)$ be a flow of $u(t,x)$; that is, $\Phi_t(x)$ satisfies

$$\begin{cases} \frac{\mathrm{d}\Phi_t(x)}{\mathrm{d}t} = u\big(t, \Phi_t(x)\big) = \int_0^{\Phi_t(x)} v(t,y)\,\mathrm{d}y, \\ \Phi_t(x)|_{t=0} = x. \end{cases} \tag{2.3.64}$$

CLAIM 1. *Let* $0 \leqslant x_1 < x_2 \leqslant 1$ *and* $t_0 > 0$. *Then there holds*

$$\Phi_t(x_1) = \Phi_t(x_2) \quad \textit{for all } t \geqslant t_0 \quad \textit{if } \Phi_{t_0}(x_1) = \Phi_{t_0}(x_2). \tag{2.3.65}$$

PROOF. The claim follows from the fact that $\partial_x u$ has an upper bound for $t \geqslant t_0$ and we are solving the ordinary differential equation (2.3.64) in the positive direction. In fact, suppose (2.3.65) fails, then we would have case (i) $\Phi_t(x_1) > \Phi_t(x_2)$ (or case (ii) $\Phi_t(x_1) < \Phi_t(x_2)$), due to the continuity of $\Phi_t(x)$ in t, for $t_2 > t > t_1 \geqslant t_0$ for some $t_2 > t_1 \geqslant t_0$ with $\Phi_{t_1}(x_1) = \Phi_{t_1}(x_2)$. Without loss of generality we set $t_1 = t_0$. From (2.3.64) and the entropy condition (2.1.2), we would then have, for case (i) for $t_2 > t > t_0$,

$$\begin{aligned} 0 &< \Phi_t(x_1) - \Phi_t(x_2) \\ &= \Phi_{t_0}(x_1) - \Phi_{t_0}(x_2) + \int_{t_0}^{t}\int_{\Phi_s(x_2)}^{\Phi_s(x_1)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant \int_{t_0}^{t} \frac{2}{s}\big(\Phi_s(x_1) - \Phi_s(x_2)\big)\,\mathrm{d}s. \end{aligned} \tag{2.3.66}$$

Then, by Gronwall's inequality, we find that $\Phi_t(x_1) = \Phi_t(x_2)$ for $t_2 \geqslant t \geqslant t_0$, a contradiction. The proof for case (ii) is the same. This proves (2.3.65). □

We obtain immediately from (2.3.65) the monotonicity

$$\Phi_t(x_1) \leqslant \Phi_t(x_2) \quad \forall t > 0, \text{ for any } 0 \leqslant x_1 < x_2 \leqslant 1. \tag{2.3.67}$$

And there holds the strip-wise energy estimate (2.2.63) of Lemma 2.2.7.

CLAIM 2. *For* $0 \leqslant x_1 < x_2 \leqslant 1$, *there holds*

$$\begin{aligned} &\Phi_t(x_2) - \Phi_t(x_1) \\ &\quad = x_2 - x_1 + t\int_{x_1}^{x_2} v_0(y)\,\mathrm{d}y + \frac{1}{2}\int_0^t\int_0^s\int_{\Phi_\tau(x_1)}^{\Phi_\tau(x_2)} v^2(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned} \tag{2.3.68}$$

PROOF. The proof is the same as (3.6) of [71]. For completeness, we reproduce its proof. Let $v^\varepsilon(t,x) = \int_{\mathbb{R}} j_\varepsilon(y)v(t,x-y)\,\mathrm{d}y$, where v can be extended by zero into $x<0$. By Lemma 2.3 of [48], we find that v^ε satisfies

$$\begin{cases} \partial_t v^\varepsilon + \partial_x\big(uv^\varepsilon\big) = \frac{1}{2}\big(v^\varepsilon\big)^2 + R_\varepsilon(t,x), \\ u(t,x) = \int_0^x v(t,y)\,\mathrm{d}y, \\ v^\varepsilon(t,x)\big|_{t=0} = v_0^\varepsilon(x), \end{cases} \tag{2.3.69}$$

where $R_\varepsilon(t,x) = \partial_x(uv^\varepsilon) - \partial_x(uv) * j_\varepsilon - \frac{1}{2}((v^\varepsilon)^2 - v^2 * j_\varepsilon)$, and $R_\varepsilon \to 0$ in $L^{1+\alpha/2}_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+)$ for all $\alpha\in[0,1)$. Integrating (2.3.69) over $[0,x]$, we obtain

$$\begin{aligned} &\partial_t \int_0^x v^\varepsilon(t,y)\,\mathrm{d}y + u(t,x)v^\varepsilon(t,x) \\ &\quad = \frac{1}{2}\int_0^x \big(v^\varepsilon(t,y)\big)^2\,\mathrm{d}y + \int_0^x R_\varepsilon(t,y)\,\mathrm{d}y. \end{aligned} \tag{2.3.70}$$

Let $u^\varepsilon(t,x) := \int_0^x v^\varepsilon(t,y)\,\mathrm{d}y$. Then by (2.3.70), we find

$$\frac{\mathrm{d}u^\varepsilon(t,\Phi_t(x))}{\mathrm{d}t} = \frac{1}{2}\int_0^{\Phi_t} \big(v^\varepsilon(t,y)\big)^2\,\mathrm{d}y + \int_0^{\Phi_t(x)} R_\varepsilon(t,y)\,\mathrm{d}y,$$

that is,

$$u^\varepsilon\big(t,\Phi_t(x)\big) = u_0^\varepsilon(x) + \frac{1}{2}\int_0^t\int_0^{\Phi_s} \big(v^\varepsilon(s,y)\big)^2\,\mathrm{d}y\,\mathrm{d}s + \int_0^t\int_0^{\Phi_s(x)} R_\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

where $u_0^\varepsilon(x) = \int_0^x v_0^\varepsilon(y)\,\mathrm{d}y$. Letting $u_0(x) = \int_0^x v_0(y)\,\mathrm{d}y$ and tending $\varepsilon\to 0$, we find

$$u\big(t,\Phi_t(x)\big) = u_0(x) + \frac{1}{2}\int_0^t\int_0^{\Phi_s(x)} v^2(s,y)\,\mathrm{d}s\,\mathrm{d}y. \tag{2.3.71}$$

Now, for $x_1<x_2$, integrating the first equation of (2.3.64) and by (2.3.71), we obtain

$$\begin{aligned} &\Phi_t(x_2) - \Phi_t(x_1) \\ &\quad = x_2 - x_1 + \int_0^t \big[u\big(s,\Phi_s(x_2)\big) - u\big(s,\Phi_s(x_1)\big)\big]\,\mathrm{d}s \\ &\quad = x_2 - x_1 + t\big[u_0(x_2) - u_0(x_1)\big] + \frac{1}{2}\int_0^t\int_0^s\int_{\Phi_\tau(x_1)}^{\Phi_\tau(x_2)} v^2(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{2.3.72}$$

which yields (2.3.68). $\square$

CLAIM 3. *For any $t > 0$, the derivative $\partial_x \Phi_t(x)$ exists and there holds*

$$\max\{0, 1 + t v_0(x)\} \leqslant \partial_x \Phi_t(x) \leqslant \left(1 + \frac{1}{2} v_0(x) t\right)^2 \tag{2.3.73}$$

for almost all $x \in [0, 1]$ and in the sense of distributions.

PROOF. The first inequality in (2.3.73) follows from (2.3.67) and (2.3.68) directly. By (2.3.68), (2.3.67) and (2.2.63), we obtain

$$\begin{aligned} 0 &\leqslant \Phi_t(x_2) - \Phi_t(x_1) \\ &\leqslant x_2 - x_1 + t \int_{x_1}^{x_2} v_0(y)\,\mathrm{d}y + \frac{t^2}{4} \int_{x_1}^{x_2} v_0^2(y)\,\mathrm{d}y \\ &= \int_{x_1}^{x_2} \left(1 + \frac{1}{2} t v_0(y)\right)^2 \mathrm{d}y. \end{aligned} \tag{2.3.74}$$

Thus, the flow $\Phi_t(x)$ is a nondecreasing continuous function with respect to x, $\Phi_t(x)$ can be differentiated with respect to x for almost all $x \in [0, 1]$ for any $t > 0$, and there holds (2.3.73). □

Step 2. We improve the above inequalities (2.3.73) from *almost everywhere* to *on almost all characteristics* and establish the continuity of $\partial_x \Phi_t(x) \in C(\mathbb{R}^+)$ in t for almost all fixed $x \in \mathbb{R}^+$.

CLAIM 4. *There exists a null set $\mathbb{B} \subset [0, 1]$ such that for any $x \in \mathbb{B}^c$ there holds*

$$\partial_x \Phi_t(x) \in C\big([0, \infty)\big). \tag{2.3.75}$$

PROOF. Take any $T > 0$. We let

$$E^T := \big\{(t, x) \in [0, T] \times [0, 1] \;\big|\; \partial_x \Phi_t(x) \text{ does not exist}\big\}.$$

Then, by Claim 3 and Fubini's theorem, we find that the measure of E^T is zero. Thus, if we denote $\mathbb{B}$ the set of points on $[0, 1]$ such that $H^1(S(x)) > 0$ for every point $x \in \mathbb{B}$, where $S(x) := \{(t, x) \mid t \in [0, T], \partial_x \Phi_t(x) \text{ does not exist}\}$ and $H^1(S(x))$ is the one-dimensional Hausdorff measure of $S(x)$, then trivially

$$\int_{\mathbb{B}} H^1\big(S(x)\big)\,\mathrm{d}x = m\big(E^T\big) = 0.$$

Thus, $m(\mathbb{B}) = 0$. So for all $T > 0$ and all $x \in [0, 1] \setminus \mathbb{B}$ with $m(\mathbb{B}) = 0$, there holds

$$H^1\big(S(x)\big) = 0. \tag{2.3.76}$$

Next, we prove that $\partial_x \Phi_t(x)$ exists for $0 \leqslant t \leqslant T$ and (2.3.75) holds for all $x \in [0,1] \setminus \mathbb{B}$, $m(\mathbb{B}) = 0$. In fact, we have, from (2.2.63) and (2.3.68),

$$\begin{aligned}
&\left| \frac{\Phi_{t_2}(x_2) - \Phi_{t_2}(x_1)}{x_2 - x_1} - \frac{\Phi_{t_1}(x_2) - \Phi_{t_1}(x_1)}{x_2 - x_1} \right| \\
&\quad \leqslant \left| (t_2 - t_1) \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} v_0(y)\,\mathrm{d}y \right| + \left| \frac{t_2^2 - t_1^2}{4(x_2 - x_1)} \int_{x_1}^{x_2} v_0^2(y)\,\mathrm{d}y \right|. \qquad (2.3.77)
\end{aligned}$$

Thus, at any point $x \in [0,1] \setminus \mathbb{B}$ which is also a Lebesgue point of $v_0(\cdot)$ with finite value $v_0(x) \neq \pm\infty$, by the Lebesgue point assumption (2.2.7) and Ascoli–Arzelà theorem, we find that there exist a continuous function $f_x(\cdot) \in C([0,T])$ and a sequence of x_n such that

$$\lim_{x_n \to x} \left| \frac{\Phi_t(x_n) - \Phi_t(x)}{x_n - x} - f_x(t) \right| = 0 \quad \text{uniformly for } t \in [0,T].$$

Thus, $\partial_x \Phi_t(x) = f_x(t)$ for $(t,x) \notin S(x)$. Moreover, by (2.3.76), for any $(t,x) \in S(x)$, there is a sequence of $\{t_n\}$ such that $t_n \to t$, and $(t_n, x) \notin S(x)$, then, we find by (2.3.77) that

$$\begin{aligned}
&\left| \frac{\Phi_t(y) - \Phi_t(x)}{y - x} - f_x(t) \right| \\
&\quad \leqslant \left| \frac{\Phi_t(y) - \Phi_t(x)}{y - x} - \frac{\Phi_{t_n}(y) - \Phi_{t_n}(x)}{y - x} \right| \\
&\qquad + \left| \frac{\Phi_{t_n}(y) - \Phi_{t_n}(x)}{y - x} - f_x(t_n) \right| + \left| f_x(t_n) - f_x(t) \right|. \qquad (2.3.78)
\end{aligned}$$

From (2.3.78), we find that

$$\lim_{y \to x} \frac{\Phi_t(y) - \Phi_t(x)}{y - x} = f_x(t).$$

With the fact that the set of non-Lebesgue points of v_0 or $v_0 = \pm\infty$ has measure zero, we prove (2.3.75) for any $T > 0$. By the diagonal process for the time $T \to \infty$, we obtain that for any $x \in [0,1] \setminus \mathbb{B}$ with $m(\mathbb{B}) = 0$, $\Phi_t(x)$ can be differentiated with respect to x and $\partial_x \Phi_t(x) \in C([0,\infty))$. □

Step 3. We establish the short-time uniqueness. Let

$$t_\gamma^1(x) := \begin{cases} -\frac{1-\gamma}{v_0(x)} & \text{if } v_0(x) < 0, \\ \infty & \text{if } v_0(x) \geqslant 0, \end{cases} \qquad \text{for a fixed } \gamma \in (0,1). \qquad (2.3.79)$$

Then on the set $D := \{(t,x) \in \mathbb{R}^+ \times [0,1] \mid t \leqslant t_\gamma^1(x)\}$, see Figure 2, and by Claim 3, there holds

$$\partial_x \Phi_t(x) \geqslant 1 + t v_0(x) \geqslant \gamma \quad \text{on } D. \qquad (2.3.80)$$

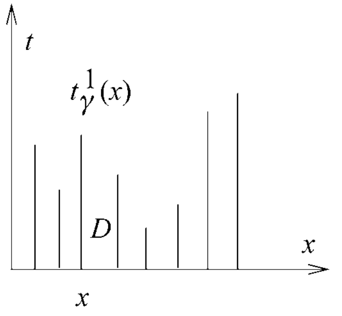

Fig. 2. The domain D.

CLAIM 5. *There holds*

$$v\big(t, \Phi_t(x)\big) = \frac{2v_0(x)}{2 + v_0(x)t} \quad \textit{a.e. on } D. \tag{2.3.81}$$

PROOF. We modify the proof of Lemma 5 in [71] for this claim. The difference is that our D may have an irregular shape while the corresponding domain in [71] is a rectangle, see Figure 2.

Let $v^\varepsilon(t,x) = \int_{\mathbb{R}} j_\varepsilon(y)v(t,x-y)\,\mathrm{d}y$ as in Claim 2. By DiPerna–Lions' folklore Lemma II.1 of [19], we find that $v^\varepsilon(t,x)$ satisfies

$$\begin{cases} \partial_t v^\varepsilon + u\,\partial_x v^\varepsilon = -\frac{1}{2}\big(v^\varepsilon\big)^2 + r_\varepsilon(t,x), \\ u(t,x) = \int_0^x v(t,y)\,\mathrm{d}y, \\ v^\varepsilon(t,x)\big|_{t=0} = v_0^\varepsilon(x), \end{cases} \tag{2.3.82}$$

where $r_\varepsilon(t,x) := u\,\partial_x v^\varepsilon - (u\,\partial_x v) * j_\varepsilon + \frac{1}{2}((v^\varepsilon)^2 - v^2 * j_\varepsilon)$ and $r_\varepsilon \to 0$ in $L^p_{\mathrm{loc}}(Q_\infty)$ for all $p \in [1, \frac{3}{2})$. We compare solutions of (2.3.82) with solutions $W^\varepsilon(t,x)$ of the problem

$$\begin{cases} \partial_t W^\varepsilon + u\,\partial_x W^\varepsilon = -\frac{1}{2}\big(W^\varepsilon\big)^2, \\ u(t,x) = \int_0^x v(t,y)\,\mathrm{d}y, \\ W^\varepsilon(t,x)\big|_{t=0} = v_0^\varepsilon(x). \end{cases} \tag{2.3.83}$$

Let $Y^\varepsilon(t,y) := v^\varepsilon(t,y) - W^\varepsilon(t,y)$. Then by (2.3.82), (2.3.83) and the method of characteristics, we find

$$Y^\varepsilon\big(t, \Phi_t(x)\big) = \int_0^t \left[-\frac{1}{2}Y^\varepsilon\big(s, \Phi_s(x)\big)\big(v^\varepsilon + W^\varepsilon\big)\big(s, \Phi_s(x)\big) + r_\varepsilon\big(s, \Phi_s(x)\big)\right]\mathrm{d}s. \tag{2.3.84}$$

We show that

$$Y^\varepsilon\big(t,\Phi_t(x)\big) \to Y\big(t,\Phi_t(x)\big) := v\big(t,\Phi_t(x)\big) - \frac{2v_0(x)}{2+v_0(x)t} \tag{2.3.85}$$

as $\varepsilon \to 0$, and pass the limit in (2.3.84). We easily obtain from (2.3.83) that

$$W^\varepsilon\big(t,\Phi_t(x)\big) = \frac{2v_0^\varepsilon(x)}{2+v_0^\varepsilon(x)t} \quad \big(\text{on } D^\alpha,\text{ see below}\big). \tag{2.3.86}$$

This solution may blow up on D, due to changes in v_0^ε from mollification. But for any small $\alpha > 0$, we can find a set $D^\alpha \subset D$ and an $\varepsilon_\alpha > 0$ such that

$$2 + v_0^\varepsilon(x)t \geqslant 1 \quad \text{on } D^\alpha \tag{2.3.87}$$

and (2.3.86) holds for all $\varepsilon \in (0, \varepsilon_\alpha]$, where $|D \setminus D^\alpha| < \alpha$. Note that D^α is similar to D in structure; i.e., if a point $(t, x_0) \in D$, then the whole vertical segment $\{(t, x_0) \mid 0 \leqslant t \leqslant t_\gamma^1(x_0)\}$ is in D. Let D_t denote the cross-section of D^α at time t. By (2.3.80) and (2.3.87), we trivially have

$$\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}^+} \left\| W^\varepsilon\big(t,\Phi_t(x)\big) - \frac{2v_0(x)}{2+v_0(x)t} \right\|_{L^2(D_t)} = 0. \tag{2.3.88}$$

By (2.3.80) and the change of variables Theorem 2 on p. 99 of [24], we find for any $t \geqslant 0$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \big\| v^\varepsilon\big(t,\Phi_t(x)\big) - v\big(t,\Phi_t(x)\big)\big\|^2_{L^2(D_t)} \\
&\quad = \lim_{\varepsilon\to 0} \big\| \big(v^\varepsilon(t,y) - v(t,y)\big)^2\, \partial_y(\Phi_t)^{-1}(y)\big\|_{L^1(\Phi_t(D_t))} \\
&\quad \leqslant \lim_{\varepsilon\to 0} \big\| \big(v^\varepsilon(t,y) - v(t,y)\big)\big\|^2_{L^2([0,K(T)])} \\
&\quad = 0.
\end{aligned} \tag{2.3.89}$$

Since $v \in L^\infty(\mathbb{R}^+, L^2)$, we hence have

$$\big\| Y^\varepsilon\big(t,\Phi_t(x)\big) - Y\big(t,\Phi_t(x)\big)\big\|_{L^2(D_t)} = 0 \quad \text{for all } t \geqslant 0,\ \text{in } L^2\big([0,T]\big) \tag{2.3.90}$$

for all $T > 0$ as $\varepsilon \to 0$. By a similar procedure, we have

$$\lim_{\varepsilon\to 0} \left\| \big((v^\varepsilon)^2 - (W^\varepsilon)^2\big)\big(s,\Phi_s(x)\big) - \left(v^2 - \left(\frac{2v_0(x)}{2+v_0(x)s}\right)^2\right)\right\|_{L^1_{\mathrm{loc}}(D^\alpha)} = 0. \tag{2.3.91}$$

While trivially

$$\lim_{\varepsilon\to 0}\big\|r_\varepsilon\big(s,\Phi_s(x)\big)\big\|_{L^1_{\mathrm{loc}}(D^\alpha)}=0. \tag{2.3.92}$$

Summing up (2.3.90)–(2.3.92) and using Fubini's theorem, we find in the limit $\varepsilon\to 0$ in almost everywhere convergence that

$$Y\big(t,\Phi_t(x)\big)=\int_0^t -\frac{1}{2}Y\big(s,\Phi_s(x)\big)\left(v\big(s,\Phi_s(x)\big)+\frac{2v_0(x)}{2+v_0(x)s}\right)\mathrm{d}s,\quad \text{a.e. } D^\alpha. \tag{2.3.93}$$

We see easily from (2.3.91) that the integrand in (2.3.93) is integrable over $[0,T]$ in s for almost all x. Thus $Y(t,\Phi_t(x))$ is continuous in t for almost all $(t,x)\in D^\alpha$. The regularity for v is $v(s,\Phi_s(x))\in L^1([0,t])$ for almost all $(t,x)\in D^\alpha$ since $v(s,\Phi_s(x))\in L^1_{\mathrm{loc}}(D^\alpha)$ by (2.3.89). Taking absolute value on both sides and using Gronwall's inequality in (2.3.93), we find

$$Y\big(t,\Phi_t(x)\big)=0\quad \text{a.e. } D^\alpha.$$

But α is arbitrary, so (2.3.81) holds. This completes the proof of Claim 5. □

CLAIM 6. *There holds*

$$\partial_x\Phi_t(x)=\left(1+\frac{1}{2}v_0(x)t\right)^2\quad \textit{a.e. on } D. \tag{2.3.94}$$

PROOF. In fact, we need only prove that

$$\partial_x u\big(t,\Phi_t(x)\big)=v\big(t,\Phi_t(x)\big)\,\partial_x\Phi_t(x), \tag{2.3.95}$$

because we can differentiate (2.3.64) with respect to x, use (2.3.95) and (2.3.81), then integrate the linear equation of $\partial_x\Phi_t(x)$ to obtain (2.3.94). The proof of (2.3.95) is easy. Let $u^\varepsilon(t,y):=\int j_\varepsilon(y-z)u(t,z)\,\mathrm{d}z$, then $\partial_x u^\varepsilon(t,\Phi_t(x))=v^\varepsilon(t,\Phi_t(x))\,\partial_x\Phi_t(x)$. Similar to the proof of the differentiation of $\Phi_t(x)$ with respect to x, we can prove that $\partial_y\Phi_t^{-1}(y)$ exists and $0\leqslant\partial_y\Phi_t^{-1}(y)\leqslant 1/\gamma$ for almost all $(t,y)\in D^*$, where

$$D^*:=\big\{(t,y)\;\big|\;0\leqslant t\leqslant t^1_\gamma(x),\ y=\Phi_t(x),\ x\in[0,1],\ t\leqslant T\big\}. \tag{2.3.96}$$

Thus, by the change of variables formula (Theorem 2 on p. 99 of [24]), we find

$$\begin{aligned}
&\int_0^1\int_0^{t^1_\gamma(x)}\big|v^\varepsilon\big(t,\Phi_t(x)\big)-v\big(t,\Phi_t(x)\big)\big|\,\partial_x\Phi_t(x)\,\mathrm{d}t\,\mathrm{d}x\\
&\quad=\iint_{D^*}\big|v^\varepsilon(t,y)-v(t,y)\big|\big|\partial_x\Phi_t\big(\Phi_t^{-1}(y)\big)\big|\big|\partial_y\Phi_t^{-1}(y)\big|\,\mathrm{d}t\,\mathrm{d}y\\
&\quad\leqslant\int_0^{K(T)}\int_0^T\big|v^\varepsilon(t,y)-v(t,y)\big|\,\mathrm{d}t\,\mathrm{d}y\to 0
\end{aligned} \tag{2.3.97}$$

as $\varepsilon \to 0$. This proves (2.3.95). $\square$

CLAIM 7. *There holds*

$$v\big(t, \Phi_t(x)\big) = \frac{2v_0(x)}{2 + v_0(x)t}, \qquad \partial_x \Phi_t(x) = \left(\frac{2 + v_0(x)t}{2}\right)^2 \tag{2.3.98}$$

for all $t \in [0, t^*(x))$, *where*

$$t^*(x) := \begin{cases} -\frac{2}{v_0(x)} & \text{if } v_0(x) < 0, \\ \infty & \text{if } v_0(x) \geqslant 0. \end{cases} \tag{2.3.99}$$

PROOF. We have from Claim 6 that

$$\partial_x \Phi_t(x)|_{t=t_\gamma^1(x)} \geqslant \left(\frac{1+\gamma}{2}\right)^2 \quad \text{for a.e. } x \in [0, 1]. \tag{2.3.100}$$

Hence, by the continuity of $\partial_x \Phi_t(x)$ for almost all x with respect to t, and by a similar proof of Claims 5 and 6, we find: For almost all $x \in [0, 1]$, there is a $t_\gamma^2(x)$, such that $t_\gamma^2(x) > t_\gamma^1(x)$, and (2.3.98) holds for all $t \leqslant t_\gamma^2(x)$. Repeating the above argument, we find that: For almost all $x \in [0, 1]$, there is a maximal $t_m(x) \in \mathbb{R}^+$, such that (2.3.98) holds for $t \leqslant t_m(x)$. If $t_m(x) < t^*(x)$ on a set of x of positive measure, then, by the arguments following (2.3.100), we find: There is a $t_{m'}(x)$ on a set of x of positive measure such that $t_m(x) < t_{m'}(x) < t^*(x)$, and (2.3.98) holds for $t \leqslant t_{m'}(x)$. This contradicts with the maximal assumption of $t_m(x)$. Hence, (2.3.98) holds for all $t < t^*(x)$. $\square$

Step 4. We establish the whole uniqueness.

It is easy to see that in order to prove that the first equality of (2.3.98) holds almost everywhere on $\mathbb{R}^+ \times [0, 1]$, we need only prove the following lemma.

LEMMA 2.3.1 (Singularity set). *The singularity set*

$$B_1 := \big\{(t, y) \bigm| \exists x \in [0, 1], y = \Phi_t(x), t^*(x) \leqslant t < \infty\big\} \tag{2.3.101}$$

has 2*D measure zero*.

To prove the lemma, we need another lemma, Lemma 2.3.2. For Lemma 2.3.2 we note trivially that there must hold $v_0(x) \leqslant -2/T$ at a point $x \in [0, 1]$ in order for the characteristic $y = \Phi_t(x)$ issued from x to have a portion belonging to B_T, where

$$B_T := \big\{(t, y) \bigm| \exists x \in [0, 1], y = \Phi_t(x), T \geqslant t \geqslant \max\big\{\varepsilon, t^*(x)\big\}\big\} \tag{2.3.102}$$

for any $T > \varepsilon > 0$. Also, from (2.3.74), we see that characteristics from a small subset of $[0, 1]$ form a small region in $[0, T] \times \mathbb{R}^+$. Now we present the lemma.

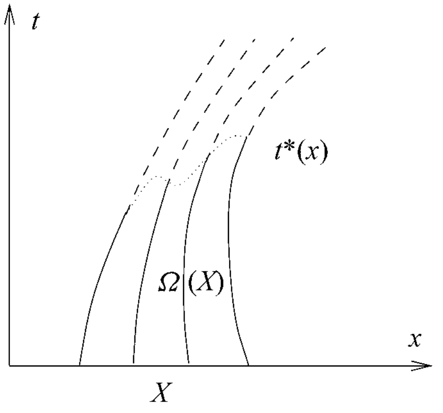

Fig. 3. Domain of integration.

LEMMA 2.3.2. *Let X be a measurable subset of* $[0,1]$ *such that* $v_0(x) \leqslant -2/T$ *for almost all* $x \in X$. *Then*

$$|X| + \iint_{\Omega(X)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s = 0, \tag{2.3.103}$$

where

$$\Omega(X) := \big\{(s,y) \bigm| y = \Phi_s(x), x \in X, 0 \leqslant s < t^*(x)\big\} \tag{2.3.104}$$

which represents the collection of characteristics issued from the set X before the solution blows up (*see Figure* 3). *Here $|X|$ represents the measure of the set X.*

PROOF. First suppose X is an interval (x_1, x_2). From Step 1 we know that the characteristics $\Phi_t(x)$ are monotone with respect to x. We make the change of variables

$$(s,y) \to (t,x) = \big(s, \Phi_s^{-1}(y)\big) \tag{2.3.105}$$

whose Jacobian is $(1 + \frac{1}{2}v_0(x)t)^2$. From Step 3 we know that the solution is given by (2.3.98) in $\Omega(X)$. So we have

$$\begin{aligned} \iint_{\Omega(X)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s &= \int_{x_1}^{x_2}\int_0^{t^*(x)} \frac{v_0(x)}{1+(1/2)v_0(x)t}\left(1+\frac{1}{2}v_0(x)t\right)^2 \mathrm{d}t\,\mathrm{d}x \\ &= \int_{x_1}^{x_2}\left[v_0(x)t^*(x) + \frac{1}{4}v_0^2(x)\big(t^*(x)\big)^2\right]\mathrm{d}x. \end{aligned} \tag{2.3.106}$$

Upon substituting the formula for $t^*(x)$ into (2.3.106), we obtain (2.3.103).

For a general measurable set X, we use a disjoint countable union of open intervals G to approximate X from the exterior arbitrarily. Let $t_T^*(x) := \min\{t^*(x), T\}$. Let

$$\Omega_T(G) := \big\{(s, y) \in \Omega(G) \mid s \leqslant T\big\}. \tag{2.3.107}$$

We have from the previous paragraph

$$\begin{aligned}
&\iint_{\Omega_T(G)} v(s, y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \int_G \left[v_0(x)t_T^*(x) + \frac{1}{4}v_0^2(x)\big(t_T^*(x)\big)^2\right]\mathrm{d}x \\
&\quad = -|X| + \int_{G\setminus X} \left[v_0(x)t_T^*(x) + \frac{1}{4}v_0^2(x)\big(t_T^*(x)\big)^2\right]\mathrm{d}x.
\end{aligned} \tag{2.3.108}$$

Sending the approximation $G \to X$, we recover (2.3.103) from (2.3.108) since the last integrand in (2.3.108) is integrable and $\Omega_T(G)$ approaches $\Omega(X)$ by the smallness notice given just before the lemma. This completes the proof of Lemma 2.3.2. $\square$

PROOF OF LEMMA 2.3.1. We need only prove that for any $T > \varepsilon > 0$, the singularity set B_T in (2.3.102) has 2D measure zero. Given any $\delta > 0$ (assume $\delta < 1/T$), we show that the singularity set B_T of (2.3.102) has measure less than $\mathrm{O}(\delta)$. By Lusin's theorem, there exists a continuous function $\psi(x)$ on [0, 1] such that the set

$$S := \big\{x \mid v_0(x) \neq \psi(x), x \in [0, 1]\big\} \tag{2.3.109}$$

has measure

$$|S| < \delta. \tag{2.3.110}$$

From (2.3.74), we see that characteristics from S form a region in $[0, T] \times \mathbb{R}^+$ of measure less than C_δ which goes to zero as $\delta \to 0$. Next we can find a finite covering $\{I_i\}_{i=1}^N$, $I_i := (a_i, b_i)$, $i = 1, \ldots, N$, of $[0, 1]$ such that the fluctuation of ψ on each I_i is less than δ; that is, the difference between the maximum ψ^1 and the minimum ψ^0 on each I_i is less than δ. Let S_i be the part of S that is in I_i. Note that B_T may not have any portion that is issued from some $I_i \setminus S_i$. Now consider the characteristics of B_T issued from some $I_i \setminus S_i$. We separate the characteristics of B_T issued from $I_i \setminus S_i$ into two portions: the tail and the torso, see Figure 4. The torso consists of the characteristics between the time interval $t \in [-\frac{2}{\psi^0}, -\frac{2}{\psi^1}]$ which has a height

$$\left|\frac{2}{\psi^0} - \frac{2}{\psi^1}\right| = \frac{2|\psi^1 - \psi^0|}{|\psi^1\psi^0|} \leqslant T^2\delta,$$

where we used $\psi^0 \leqslant -2/T$ and $\psi^1 \leqslant -2/T + \delta \leqslant -1/T$. The width of the torso portion sums up over i to a finite number independent of δ. So the torso portion has measure less

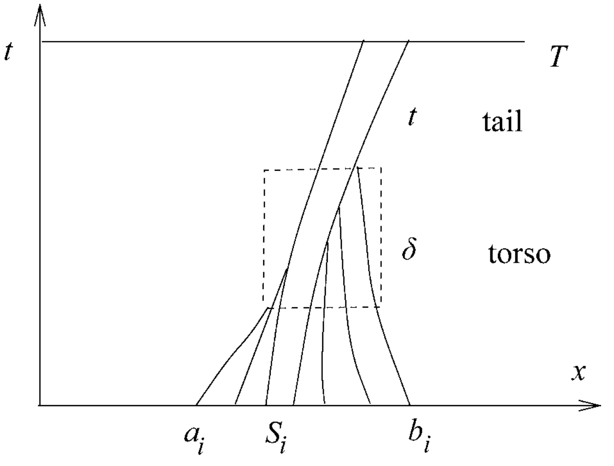

Fig. 4. The torso and tail of a part of a singular set.

than $\mathrm{O}(\delta)$. The tail portion is for time $t > -2/\psi^1$ which we show to be small next. First, apparently, we need $T > -2/\psi^1$ for the tail to be in B_T. Otherwise, the tail is beyond the time T. Now for $t \in [-\frac{2}{\psi^1}, T]$ and from integrating (2.3.64), we have

$$\begin{aligned} 0 &< \Phi_t(b_i) - \Phi_t(a_i) \\ &= |I_i| + \int_0^t \int_{\Phi_s(a_i)}^{\Phi_s(b_i)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= |I_i| + \iint_{\Omega_t(I_i)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s + \iint_{\Omega_c} v(s,y)\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{2.3.111}$$

where $\Omega_t(I_i)$ is the same as in (2.3.107) at t and

$$\Omega_c := \bigl\{(s,y) \bigm| y = \Phi_s(x), t^*(x) < s < t, x \in I_i\bigr\}.$$

Using Lemma 2.3.2 for $X = I_i \setminus S_i$ in (2.3.111) and noting $\Omega_t(X) = \Omega(X)$ for this X, we have

$$\begin{aligned} &\Phi_t(b_i) - \Phi_t(a_i) \\ &\quad = |S_i| + \iint_{\Omega_t(S_i)} v(s,y)\,\mathrm{d}y\,\mathrm{d}s + \iint_{\Omega_c} v(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad \leqslant |S_i| + \iint_{\Omega_T(S_i)} \bigl|v(s,y)\bigr|\,\mathrm{d}y\,\mathrm{d}s + C_m \int_0^t \bigl(\Phi_s(b_i) - \Phi_s(a_i)\bigr)\,\mathrm{d}s, \end{aligned} \tag{2.3.112}$$

where $C_m = C/t_m$ is from the entropy condition $v < C/t$ and t_m depends on the (negative) minimum of ψ on $[0,1]$. By the barrier ε in B_T of (2.3.102), we can take $t_m = \varepsilon$. From (2.3.112) and Gronwall's inequality, we obtain that

$$\Phi_t(b_i) - \Phi_t(a_i) \leqslant C\left(|S_i| + \iint_{\Omega_T(S_i)} \left|v(s,y)\right| \mathrm{d}y\, \mathrm{d}s\right). \tag{2.3.113}$$

Summing over i, we see that the tail has a size which goes to zero as δ goes to zero. Thus the total measure of B_T is less than a quantity which goes to zero as $\delta \to 0$. This completes the proof of Lemma 2.3.1. □

So the proof of the uniqueness of dissipative weak solutions is complete. □

PROOF OF THE EXPLICIT FORMULA. From the proof of uniqueness, we see that a distributional solution has to be given by the formula (2.3.98) on B_1^c, where B_1 has measure zero. Furthermore, from (2.3.98) and Lemma 2.3.1, there holds

$$\partial_x \Phi_t(x) = \left(1 + \frac{1}{2} v_0(x) t\right)^2 \mathbb{1}_{\{2+v_0(x)t>0\}} \quad \text{a.e.} \tag{2.3.114}$$

From this and the absolute continuity (2.3.74), we see that (2.2.56) holds for any dissipative weak solution. We then have the following lemma.

LEMMA 2.3.3 (Flow inverse). *The inverse $\Phi_t^{-1}(y)$ exists from B_1^c to $[0,1]$ for each $t \geqslant 0$.*

PROOF. Fix a point $(t,y) \in B_1^c$. Suppose there are two numbers $a < b$ in $[0,1]$ such that $\Phi_t(a) = \Phi_t(b) = y$. From (2.2.56), we have

$$\int_a^b \left(1 + \frac{1}{2} v_0(z) t\right)^2 \mathbb{1}_{\{1+(1/2)v_0(z)t \geqslant 0\}} \, \mathrm{d}z = 0$$

which implies that

$$1 + \frac{1}{2} v_0(z) t \leqslant 0 \quad \text{a.e. } z \in (a,b). \tag{2.3.115}$$

Take any point $c \in (a,b)$ such that (2.3.115) holds. Then (t,y) belongs to B_1 through the realization of the characteristic $y = \Phi_t(c)$, a contradiction. This completes the proof of Lemma 2.3.3. □

So the explicit formula v of Theorem 2.1.3 is established. □

2.4. *Conservative solutions*

We prove Theorem 2.1.4 in this subsection.

2.4.1. *Approximate solutions.* We use (2.2.6) to approximate the initial data and construct the approximate solutions as follows

$$v^n(t,x) = \frac{2v_i^n}{2+tv_i^n}, \quad \frac{1}{n}\sum_{j=1}^{i-1}\left(1+\frac{1}{2}v_j t\right)^2 \leqslant x < \frac{1}{n}\sum_{j=1}^{i}\left(1+\frac{1}{2}v_j t\right)^2,$$

where $i = 1, 2, \ldots, n$. We calculate u^n from $\partial_x u^n = v^n$ and $u^n(t,0) = 0$.

2.4.2. *Primitive estimates.* Corresponding to Lemma 2.2.1, we have the following lemmas.

LEMMA 2.4.1 (Primitive estimates). *For all $p \in [2,3)$, $T > 0$ and $R > 0$, the approximate solution sequence $\{v^n, u^n\}$ constructed above satisfies the estimates*

$$\text{(b)} \quad \left\|v^n(t,\cdot)\right\|_{L^2(\mathbb{R}^+)} = \left\|v_0^n\right\|_{L^2([0,1])} \quad a.e.\ t > 0,$$

$$\text{(c)} \quad \left\|v^n\right\|^p_{L^p([0,T]\times\mathbb{R}^+)} \leqslant C_{T,p}\left\|v_0^n\right\|^2_{L^2([0,1])}. \tag{2.4.116}$$

Moreover, $\{u^n(t,x)\}$ are uniformly bounded in $W^{1,p}_{\mathrm{loc}}(Q_\infty)$.

LEMMA 2.4.2 (Basic precompactness). *It holds for conservative solutions without change.*

2.4.3. *Flow map.* We approach strong precompactness through the flow map, since the argument in Section 2.2.3 does not seem to extend to the conservative solutions. We let $\{\Phi_t^n(x)\}$ be the special flow of $u^n(t,x)$

$$\Phi_t^n(x) = \int_0^x \left(1+\frac{1}{2}v_0^n(y)t\right)^2 \mathrm{d}y. \tag{2.4.117}$$

Then by the construction of the approximate solutions, we find that for any $x_1 < x_2$ in $[0,1]$ there holds

$$\int_{\Phi_t^n(x_1)}^{\Phi_t^n(x_2)} \left(v^n(t,y)\right)^2 \mathrm{d}y = \int_{x_1}^{x_2} \left(v_0^n(y)\right)^2 \mathrm{d}y. \tag{2.4.118}$$

Moreover, Lemma 2.2.6 holds for conservative solutions when (2.2.56) is changed into

$$\Phi_t(x) = \int_0^x \left(1+\frac{1}{2}v_0(y)t\right)^2 \mathrm{d}y. \tag{2.4.119}$$

Clearly uniqueness of the flow map fails for conservative weak solutions (v,u). An example is when all characteristics come to one singularity point and emerge in two different orders. If characteristics are required to preserve the energy between them, then the flow map is unique.

LEMMA 2.4.3 (Uniqueness of conservative flow maps). *Let (v, u, Φ) be any conservative weak solution by Definition* 2.1.2. *Then $\Phi_t(x)$ is uniquely determined by the initial data given in* (2.4.119).

PROOF. By taking $x_1 = 0$ and $x_2 = x$ in (2.3.68) which holds for conservative solutions, we have for $x \geqslant 0$ that

$$\Phi_t(x) = x + t \int_0^x v_0(y)\,\mathrm{d}y + \frac{1}{2} \int_0^t \int_0^s \int_0^{\Phi_\tau(x)} v^2(\tau, y)\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}s. \tag{2.4.120}$$

By the local energy conservation of (c3) of Definition 2.1.2, we obtain (2.4.119). □

For Lemma 2.3.1, we have the following one.

LEMMA 2.4.4. *The set*

$$B_2 := \big\{(t, y) \bigm| y = \Phi_t(x),\ t = t^*(x) \in \mathbb{R}^+,\ x \in [0, 1]\big\} \tag{2.4.121}$$

has 2*D Lebesgue measure zero for the flow $\Phi_t(x)$ constructed in Lemma* 2.2.6 *for the flow map* (2.4.117).

The proof of Lemma 2.4.4 seems obvious. For completeness we point out that it can be proved similar to that of Lemma 2.3.1. In place of (2.2.56), we use (2.4.119) and we do not need to handle the tail portion here.

2.4.4. *Strong precompactness.* By the construction of the approximate solution sequence $\{(v^n, u^n)\}$, we find that there holds

$$\partial_t v^n + \partial_x \big(u^n v^n\big) = \frac{1}{2} \big(v^n\big)^2 \tag{2.4.122}$$

in the sense of distributions. Thus, by (c) of Lemma 2.4.1, and an argument similar to (2.3.69)–(2.3.71) and (2.4.118), we find that

$$\begin{aligned} u^n\big(t, \Phi_t^n(x)\big) &= u_0^n(x) + \frac{1}{2} \int_0^t \int_0^{\Phi_s^n(x)} \big(v^n(s, y)\big)^2\,\mathrm{d}y\,\mathrm{d}s \\ &= u_0^n(x) + \frac{t}{2} \int_0^x \big(v_0^n\big)^2(y)\,\mathrm{d}y. \end{aligned} \tag{2.4.123}$$

Hence, by Lemma 2.2.2 and the fact that $\Phi_t^n(x) \to \Phi_t(x)$ uniformly on every compact subset of $\mathbb{R}^+ \times \mathbb{R}^+$ as $n \to \infty$, we find, by tending $n \to \infty$ in (2.4.123), that

$$u\big(t, \Phi_t(x)\big) = u_0(x) + \frac{t}{2} \int_0^x v_0^2(y)\,\mathrm{d}y. \tag{2.4.124}$$

Thus, for a.e. $x \in [0,1]$, $\forall t > 0$, there holds

$$\partial_x u\big(t, \Phi_t(x)\big) = v_0(x) + \frac{t}{2} v_0^2(x). \tag{2.4.125}$$

Let us recall from (2.3.96), $D^* := \{(t, y) \mid 0 \leqslant t \leqslant t_\mu^1(x), y = \Phi_t(x), x \in [0,1]\}$, but here we use

$$t_\mu^1(x) := \begin{cases} -\frac{2(1-\mu)}{v_0(x)} & \text{if } v_0(x) < 0, \\ \infty & \text{if } v_0(x) \geqslant 0, \end{cases} \quad \text{for a fixed } \mu \in (0,1). \tag{2.4.126}$$

Thus, by (2.4.119), we find that $\Phi_t^{-1}(y)$ exists for $(t, y) \in D^*$. Moreover, for almost all $(t, y) \in D^*$, there holds

$$0 \leqslant \partial_y \Phi_t^{-1}(y) \leqslant \frac{1}{\mu}. \tag{2.4.127}$$

Thus, by a similar proof of (2.3.95), we find that

$$\partial_x u\big(t, \Phi_t(x)\big) = \partial_y u\big(t, \Phi_t(x)\big)\, \partial_x \Phi_t(x) = \bar{v}\big(t, \Phi_t(x)\big)\, \partial_x \Phi_t(x). \tag{2.4.128}$$

On the other hand, by (2.4.119), for a.e. $x \in [0,1]$, we have

$$\partial_x \Phi_t(x) = \left(1 + \frac{1}{2} v_0(x) t\right)^2. \tag{2.4.129}$$

Thus, by summing up (2.4.125)–(2.4.129), we find

$$\bar{v}\big(t, \Phi_t(x)\big) = \frac{2 v_0(x)}{2 + v_0(x) t} \quad \text{a.e. } x \in [0,1], t < t_\mu^1(x). \tag{2.4.130}$$

Then by taking t derivative to both sides of (2.4.130), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{v}\big(t, \Phi_t(x)\big) = -\frac{1}{2} \bar{v}^2\big(t, \Phi_t(x)\big). \tag{2.4.131}$$

But, by (2.4.122), we find $\bar{v}(t, y)$ also satisfies

$$\partial_t \bar{v} + \partial_x (u \bar{v}) = \frac{1}{2} \overline{v^2}. \tag{2.4.132}$$

Thus, again by Lemma 2.3 of [48], we find

$$\partial_t \bar{v}^\varepsilon + \partial_x \big(u \bar{v}^\varepsilon\big) = \frac{1}{2} \big(\overline{v^2}\big)^\varepsilon + R_\varepsilon, \tag{2.4.133}$$

where $f^\varepsilon(t,\cdot)=f(t,\cdot)*j_\varepsilon$, $R_\varepsilon\to 0$ in $L^{1+\alpha/2}_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+)$ for all $\alpha\in[0,1)$. Thus

$$\frac{\mathrm{d}\bar{v}^\varepsilon(t,\Phi_t(x))}{\mathrm{d}t}=\left(\frac{1}{2}\big(\overline{v^2}\big)^\varepsilon-\bar{v}^\varepsilon\cdot\bar{v}\right)\big(t,\Phi_t(x)\big)+R_\varepsilon\big(t,\Phi_t(x)\big)$$

which implies

$$\begin{aligned}\bar{v}^\varepsilon\big(t,\Phi_t(x)\big)\\ &=v_0^\varepsilon(x)+\int_0^t\left[\left(\frac{1}{2}\big(\overline{v^2}\big)^\varepsilon-\bar{v}^\varepsilon\cdot\bar{v}\right)\big(s,\Phi_s(x)\big)+R_\varepsilon\big(s,\Phi_s(x)\big)\right]\mathrm{d}s.\end{aligned}\tag{2.4.134}$$

While by (2.4.127), we find easily that

$$\lim_{\varepsilon\to 0}\int_0^1\int_0^{t_\mu^1(x)}\left|\left(\left(\frac{1}{2}\big(\overline{v^2}\big)^\varepsilon-\bar{v}^\varepsilon\cdot\bar{v}\right)-\left(\frac{1}{2}\overline{v^2}-\bar{v}^2\right)\right)\big(s,\Phi_s(x)\big)\right|\mathrm{d}x\,\mathrm{d}s=0$$

and

$$\lim_{\varepsilon\to 0}\int_0^1\int_0^{t_\mu^1(x)}\left|R_\varepsilon\big(s,\Phi_s(x)\big)\right|\mathrm{d}s\,\mathrm{d}x=0.$$

Then, by summing up above, we find by tending $\varepsilon\to 0$ in (2.4.134) that

$$\bar{v}\big(t,\Phi_t(x)\big)=v_0(x)+\int_0^t\left(\frac{1}{2}\overline{v^2}-\bar{v}^2\right)\big(s,\Phi_s(x)\big)\,\mathrm{d}s,\quad x\in[0,1],\ t\leqslant t_\mu^1(x).\tag{2.4.135}$$

By differentiating (2.4.135) with respect to t, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}\big(t,\Phi_t(x)\big)=\left(\frac{1}{2}\overline{v^2}-\bar{v}^2\right)\big(t,\Phi_t(x)\big).\tag{2.4.136}$$

Thus, by summing up (2.4.131) and (2.4.136), we find

$$\overline{v^2}\big(t,\Phi_t(x)\big)=\bar{v}^2\big(t,\Phi_t(x)\big),\quad x\in[0,1],\ t\leqslant t_\mu^1(x).\tag{2.4.137}$$

Sending $\mu\to 0$, we find that (2.4.137) holds for all $t<t^*(x)$. Thus, the associated Young measures $\mu(t,x,\mathrm{d}\lambda)$ of $\{v^n(t,y)\}$ (see Lemma 2.2.3) has the property

$$\mu\big(t,\Phi_t(x),\mathrm{d}\lambda\big)=\delta_{\bar{v}(t,\Phi_t(x))}(\lambda),\quad x\in[0,1],\ t<t^*(x).\tag{2.4.138}$$

Exactly as the proof of (2.4.138), we can also prove that

$$\mu\big(t, \Phi_t(x), \mathrm{d}\lambda\big) = \delta_{\bar{v}(t,\Phi_t(x))}(\lambda), \quad x \in [0,1],\ t > t^*(x). \tag{2.4.139}$$

Now, notice that the map $\Phi_t(x) : x \to \Phi_t(x)$, maps a null set to a null set (see (2.4.119)), and the singular set is a null set by Lemma 2.4.4. So, by summing up (2.4.138) and (2.4.139), we in fact have established that

$$\mu(t, y, \mathrm{d}\lambda) = \delta_{\bar{v}(t,y)}(\lambda) \quad \text{a.e. } (t, y) \in Q_\infty. \tag{2.4.140}$$

2.4.5. *Existence.* With (2.4.140), and exactly the same proof as the existence of dissipative weak solution of (2.1.1) (see Section 2.2.5 for more details), we can prove the existence of conservative weak solutions to (2.1.1). Moreover, there still holds the regularity $v(t,x) \in C([0,\infty), L^q(\mathbb{R}^+)) \cap L^p_{\mathrm{loc}}(Q_\infty)$ for all $q < 2$ and $p < 3$.

The almost everywhere continuity of $v(t,x)$ from $[0,\infty)$ to $L^2(\mathbb{R}^+)$ follows from its weak continuity and the almost everywhere norm equality of the energy.

Corresponding to Lemma 2.2.7, we obtain the local energy conservation from L^2 strong precompactness of $\{v^n\}$, the uniform precompactness of Φ^n_t and (2.4.118).

2.4.6. *Uniqueness.* The entire uniqueness of the triplet (v, u, Φ_t) follows from the uniqueness of Φ_t (see (2.4.119)) and

$$u\big(t, \Phi_t(x)\big) = u_0(x) + \frac{t}{2}\int_0^x v_0^2(y)\,\mathrm{d}y$$

derived from (2.3.71) and local energy conservation. This completes the proof of Theorem 2.1.4.

2.5. *The uniqueness of solution to the ODE*

PROPOSITION A (Uniqueness of dissipative flow). *The solution to* (2.2.55) *is unique for any* (v, u) *satisfying the regularity items* (d1) *and* (d3)–(d5) *of Definition* 2.1.1.

We remark that we find it is interesting to establish the uniqueness of the flow for a given rough (v, u), independent of (v, u) being a solution to the equations. Compare this result with that of DiPerna and Lions [19].

PROOF OF PROPOSITION A. The existence of a flow is trivial since u is continuous. For the uniqueness, we suppose, on the contrary, that there are two different flows $\Phi^1_t(x)$ and $\Phi^2_t(x)$ in the time interval $[0, T]$ starting from some point x. Then we have

$$\Phi^1_t(x) = \Phi^2_t(x) \quad \text{for } t = \tau_1, \qquad \Phi^1_t(x) > \Phi^2_t(x) \quad \text{for } \tau_1 < t < \tau_2, \tag{2.5.141}$$

for some $\tau_1 \in [0, T)$ and some $\tau_2 \in (\tau_1, T]$. For simplicity we set $\tau_1 = 0$. Then we find from (2.2.55) that for $t \leqslant \tau_2$,

$$\begin{aligned}
0 &\leqslant \Phi_t^1(x) - \Phi_t^2(x) \\
&= \int_0^t \int_{\Phi_s^2(x)}^{\Phi_s^1(x)} v(s, y)\, \mathrm{d}y\, \mathrm{d}s \\
&\leqslant t \max_{0 \leqslant s \leqslant t} \big(\Phi_s^1(x) - \Phi_s^2(x)\big)^{1/2} \|v\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^+))},
\end{aligned}$$

from which we can easily derive that for $t \leqslant \tau_2$,

$$0 \leqslant \max_{0 \leqslant s \leqslant t} \big[\Phi_s^1(x) - \Phi_s^2(x)\big] \leqslant t^2 \|v\|^2_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^+))} =: t^2 C(T)^2.$$

Thus by (2.2.55) again, we have for $t \leqslant \tau_2$,

$$\begin{aligned}
0 &\leqslant \Phi_t^1(x) - \Phi_t^2(x) \\
&= \int_0^t \int_{\Phi_s^2(x)}^{\Phi_s^1(x)} v(s, y)\, \mathrm{d}y\, \mathrm{d}s \\
&\leqslant \sup_{0 \leqslant s \leqslant t} \big\| v(s, \cdot) \big\|_{L^2(E_s(x))} \int_0^t \big(C(T)^2 s^2\big)^{1/2} \mathrm{d}s \\
&\leqslant C(T) \sup_{0 \leqslant s \leqslant t} \big\| v(s, \cdot) \big\|_{L^2(E_s(x))} t^2,
\end{aligned} \tag{2.5.142}$$

where $E_s(x) = \{y \mid \Phi_s^2(x) \leqslant y \leqslant \Phi_s^1(x)\}$. We comment in passing that more iterations can reduce the constant $C(T)$ in (2.5.142) to $C(T)/4$. Next, by (d5) of Definition 2.1.1 and (2.5.142), we find for $t \leqslant \tau_2$,

$$\begin{aligned}
0 &\leqslant \Phi_t^1(x) - \Phi_t^2(x) \\
&= \int_0^\varepsilon \int_{\Phi_s^2(x)}^{\Phi_s^1(x)} v(s, y)\, \mathrm{d}y\, \mathrm{d}s + \int_\varepsilon^t \int_{\Phi_s^2(x)}^{\Phi_s^1(x)} v(s, y)\, \mathrm{d}y\, \mathrm{d}s \\
&\leqslant C(T) \sup_{0 \leqslant s \leqslant \varepsilon} \big\| v(s, \cdot) \big\|_{L^2(E_s(x))} \varepsilon^2 + \int_0^t \psi(\varepsilon, s)\big(\Phi_s^1(x) - \Phi_s^2(x)\big)\, \mathrm{d}s,
\end{aligned} \tag{2.5.143}$$

where

$$\psi(\varepsilon, s) = \begin{cases} \frac{2}{s}, & s \geqslant \varepsilon, \\ 0, & s < \varepsilon. \end{cases}$$

Then, by Gronwall's inequality, we find for $t \in [0, \tau_2]$ that

$$0 \leqslant \Phi_t^1(x) - \Phi_t^2(x) \leqslant C(T)T^2 \sup_{0 \leqslant s \leqslant \varepsilon} \big\| v(s, \cdot) \big\|_{L^2(E_s(x))}. \tag{2.5.144}$$

We claim a one-sided continuity

$$\lim_{t \to \tau_1+} \int v^2(t, x)\, \mathrm{d}x = \int v^2(\tau_1, x)\, \mathrm{d}x, \qquad \tau_1 \geqslant 0.$$

In fact, from (d1) and (d4), we have

$$v(s, \cdot) \rightharpoonup v_0(\cdot) \quad \text{weakly in } L^2\big(\mathbb{R}^+\big), \tag{2.5.145}$$

and

$$v(s, x) \to v_0(x) \quad \text{a.e. } x \in \mathbb{R}^+ \tag{2.5.146}$$

as $s \to 0+$. From this almost everywhere convergence and Fatou's lemma, we have

$$\int_{\mathbb{R}^+} v_0^2(x)\, \mathrm{d}x \leqslant \lim_{s \to 0+} \int_{\mathbb{R}^+} v^2(s, x)\, \mathrm{d}x.$$

But by (d3) of Definition 2.1.1, we have

$$\lim_{s \to 0} \big\| v(s, \cdot) \big\|_{L^2(\mathbb{R}^+)} \leqslant \|v_0\|_{L([0,1])}.$$

So we have

$$\int_{\mathbb{R}^+} v_0^2(x)\, \mathrm{d}x = \lim_{t \to 0+} \int_{\mathbb{R}^+} v^2(t, x)\, \mathrm{d}x.$$

Thus, by Theorem 8 on p. 11 of [23], we in fact have

$$v(s, \cdot) \to v_0(\cdot) \quad \text{strongly in } L^2\big(\mathbb{R}^+\big) \text{ as } s \to 0+. \tag{2.5.147}$$

Hence, by Dunford–Pettis' theorem, there holds

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant s \leqslant \varepsilon} \big\| v(s, \cdot) \big\|_{L^2(E_s(x))} = 0,$$

since the length of $E_s(x)$ satisfies $|E_s(x)| = |\{y \mid \Phi_s^2(x) \leqslant y \leqslant \Phi_s^1(x)\}| \leqslant C(T)\varepsilon^2$. Thus, by tending $\varepsilon \to 0$ in (2.5.144), we prove that

$$\Phi_t^1(x) - \Phi_t^2(x) = 0 \quad \text{for } 0 < t \leqslant \tau_2, \tag{2.5.148}$$

a contradiction with (2.5.141). And this completes the proof of Proposition A. $\square$

By modifying the method in the proof of Proposition A, we can prove the following more general Proposition B.

PROPOSITION B. *Let* $(v(t,x),u(t,x)) \in L^\infty(\mathbb{R}^+, L^p(\mathbb{R})) \cap L^\infty(\mathbb{R}^+, W^{1,p}(\mathbb{R}))$ *for some* $1 \leqslant p < \infty$, $\lim_{t\to 0} \|v(t,\cdot) - v_0(\cdot)\|_{L^p} = 0$ *and*

$$v(t,x) \leqslant \frac{p}{t}. \tag{2.5.149}$$

Then the solution to (2.2.55) *is unique*.

PROOF. For convenience, we use the same notation as that in the proof of Proposition A. Firstly, by exactly the same proof of (2.5.142), we find

$$0 \leqslant \Phi_t^1(x) - \Phi_t^2(x) \leqslant C(T) \sup_{0\leqslant s\leqslant t} \left\| v(s,\cdot) \right\|_{L^p(E_s(x))} t^p. \tag{2.5.150}$$

Thus, by (2.5.150), we find for $t \leqslant \tau_2$ that

$$\begin{aligned} 0 &\leqslant \Phi_t^1(x) - \Phi_t^2(x) \\ &\leqslant C(T) \sup_{0\leqslant s\leqslant \varepsilon} \left\| v(s,\cdot) \right\|_{L^p(E_s(x))} \varepsilon^p + \int_0^t \psi'(\varepsilon,s)\big(\Phi_s^1(x) - \Phi_s^2(x)\big)\,\mathrm{d}s, \end{aligned} \tag{2.5.151}$$

where

$$\psi'(\varepsilon,s) = \begin{cases} \frac{p}{s}, & s \geqslant \varepsilon, \\ 0, & s < \varepsilon. \end{cases} \tag{2.5.152}$$

Then, by Gronwall's inequality, we find for $t \in [0,\tau_2]$ that

$$0 \leqslant \Phi_t^1(x) - \Phi_t^2(x) \leqslant C(T) T^p \sup_{0\leqslant s\leqslant \varepsilon} \left\| v(s,\cdot) \right\|_{L^p(E_s(x))}. \tag{2.5.153}$$

Since $\lim_{t\to 0} \|v(t,\cdot) - v_0(\cdot)\|_{L^p} = 0$, we can now use the proof of Proposition A to complete the proof of Proposition B. □

REMARK 2.5.1. The condition that $\lim_{t\to 0} \|v(t,\cdot) - v_0(\cdot)\|_{L^p} = 0$ is crucial in the proof of the uniqueness of solution to (2.2.55), see the example in the Introduction (before Definition 2.1.2). By requiring v to be a solution to (2.1.1), we can relax the condition $v \leqslant 2/t$ to $v \leqslant C/t$ for any $C \geqslant 2$.

3. Rarefactive solutions to (0.0.1)

3.1. *Introduction*

In this section we use the generalized compensated compactness ([27] or [64]), compensated compactness in L^p ([59] and references therein and [56]), the latest developments in the Young measure method of Lions [49] and Joly, Métivier and Rauch [39], and the techniques used in our earlier paper [70] to establish the global existence of weak solutions for (0.0.1). We first establish the global existence of smooth solutions to a viscously perturbed equation for general $c(u)$ with general data. We then present our discovery of an invariant region in the phase space for a monotone $c(u)$. In the invariant region, we show that smooth data evolve smoothly forever; and for weak data we show that both the viscous approximation and the data regularization yield global weak solutions for (0.0.1) by using the aforementioned Young measure and related theories.

Before we present precisely our results, let us introduce some notation: $\mathbb{R}^+ = (0, \infty)$, H^k are Sobolev spaces and Lip stands for Lipschitz. We use

$$R := \partial_t u + c(u)\,\partial_x u, \qquad S := \partial_t u - c(u)\,\partial_x u, \qquad \tilde{c}(\cdot) := \frac{1}{4}\ln c(\cdot), \tag{3.1.1}$$

so that $\tilde{c}'(u) = c'(u)/[4c(u)]$. We use ":=" for definition.

DEFINITION 3.1.1. We call $u(t,x)$ an admissible weak solution of (0.0.1) if

(1) $u(t,x) \in L^\infty(\mathbb{R}^+, H^1_{\mathrm{loc}}(\mathbb{R})) \cap \mathrm{Lip}([0,\infty), L^2_{\mathrm{loc}}(\mathbb{R}))$ and

$$\int_{\mathbb{R}} \left(|\partial_t u|^2 + \left|c(u)\,\partial_x u\right|^2\right) \mathrm{d}x \leqslant \int_{\mathbb{R}} \left(|u_1|^2 + \left|c(u_0)\,\partial_x u_0\right|^2\right) \mathrm{d}x; \tag{3.1.2}$$

(2) for all test functions $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, there holds

$$\iint_{\mathbb{R}^+\times\mathbb{R}} \left(\partial_t\varphi\,\partial_t u - \partial_x\varphi\, c^2(u)\,\partial_x u - \varphi c'(u)c(u)(\partial_x u)^2\right) \mathrm{d}x\,\mathrm{d}t = 0; \tag{3.1.3}$$

(3) for any convex function $\eta(\cdot)$ with $\eta''(\cdot) \in C_c^\infty(\mathbb{R})$, there hold

$$\begin{aligned} &\partial_t\eta(R) - c(u)\,\partial_x\eta(R) - \tilde{c}'(u)\eta'(R)\left(R^2 - S^2\right) \leqslant 0,\\ &\partial_t\eta(S) + c(u)\,\partial_x\eta(S) - \tilde{c}'(u)\eta'(S)\left(S^2 - R^2\right) \leqslant 0; \end{aligned} \tag{3.1.4}$$

(4) $u(t,x) \to u_0(x)$ in $\mathrm{Lip}([0,\infty), L^2(\mathbb{R}))$ and $\partial_t u(t,x) \to u_1(x)$ in the distributional sense as $t \to 0+$.

We explain the motivation of (3.1.4). As can be seen from [29], if $u(t,x)$ is a smooth solution of (0.0.1), we can write (0.0.1) as the system of first-order equations

$$\begin{cases} \partial_t R - c\,\partial_x R = \tilde{c}'(u)\big(R^2 - S^2\big), \\ \partial_t S + c\,\partial_x S = \tilde{c}'(u)\big(S^2 - R^2\big), \\ \partial_x u = \frac{R-S}{2c(u)}. \end{cases} \tag{3.1.5}$$

Similar to the viscous approximation of [37], we consider

$$\begin{cases} \partial_t R_\varepsilon - c(u_\varepsilon)\,\partial_x R_\varepsilon = \tilde{c}'(u_\varepsilon)\big(R_\varepsilon^2 - S_\varepsilon^2\big) + \varepsilon\,\partial_x^2 R_\varepsilon, \\ \partial_t S_\varepsilon + c(u_\varepsilon)\,\partial_x S_\varepsilon = \tilde{c}'(u_\varepsilon)\big(S_\varepsilon^2 - R_\varepsilon^2\big) + \varepsilon\,\partial_x^2 S_\varepsilon, \\ \partial_x u_\varepsilon = \frac{R_\varepsilon - S_\varepsilon}{2c(u_\varepsilon)}, \end{cases} \tag{3.1.6}$$

where $\varepsilon > 0$. We will see that (3.1.6) has global smooth solutions with general Cauchy data. So for any convex function $\eta(\cdot)$, we multiply $\eta(R_\varepsilon)$ to both sides of the first equation of (3.1.6) to obtain

$$\begin{aligned} &\partial_t \eta(R_\varepsilon) - c\,\partial_x \eta(R_\varepsilon) - \tilde{c}'(u_\varepsilon)\eta'(R_\varepsilon)\big(R_\varepsilon^2 - S_\varepsilon^2\big) \\ &\quad = \varepsilon\big(\partial_x^2 \eta(R_\varepsilon) - \eta''(R_\varepsilon)(\partial_x R_\varepsilon)^2\big) \\ &\quad \leqslant \varepsilon\,\partial_x^2 \eta(R_\varepsilon). \end{aligned} \tag{3.1.7}$$

Passing $\varepsilon \to 0$ in (3.1.7), we obtain the first inequality of (3.1.4). It is similar to obtain the second of (3.1.4).

In the sequel, we always assume that

$$0 < C_1 \leqslant c(\cdot) \leqslant C_2 \quad \text{and} \quad \big|c^{(l)}(\cdot)\big| \leqslant M_l \quad \text{for all } l \geqslant 1, \tag{3.1.8}$$

for some positive constants C_1, C_2 and M_l. We also assume in most cases that

$$c'(\cdot) \geqslant 0. \tag{3.1.9}$$

In some cases we even assume

$$c'(u) \geqslant C_M > 0, \quad u \in [-M, M], \tag{3.1.10}$$

for some positive constant C_M and any $M > 1$.

Now, we state precisely our main results of the chapter.

THEOREM 3.1.2 (Local classical solutions). *Let $u_0 \in H^{k+1}(\mathbb{R})$ and $u_1 \in H^k(\mathbb{R})$ be compactly supported for some $k \geqslant 1$. Then there exists a $T^* \in (0, \infty]$ such that* (0.0.1) *has* *a unique solution* $u(t,x) \in L^\infty([0,T], H^{k+1}(\mathbb{R})) \cap \mathrm{Lip}([0,T], H^k(\mathbb{R}))$ *for any positive* $T < T^*$, *and*

$$\overline{\lim_{t\to T^*}}\left(\|\partial_t u(t,\cdot)\|_{L^\infty} + \|\partial_x u(t,\cdot)\|_{L^\infty}\right) = \infty$$

if $T^* < \infty$.

Let $R_0 := u_1 + c(u_0)\,\partial_x u_0$ and $S_0 := u_1 - c(u_0)\,\partial_x u_0$.

THEOREM 3.1.3 (Global rarefactive classical solutions). *Assume* $u_0 \in H^{k+1}(\mathbb{R})$ *and* $u_1 \in H^k(\mathbb{R})$ *be compactly supported for some* $k \geqslant 1$. *Assume further that* $c' \geqslant 0$, $R_0 \leqslant 0$ *and* $S_0 \leqslant 0$. *Then* (0.0.1) *has a global solution* $u(t,x) \in L^\infty(\mathbb{R}^+, H^{k+1}(\mathbb{R})) \cap \mathrm{Lip}([0,\infty), H^k(\mathbb{R}))$.

REMARK 3.1.4. From the proof of Theorem 1 of [29], we find that there exists an example where blow-up occurs in a solution of (0.0.1) with a smooth initial datum satisfying $c' \geqslant 0$, $R_0 \leqslant 0$, $S_0 > 0$.

THEOREM 3.1.5 (Global rarefactive L^p solutions). *Assume* $c' \geqslant 0$, $R_0 \leqslant 0$, $S_0 \leqslant 0$ *and* $(R_0, S_0) \in L^p(\mathbb{R})$ *with compact support for some* $p > 3$. *Then* (0.0.1) *has a global admissible weak solution in the sense of Definition* 3.1.1. *The solution can be obtained either through initial data mollification or vanishing viscosity. Moreover, there holds* $(R,S)(t,x) \in L^\infty(\mathbb{R}^+, L^p(\mathbb{R}))$. *Furthermore, if* (3.1.10) *holds, then* $\partial_x u(t,x) \in L^{p+1}([0,T]\times\mathbb{R})$ *for any* $T > 0$.

THEOREM 3.1.6 (Global rarefactive L^2 solutions). *Assume* (3.1.10) *holds and* $R_0 \leqslant 0$, $S_0 \leqslant 0$, $(R_0, S_0) \in L^2(\mathbb{R})$ *with compact support. Then* (0.0.1) *has a global admissible weak solution in the sense of Definition* 3.1.1. *Moreover, there holds* $\partial_x u \in L^{2+\alpha}([0,T]\times\mathbb{R})$ *for any* $\alpha < 1$ *and* $T < \infty$. *Furthermore there exists a constant* $C > 0$ *such that*

$$-\frac{C}{t} \leqslant R(t,x) \leqslant 0, \qquad -\frac{C}{t} \leqslant S(t,x) \leqslant 0, \quad t \in (0,1], x \in \mathbb{R}, \tag{3.1.11}$$

and (R,S) *remain bounded from below by* $-C$ *for all time* $t > 1$.

We comment on Theorems 3.1.5 and 3.1.6. The L^p regularity ($p > 3$) of the solutions in Theorem 3.1.5 is such that the quadratic nonlinearity of the equation is under control. In Theorem 3.1.6, we rely on the positivity of c' to derive the nonlinear desingularization (3.1.11) to control the quadratic nonlinearity.

We note that the wave speed $c(u)$ in (1.1.4) does not have the monotonicity property required in Theorems 3.1.3–3.1.6. It is our wish to remove the monotonicity condition in the future.

REMARK 3.1.7. Theorems 3.1.3–3.1.6 hold similarly when the signs of c', R_0 and S_0 are reversed.

3.2. *The viscous approximation*

In this subsection, we establish the global existence of smooth solutions to (3.1.6) with general initial data for general $c(u)$. For convenience, we will just write (R, S) as $(R_\varepsilon, S_\varepsilon)$. So, we consider the problem

$$\begin{cases} \partial_t R - \partial_x\big(c(u)R\big) = -\tilde{c}'(u)(R-S)^2 + \varepsilon\,\partial_x^2 R, \\ \partial_t S + \partial_x\big(c(u)S\big) = -\tilde{c}'(u)(R-S)^2 + \varepsilon\,\partial_x^2 S, \\ \partial_x u = \frac{R-S}{2c(u)}, \\ \lim_{x\to-\infty} u(t,x) = 0, \\ (R,S)|_{t=0} = (R_0,S_0)(x), \end{cases} \tag{3.2.12}$$

where $(R_0, S_0)(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ are given initial data and $\varepsilon > 0$.

LEMMA 3.2.1 (Solutions of the viscous system with smooth data). *For given* $(R_0, S_0)(x) \in C_c^\infty(\mathbb{R})$, *problem* (3.2.12) *has a global smooth solution* $(R, S, u)(t, x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$ *satisfying the identity*

$$\begin{aligned} &\int \big(R^2 + S^2\big)(t,x)\,\mathrm{d}x + 2\varepsilon \int_0^t\!\!\int (\partial_x R)^2 + (\partial_x S)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int (R_0)^2 + (S_0)^2\,\mathrm{d}x. \end{aligned} \tag{3.2.13}$$

One can check the Appendix of [73] for details.

REMARK 3.2.2. It is interesting to compare this existence result with the blow-up phenomena established by Fujita [26] for

$$\partial_t V = \frac{1}{2}V^2 + \varepsilon\,\partial_x^2 V,$$

with any initial function $V_0(x)$ such that $V_0(x_0) > 0$ for some point x_0.

LEMMA 3.2.3 (Solutions of the viscous system with general data). *For given* $(R_0, S_0)(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, *problem* (3.2.12) *has a global weak solution* $(R, S, u)(t, x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$ *satisfying the inequalities*

$$\begin{aligned} &\int \big(R_\varepsilon^2 + S_\varepsilon^2\big)(t,x)\,\mathrm{d}x + 2\varepsilon \int_0^t\!\!\int (\partial_x R_\varepsilon)^2 + (\partial_x S_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \int (R_0)^2 + (S_0)^2\,\mathrm{d}x, \end{aligned} \tag{3.2.14}$$

$$\big\|R_\varepsilon(t,\cdot)\big\|_{L^1(\mathbb{R})} + \big\|S_\varepsilon(t,\cdot)\big\|_{L^1(\mathbb{R})} \leqslant C\bigg(T, \frac{1}{\varepsilon}\bigg), \quad t \in [0,T], \tag{3.2.15}$$

for any $T > 0$, where $C(T, 1/\varepsilon)$ depends also on the $L^1 \cap L^2(\mathbb{R})$ norm of (R_0, S_0).

SKETCH OF THE PROOF. Let

$$R_0^\sigma = (R_0\chi_\sigma) * j_\sigma, \qquad S_0^\sigma = (S_0\chi_\sigma) * j_\sigma, \tag{3.2.16}$$

where $\sigma \to 0+$, $j_\sigma(x)$ is standard Friedrichs' mollifier, and $\chi_\sigma(x) = \chi(x\sigma)$ is the scaled cut-off function where $\chi(x) = 1$ on $[-1, 1]$ and zero otherwise. Thus, by Lemma 3.2.1, we find that (3.2.12) has a global smooth solution $(R_\sigma, S_\sigma, u_\sigma)$ with this initial data (R_0^σ, S_0^σ). The solutions satisfy the energy estimate. We can use Lions–Aubin lemma (see [65], pp. 270–271, for example) to establish the strong $L^2([0, T] \times \mathbb{R})$ precompactness of (R_σ, S_σ) to show that we have a weak solution in the limit $\sigma \to 0$. We can use the heat kernel representation of the solution to establish its $C^\infty((0,\infty) \times \mathbb{R})$ regularity.

Using the heat kernel $E(t, x) = (4\pi\varepsilon t)^{-1/2}\mathrm{e}^{-|x|^2/(4\varepsilon t)}$ and (3.2.12), we find

$$\begin{aligned} R(t, x) = \int E(t, x-y)R_0(y)\,\mathrm{d}y + \int_0^t\!\!\int c(u)R(s, y)\,\partial_x E(t-s, x-y)\,\mathrm{d}y\,\mathrm{d}s \\ - \int_0^t\!\!\int \tilde{c}'(u)(R-S)^2(s, y)E(t-s, x-y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Taking the L^1 norm, we have

$$\begin{aligned} \big\|R(t,\cdot)\big\|_{L^1(\mathbb{R})} \leqslant \|R_0\|_{L^1(\mathbb{R})} + Ct\big(\|R_0\|^2_{L^2(\mathbb{R})} + \|S_0\|^2_{L^2(\mathbb{R})}\big) \\ + C_2\varepsilon^{-1/2}\int_0^t \big\|R(s,\cdot)\big\|_{L^1(\mathbb{R})}(t-s)^{-1/2}\,\mathrm{d}s, \end{aligned} \tag{3.2.17}$$

from which the L^1 bound follows, first by choosing a short time interval so that the last term in (3.2.17) is less than half of the left-hand side of (3.2.17), and then the same time interval can be repeated forward indefinitely. This completes the proof of Lemma 3.2.3. □

LEMMA 3.2.4 (Precompactness of u_ε). *Let $(R_0, S_0) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a subsequence $\{\varepsilon_j\}$ such that $\{u_{\varepsilon_j}\}$ of the solutions of* (3.2.12) *is uniformly convergent in every compact subset of* $[0,\infty) \times \mathbb{R}$.

PROOF. Firstly, we let $F(u_\varepsilon) := 2\int_0^{u_\varepsilon} c(\xi)\,\mathrm{d}\xi$. By the third equation of (3.2.12), we find

$$\partial_x F(u_\varepsilon) = R_\varepsilon - S_\varepsilon \quad \text{are uniformly bounded in } L^\infty\big(\mathbb{R}^+, L^2(\mathbb{R})\big). \tag{3.2.18}$$

While, by the fourth assumption of (3.2.12), the estimate (3.2.15) and (3.2.18), we have

$$F(u_\varepsilon) = \int_{-\infty}^x (R_\varepsilon - S_\varepsilon)(t, y)\,\mathrm{d}y. \tag{3.2.19}$$

Then, by a simple calculation, we find

$$\begin{aligned}\partial_t F(u_\varepsilon) &= \int_{-\infty}^{x} \partial_t (R_\varepsilon - S_\varepsilon)(t, y)\,\mathrm{d}y \\ &= c(R_\varepsilon + S_\varepsilon) + \varepsilon(\partial_x R_\varepsilon - \partial_x S_\varepsilon).\end{aligned} \tag{3.2.20}$$

Thus for any $T > 0$, $t_1, t_2 \in \mathbb{R}^+$, with $t_1 < t_2 \leqslant T$, we find

$$\begin{aligned}&\big\| F\big(u_\varepsilon(t_1,\cdot)\big) - F\big(u_\varepsilon(t_2,\cdot)\big)\big\|_{L^2} \\ &\quad = \left\| \int_{t_1}^{t_2} \partial_s F\big(u_\varepsilon(s,\cdot)\big)\,\mathrm{d}s \right\|_{L^2} \\ &\quad \leqslant \int_{t_1}^{t_2} \big\| \big(c(R_\varepsilon + S_\varepsilon) + \varepsilon\,\partial_x(R_\varepsilon - S_\varepsilon)\big)(s,\cdot)\big\|_{L^2}\,\mathrm{d}s \\ &\quad \leqslant C\big(|t_1 - t_2| + \varepsilon^{1/2}|t_1 - t_2|^{1/2}\big)\big(\|R_0\|_{L^2} + \|S_0\|_{L^2}\big).\end{aligned} \tag{3.2.21}$$

On the other hand, by Rellich's theorem $W^{1,2}_{\mathrm{loc}}(\mathbb{R}) \hookrightarrow\hookrightarrow C_{\mathrm{loc}}(\mathbb{R}) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R})$ and Lions–Aubin lemma [65], pp. 270–271, we find that, for any $\eta > 0$, there exists a constant $C_\eta > 0$ such that the inequality holds

$$\begin{aligned}&\big\| F\big(u_\varepsilon(t_1,\cdot)\big) - F\big(u_\varepsilon(t_2,\cdot)\big)\big\|_{L^\infty_{\mathrm{loc}}} \\ &\quad \leqslant 2\eta \sup_t \big\| F\big(u_\varepsilon(t,\cdot)\big)\big\|_{W^{1,2}_{\mathrm{loc}}} + C_{\eta,T}|t_1 - t_2|^{1/2}.\end{aligned} \tag{3.2.22}$$

Now, by a diagonal process, we can extract a subsequence $\{F(u_{\varepsilon_j}(t_\nu,\cdot))\}$ of $\{F(u_\varepsilon)\}$, which converges in L^∞_{loc} for t in the rational number subset $\{t_\nu \mid \nu \in \mathbb{N}\}$ of $\mathbb{R}^+$. For any $t_0 \in [0, T]$ and $\delta > 0$, there is some $t_\nu \in \{t_\nu \mid \nu \in \mathbb{N}\}$ such that $|t_0 - t_\nu| < \delta$, and

$$\begin{aligned}&\big\| \big(F(u_{\varepsilon_j}) - F(u_{\varepsilon_k})\big)(t_0,\cdot)\big\|_{L^\infty_{\mathrm{loc}}} \\ &\quad \leqslant \big\| F\big(u_{\varepsilon_j}(t_0,\cdot)\big) - F\big(u_{\varepsilon_j}(t_\nu,\cdot)\big)\big\|_{L^\infty_{\mathrm{loc}}} + \big\| F\big(u_{\varepsilon_j}(t_\nu,\cdot)\big) - F\big(u_{\varepsilon_k}(t_\nu,\cdot)\big)\big\|_{L^\infty_{\mathrm{loc}}} \\ &\qquad + \big\| F\big(u_{\varepsilon_k}(t_0,\cdot)\big) - F\big(u_{\varepsilon_k}(t_\nu,\cdot)\big)\big\|_{L^\infty_{\mathrm{loc}}}.\end{aligned} \tag{3.2.23}$$

This shows that $\{F(u_{\varepsilon_j})\}$ is uniformly convergent on any compact subset of $[0,\infty) \times \mathbb{R}$. While by (3.1.8), we find

$$\begin{aligned}\big|F(u_{\varepsilon_j}) - F(u_{\varepsilon_k})\big| &= \left| \int_0^1 F'\big(\tau u_{\varepsilon_j} + (1-\tau)u_{\varepsilon_k}\big)(u_{\varepsilon_j} - u_{\varepsilon_k})\,\mathrm{d}\tau \right| \\ &\geqslant c|u_{\varepsilon_j} - u_{\varepsilon_k}|.\end{aligned}$$

This completes the proof of Lemma 3.2.4. □

3.3. *Estimates*

We specialize in the case $c' \geqslant 0$, $R_0(x) \leqslant 0$ and $S_0(x) \leqslant 0$ in this subsection and establish more estimates for $R_\varepsilon(t,x)$ and $S_\varepsilon(t,x)$ of the solution of (3.1.6) with data $(R_0, S_0)(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

LEMMA 3.3.1 (Invariant region). *Assume* (3.1.8) *and* $c' \geqslant 0$. *Let* $R_0(x) \leqslant 0$, $S_0(x) \leqslant 0$, $(R_0(x), S_0(x)) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. *Then* $R_\varepsilon(t,x) \leqslant 0$ *and* $S_\varepsilon(t,x) \leqslant 0$.

PROOF. In fact, if $c'(u) \geqslant 0$, by the first equation of (3.1.6), we immediately have

$$\begin{cases} \partial_t R - c(u)\,\partial_x R \leqslant \tilde{c}'(u) R^2 + \varepsilon\,\partial_{xx} R, \\ R|_{t=0} = R_0 \leqslant 0. \end{cases} \tag{3.3.24}$$

Next, we claim that

$$R(t,x) \leqslant 0. \tag{3.3.25}$$

To prove (3.3.25), we denote $-\mathrm{e}^{-Ct} R(t,x) =: f(t,x)$, where C is an upper bound of $|\tilde{c}'(u)R|$ in $[0,T] \times \mathbb{R}$ for any $T > 0$, mollifying the data as in (3.2.16) if necessary for a finite C to exist. By (3.3.24), $f(t,x)$ satisfies the inequality

$$\partial_t f - c(u)\,\partial_x f \geqslant \big(-C + \tilde{c}'(u) R\big) f + \varepsilon\,\partial_{xx} f. \tag{3.3.26}$$

On the other hand, we know that

$$\lim_{|x|\to\infty} \big| f(t,x) \big| = 0. \tag{3.3.27}$$

If (3.3.25) does not hold, then $f(t,x)$ must attain its negative minimum in $[0,T] \times \mathbb{R}$ at some point (t_0, x_0), $(t_0 > 0)$, where

$$\partial_t f \leqslant 0, \qquad \partial_x f = 0, \qquad \partial_{xx} f \geqslant 0, \qquad \big(-C + \tilde{c}'(u) R\big) f > 0. \tag{3.3.28}$$

Trivially, (3.3.28) contradicts (3.3.26). Thus (3.3.25) holds. Exactly as above, we can prove $S(t,x) \leqslant 0$ if $S_0(x) \leqslant 0$. This completes the proof of Lemma 3.3.1. □

LEMMA 3.3.2 (Estimates). *Assume* (3.1.8) *and* $c' \geqslant 0$. *Let* $R_0(x) \leqslant 0$, $S_0(x) \leqslant 0$, $(R_0(x), S_0(x)) \in L^p(\mathbb{R}) \cap L^1(\mathbb{R})$ *with* $p > 2$. *Then*

(a) (L^p estimate) *there holds*

$$\|R_\varepsilon\|^p_{L^p(\mathbb{R})} + \|S_\varepsilon\|^p_{L^p(\mathbb{R})} \leqslant \|R_0\|^p_{L^p(\mathbb{R})} + \|S_0\|^p_{L^p(\mathbb{R})}, \tag{3.3.29}$$

(b) (L^{p+1} estimate) *there holds*

$$\int_0^\infty \int_{\mathbb{R}} c'(u_\varepsilon) |\partial_x u_\varepsilon|^{p+1} \,\mathrm{d}x\,\mathrm{d}t \leqslant K_p \int_{\mathbb{R}} \big(|R_0|^p + |S_0|^p\big)\,\mathrm{d}x \tag{3.3.30}$$

for some constant K_p.

PROOF. By Lemma 3.3.1, and for convenience, we let $R'_\varepsilon(t,x) := -R_\varepsilon(t,x) \geqslant 0$, $S'_\varepsilon(t,x) := -S_\varepsilon(t,x) \geqslant 0$. Then, by the first equation of (3.1.6), we have

$$\partial_t R'_\varepsilon - c\,\partial_x R'_\varepsilon = -\tilde{c}'\big(\big(R'_\varepsilon\big)^2 - \big(S'_\varepsilon\big)^2\big) + \varepsilon\,\partial_{xx} R'_\varepsilon. \tag{3.3.31}$$

Next, we multiply $(R'_\varepsilon)^{p-1}$ to both sides of the above equation to yield

$$\begin{aligned}&\frac{1}{p}\big\{\partial_t\big(R'_\varepsilon\big)^p - c\,\partial_x\big(R'_\varepsilon\big)^p\big\}\\ &\quad= -\tilde{c}'\big(\big(R'_\varepsilon\big)^{p+1} - \big(R'_\varepsilon\big)^{p-1}\big(S'_\varepsilon\big)^2\big) + \varepsilon\big(R'_\varepsilon\big)^{p-1}\,\partial_{xx} R'_\varepsilon.\end{aligned} \tag{3.3.32}$$

Then, by the third equation of (3.1.6), we find

$$\begin{aligned}&\frac{1}{p}\big\{\partial_t\big(R'_\varepsilon\big)^p - \partial_x\big(c\big(R'_\varepsilon\big)^p\big)\big\} - \varepsilon\big(R'_\varepsilon\big)^{p-1}\,\partial_{xx} R'_\varepsilon\\ &\quad= -\left(\frac{1}{4} - \frac{1}{2p}\right)\frac{c'}{c}\big(R'_\varepsilon\big)^{p+1} + \frac{c'}{4c}\big(R'_\varepsilon\big)^{p-1}\big(S'_\varepsilon\big)^2 - \frac{c'}{2pc}S'_\varepsilon\big(R'_\varepsilon\big)^p.\end{aligned} \tag{3.3.33}$$

Exactly as the above procedure for $S'_\varepsilon(t,x)$, we have

$$\begin{aligned}&\frac{1}{p}\big\{\partial_t\big(S'_\varepsilon\big)^p + \partial_x\big(c\big(S'_\varepsilon\big)^p\big)\big\} - \varepsilon\big(S'_\varepsilon\big)^{p-1}\,\partial_{xx} S'_\varepsilon\\ &\quad= -\left(\frac{1}{4} - \frac{1}{2p}\right)\frac{c'}{c}\big(S'_\varepsilon\big)^{p+1} + \frac{c'}{4c}\big(S'_\varepsilon\big)^{p-1}\big(R'_\varepsilon\big)^2 - \frac{c'}{2pc}R'_\varepsilon\big(S'_\varepsilon\big)^p.\end{aligned} \tag{3.3.34}$$

Thus, by summing up (3.3.33) and (3.3.34), and integrating over $\mathbb{R}$, we have

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(R'_\varepsilon\big)^p + \big(S'_\varepsilon\big)^p\,\mathrm{d}x\\ &\qquad+ \varepsilon(p-1)\int_{\mathbb{R}}\big(R'_\varepsilon\big)^{p-2}\big(\partial_x R'_\varepsilon\big)^2 + \big(S'_\varepsilon\big)^{p-2}\big(\partial_x S'_\varepsilon\big)^2\,\mathrm{d}x\\ &\quad= -\left(\frac{1}{4} - \frac{1}{2p}\right)\int_{\mathbb{R}}\frac{c'}{c}\big(\big(R'_\varepsilon\big)^2 - \big(S'_\varepsilon\big)^2\big)\big(\big(R'_\varepsilon\big)^{p-1} - \big(S'_\varepsilon\big)^{p-1}\big)\,\mathrm{d}x\\ &\qquad- \frac{1}{2p}\int_{\mathbb{R}}\frac{c'}{c}R'_\varepsilon S'_\varepsilon\big(R'_\varepsilon - S'_\varepsilon\big)\big(\big(R'_\varepsilon\big)^{p-2} - \big(S'_\varepsilon\big)^{p-2}\big)\,\mathrm{d}x \leqslant 0.\end{aligned} \tag{3.3.35}$$

Hence, by integrating the above inequality over $[0,t]$, we find (3.3.29). Moreover, by (3.3.29) and (3.3.35) and the identity

$$\begin{aligned}
&\big((R'_\varepsilon)^2-(S'_\varepsilon)^2\big)\big((R'_\varepsilon)^{p-1}-(S'_\varepsilon)^{p-1}\big)\\
&\quad=\big(R'_\varepsilon-S'_\varepsilon\big)^2\big((R'_\varepsilon)^{p-1}+(S'_\varepsilon)^{p-1}\big)\\
&\qquad+2R'_\varepsilon S'_\varepsilon\big(R'_\varepsilon-S'_\varepsilon\big)\big((R'_\varepsilon)^{p-2}-(S'_\varepsilon)^{p-2}\big),
\end{aligned}\tag{3.3.36}$$

we find

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}}\frac{c'}{c}\big\{(R_\varepsilon-S_\varepsilon)^2\big(|R_\varepsilon|^{p-1}+|S_\varepsilon|^{p-1}\big)\\
&\qquad\qquad+C_pR_\varepsilon S_\varepsilon\big(|R_\varepsilon|-|S_\varepsilon|\big)\big(|R_\varepsilon|^{p-2}-|S_\varepsilon|^{p-2}\big)\big\}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant K'_p\big(\|R_0\|^p_{L^p(\mathbb{R})}+\|S_0\|^p_{L^p(\mathbb{R})}\big),
\end{aligned}\tag{3.3.37}$$

where $C_p=2(p-1)/(p-2)$ and $K'_p=4/(p-2)$ for $p>2$. Since both terms in the integrand in (3.3.37) are nonnegative, (3.3.37) implies

$$\int_0^\infty\!\!\int_{\mathbb{R}}\frac{c'}{c}(R_\varepsilon-S_\varepsilon)^2\big(|R_\varepsilon|^{p-1}+|S_\varepsilon|^{p-1}\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant K'_p\int_{\mathbb{R}}|R_0|^p+|S_0|^p\,\mathrm{d}x.\tag{3.3.38}$$

Using the third equation of (3.2.12) we obtain (3.3.30) from (3.3.38). □

3.4. *Classical solutions*

Consider the problem

$$\begin{cases}
\partial_tR-\partial_x\big(c(u)R\big)=-\tilde c'(u)(R-S)^2,\\
\partial_tS+\partial_x\big(c(u)S\big)=-\tilde c'(u)(R-S)^2,\\
\partial_xu=\frac{R-S}{2c(u)},\\
\lim_{x\to-\infty}u(t,x)=0,\\
(R,S)|_{t=0}=(R_0,S_0)(x),
\end{cases}\tag{3.4.39}$$

where $(R_0,S_0)(x)\in H^k(\mathbb{R})\cap L^1(\mathbb{R})$, $k\geqslant1$, are given initial data.

LEMMA 3.4.1 (Local classical solution). *Assume $(R_0,S_0)(x)\in H^k(\mathbb{R})\cap L^1(\mathbb{R})$, $k\geqslant1$. Then there exists a $T^*\in(0,\infty]$ such that problem (3.4.39) has a unique solution $(R(t,x),S(t,x))\in L^\infty([0,T],H^k(\mathbb{R}))$ for any positive $T<T^*$, and*

$$\varlimsup_{t\to T^*}\big(\|R(t,\cdot)\|_{L^\infty}+\|S(t,\cdot)\|_{L^\infty}\big)=\infty\tag{3.4.40}$$

if $T^* < \infty$.

REMARK 3.4.2. We remark that the catastrophe in (3.4.40) is associated with the type of blow-up familiar from ordinary differential equation theory rather than associated with the formation of shock waves in the context of systems of conservation laws, see Chapter 2 of Majda [51].

PROOF OF LEMMA 3.4.1. From standard theory on hyperbolic systems of equations (see [51], for example), we know we can find a positive constant T such that problem (3.4.39) has a unique solution $(R(t,x), S(t,x)) \in L^\infty([0,T], H^k(\mathbb{R}))$. The solution satisfies the estimate

$$\begin{aligned}
&\|R\|_{L^\infty([0,T],H^k(\mathbb{R}))} + \|S\|_{L^\infty([0,T],H^k(\mathbb{R}))} \\
&\quad \leqslant C\big(k, \|R_0\|_{H^k(\mathbb{R})}, \|S_0\|_{H^k(\mathbb{R})}\big).
\end{aligned} \tag{3.4.41}$$

We note that although the equations in (3.4.39) do not consist of a *bona fide* symmetric hyperbolic system, the equivalent alternative form of using $2\,\partial_t u = R + S$ does and the standard theory on hyperbolic systems applies.

Next, we claim that

$$\big\|(R,S)\big\|_{L^\infty([0,T]\times\mathbb{R})} < \infty \quad \text{implies} \quad \big\|(R,S)\big\|_{L^\infty([0,T],H^k(\mathbb{R}))} < \infty. \tag{3.4.42}$$

In fact, by taking ∂_x^l to both sides of the first equation of (3.4.39) for $l \leqslant k$, we find

$$\begin{aligned}
&\partial_t\big(\partial_x^l R\big) - c(u)\,\partial_x\big(\partial_x^l R\big) \\
&\quad = \partial_x^l\big(c(u)\,\partial_x R\big) - c(u)\,\partial_x^{l+1} R + \partial_x^l\big(\tilde{c}'(u)\big(R^2 - S^2\big)\big).
\end{aligned} \tag{3.4.43}$$

By multiplying $\partial_x^l R$ to the above equation and integrating over $\mathbb{R}$, we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{1}{2}\big(\partial_x^l R\big)^2\,\mathrm{d}x + \int_{\mathbb{R}} \frac{1}{2} c'(u)\,\partial_x u\,\big(\partial_x^l R\big)^2\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \big[\partial_x^l\big(c(u)\,\partial_x R\big) - c(u)\,\partial_x^{l+1} R\big]\,\partial_x^l R\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} \partial_x^l\big[\tilde{c}'(u)\big(R^2 - S^2\big)\big]\,\partial_x^l R\,\mathrm{d}x.
\end{aligned} \tag{3.4.44}$$

Consider the case $k = 1$ first. If $\|(R,S)\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$, then by (3.4.44), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\partial_x R)^2(t,x)\,\mathrm{d}x \leqslant C_T \int_{\mathbb{R}} \big((\partial_x R)^2 + (\partial_x S)^2\big)(t,x)\,\mathrm{d}x + C_T. \tag{3.4.45}$$

Exactly as the proof of (3.4.45), we can prove a similar inequality for the integral of $(\partial_x S)^2$. Summing up the two inequalities and using Gronwall's inequality, we prove that (3.4.42) holds for $k = 1$.

Consider the case $k \geqslant 2$. By Moser-type calculus inequality (see p. 43 of [51]), we find

$$\begin{aligned}
&\big\|\big[\partial_x^k\big(c(u)\,\partial_x R\big) - c(u)\,\partial_x^{k+1} R\big](t,\cdot)\big\|_{L^2} \\
&\quad \leqslant C_k\big[\big\|\partial_x c(u)(t,\cdot)\big\|_{L^\infty}\big\|\partial_x^k R(t,\cdot)\big\|_{L^2} + \big\|\partial_x R(t,\cdot)\big\|_{L^\infty}\big\|\partial_x^k c(u)(t,\cdot)\big\|_{L^2}\big].
\end{aligned} \tag{3.4.46}$$

Since

$$\begin{aligned}
\big\|\partial_x^k c(u)(t,\cdot)\big\|_{L^2} &= \big\|\partial_x^{k-1}\big(c'(u)\,\partial_x u\big)\big\|_{L^2} \\
&\leqslant C\big(\|\partial_x u\|_{H^{k-1}} + \big\|\partial_x^{k-1} c'(u)\big\|_{L^2}\big) \\
&\leqslant \cdots \leqslant C_k\|\partial_x u\|_{H^{k-1}},
\end{aligned}$$

we further obtain that

$$\begin{aligned}
&\big\|\big[\partial_x^k\big(c(u)\,\partial_x R\big) - c(u)\,\partial_x^{k+1} R\big](t,\cdot)\big\|_{L^2} \\
&\quad \leqslant C_k\big[\big\|\partial_x^k R(t,\cdot)\big\|_{L^2} + \big\|\partial_x R(t,\cdot)\big\|_{L^\infty}\big\|\partial_x u(t,\cdot)\big\|_{H^{k-1}}\big].
\end{aligned} \tag{3.4.47}$$

Similarly, we obtain that

$$\begin{aligned}
&\big\|\big(\partial_x^k\big(\tilde c'(u)\big(R^2 - S^2\big)\big)\big)(t,\cdot)\big\|_{L^2} \\
&\quad \leqslant C_k\big\{\big\|\tilde c'(u)(t,\cdot)\big\|_{L^\infty}\big\|\partial_x^k\big(R^2 - S^2\big)\big\|_{L^2} \\
&\qquad\qquad + \big\|\partial_x^k\big(\tilde c'(u)\big)(t,\cdot)\big\|_{L^2}\big\|\big(R^2 - S^2\big)(t,\cdot)\big\|_{L^\infty}\big\} \\
&\quad \leqslant C_k\big\{\big\|\partial_x^k R(t,\cdot)\big\|_{L^2} + \big\|\partial_x^k S(t,\cdot)\big\|_{L^2} + \big\|\partial_x u(t,\cdot)\big\|_{H^{k-1}}\big\}.
\end{aligned} \tag{3.4.48}$$

By summing up (3.4.44), (3.4.47) and (3.4.48), we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big\|R(t,\cdot)\big\|_{H^k}^2 \\
&\quad \leqslant C_k\big\{\big\|R(t,\cdot)\big\|_{H^k} + \big\|S(t,\cdot)\big\|_{H^k} \\
&\qquad\qquad + \big(1 + \|\partial_x R\|_{L^\infty}\big)\big\|\partial_x u(t,\cdot)\big\|_{H^{k-1}}\big\}\big\|R(t,\cdot)\big\|_{H^k}.
\end{aligned} \tag{3.4.49}$$

Exactly as the above procedure, we can prove a similar inequality for $\|S(t,\cdot)\|_{H^k}^2$.

On the other hand, by differentiating the first equation of (3.4.39) once, we find

$$\partial_t\partial_x R - c(u)\,\partial_x(\partial_x R) - c'(u)\,\partial_x u\,\partial_x R = \partial_x\big(\tilde c'(u)\big(R^2 - S^2\big)\big). \tag{3.4.50}$$

Thus, by the characteristic method, we obtain

$$\big\|\partial_x R(t,\cdot)\big\|_{L^\infty} \leqslant \|\partial_x R_0\|_{L^\infty} + C\int_0^t \big(\|\partial_x R\|_{L^\infty} + \|\partial_x S\|_{L^\infty} + 1\big)\,\mathrm{d}s. \tag{3.4.51}$$

Exactly as the above we can prove a similar inequality for the L^∞ norm of $\partial_x S$. Summing up the two inequalities we find, by applying Gronwall's inequality, that

$$\left\|\partial_x R(t,\cdot)\right\|_{L^\infty} + \left\|\partial_x S(t,\cdot)\right\|_{L^\infty} \leqslant C_T, \quad t \in [0,T], \tag{3.4.52}$$

where $C_T = C(T, \|R\|_{L^\infty}, \|S\|_{L^\infty}, \|\partial_x R_0\|_{L^\infty}, \|\partial_x S_0\|_{L^\infty})$ is a constant depending only on the variables. The quantities $(\|\partial_x R_0\|_{L^\infty}, \|\partial_x S_0\|_{L^\infty})$ are finite since $(R_0, S_0) \in H^2(\mathbb{R})$.

Thus by summing up (3.4.49), (3.4.52) and Gronwall's inequality, we prove (3.4.41). The proof of Lemma 3.4.1 is complete. □

LEMMA 3.4.3 (Global classical solution). *Assume $c' \geqslant 0$, $R_0 \leqslant 0$, $S_0 \leqslant 0$. Then problem* (3.4.39) *has a global solution $(R,S)(t,x) \in L^\infty(\mathbb{R}^+, H^k(\mathbb{R}))$ provided that $(R_0,S_0) \in H^k(\mathbb{R}) \cap L^1(\mathbb{R})$, $k \geqslant 1$.*

PROOF. By Lemma 3.3.2 (with $\varepsilon = 0$), if $c' \geqslant 0$, $R_0 \leqslant 0$, $S_0 \leqslant 0$, we have (3.3.29) holds for all $p > 2$. By taking p to infinity, we find that $\|R\|_{L^\infty} + \|S\|_{L^\infty}$ is bounded by $\|R_0\|_{L^\infty} + \|S_0\|_{L^\infty}$ which is finite since $(R_0,S_0) \in H^1(\mathbb{R})$. By Lemma 3.4.1, we complete the proof of Lemma 3.4.3. □

PROOF OF THEOREMS 3.1.2 AND 3.1.3. We show that the solutions of Lemmas 3.4.1–3.4.3 yield solutions for (0.0.1). From the conditions of Theorems 3.1.2 and 3.1.3, we have $(R_0,S_0) \in H^k(\mathbb{R})$, $k \geqslant 1$, and both have compact supports. Then by Lemmas 3.1.4 and 3.1.5, (3.4.39) has a solution $(R,S) \in L^\infty([0,T], H^k(\mathbb{R}))$, where T is finite or infinite accordingly. We already have $R - S = 2c(u)\,\partial_x u$, so we only need

$$R + S = 2\,\partial_t u. \tag{3.4.53}$$

It can be shown easily that the solution (R,S) has compact support for each time $t \geqslant 0$. From this and the third and fourth equations of (3.4.39), we see that u is supported in $[C_t, \infty)$ in x for each time $t \geqslant 0$ where C_t is some function of t. Now multiplying the third equation of (3.4.39) by $2c(u)$, taking t derivative and using $\partial_t(2c(u)\,\partial_x u) = \partial_x(2c(u)\,\partial_t u)$, we find

$$\partial_x\left\{2c(u)\,\partial_t u - c(u)(R+S)\right\} = 0. \tag{3.4.54}$$

The support property and (3.4.54) implies (3.4.53). Hence by summing up the first two equations of (3.4.39) and (3.4.53), we find that $u(t,x)$ is indeed a solution of (0.0.1). □

3.5. *Precompactness*

With the above preparation, we can now prove the precompactness of the solution sequence $\{R_\varepsilon(t,x), S_\varepsilon(t,x)\}$, obtained either through the viscous approximation or the mollification of the initial data, in $L^2([0,T] \times \mathbb{R})$ for any $T < \infty$ for initial data $(R_0,S_0) \in L^p(\mathbb{R}) \cap L^1(\mathbb{R})$ with $p > 3$, by applying Young measure theory [63,68] and the ideas used by the authors in the proof of global existence of weak solutions to (2.1.1) in [70] and the ideas used by Lions [49] in the proof of the global existence of weak solutions to multi-dimensional compressible Navier–Stokes equation and by Joly, Métivier and Rauch [39] in the rigorous justification of weakly nonlinear geometric optics for a semilinear wave equation.

Let $(R_0, S_0) \in L^2 \cap L^1$. We now focus on the viscous regularization. The treatment for the initial data mollification is similar. By Lemmas 3.2.3 and 1.3.1, there exist three families of Young measures $\nu^1_{t,x}(\xi)$, $\nu^2_{t,x}(\eta)$ and $\mu_{t,x}(\xi,\eta)$ which are associated with $\{R_\varepsilon\}$, $\{S_\varepsilon\}$ and $\{R_\varepsilon, S_\varepsilon\}$, respectively. Moreover, by modifying the proof of Lemma 3 in [70], we can prove the following lemma.

LEMMA 3.5.1 (Time-distinguished Young measures). *There exist a subsequence of the solution sequence $\{R_\varepsilon(t,x), S_\varepsilon(t,x)\}$, for convenience, we still denote it by $\{R_\varepsilon(t,x), S_\varepsilon(t,x)\}$, and three families of Young measures $\nu^1_{t,x}(\xi)$ and $\nu^2_{t,x}(\eta)$ on $\mathbb{R}$ and $\mu_{t,x}(\xi,\eta)$ on $\mathbb{R}^2$ such that for all continuous functions $f(\lambda) \in C_c^\infty(\mathbb{R})$, $\psi(x) \in C_c^\infty(\mathbb{R})$, $g(\xi,\eta) \in C_c^\infty(\mathbb{R}^2)$ and $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, there hold*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f\big(R_\varepsilon(t,x)\big)\psi(x)\,\mathrm{d}x = \iint_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x,\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f\big(S_\varepsilon(t,x)\big)\psi(x)\,\mathrm{d}x = \iint_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)\,\mathrm{d}\nu^2_{t,x}(\eta)\,\mathrm{d}x
\end{aligned} \tag{3.5.55}$$

uniformly in every compact subset of $[0,\infty)$*, and*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{R}} g\big(R_\varepsilon(t,x), S_\varepsilon(t,x)\big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^\infty\int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}} g(\xi,\eta)\varphi(t,x)\,\mathrm{d}\mu_{t,x}(\xi,\eta)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.5.56}$$

Moreover,

$$\begin{aligned}
&t\in[0,\infty)\mapsto \iint_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x \quad \textit{is continuous},\\
&t\in[0,\infty)\mapsto \iint_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)\,\mathrm{d}\nu^2_{t,x}(\eta)\,\mathrm{d}x \quad \textit{is continuous}.
\end{aligned} \tag{3.5.57}$$

In the sequel, we use the notation

$$\overline{g(R,S)} = \int_{\mathbb{R}} g(\xi,\eta)\,\mathrm{d}\mu_{t,x}(\xi,\eta).$$

With Lemmas 1.3.1, 3.2.3 and 3.5.1, we can prove the decoupling of the Young measure $\mu_{t,x}(\xi,\eta)$ into the tensor product of the Young measures $\nu^1_{t,x}(\xi)$ and $\nu^2_{t,x}(\eta)$.

LEMMA 3.5.2 (Decoupling of the Young measure). *Let* $\{R_\varepsilon, S_\varepsilon\}$ *be the solutions to* (3.2.12) *with data* $(R_0, S_0) \in L^2 \cap L^1(\mathbb{R})$. *Then the Young measures* $\nu^1_{t,x}(\xi)$, $\nu^2_{t,x}(\eta)$ *and* $\mu_{t,x}(\xi,\eta)$ *have the property*

$$\mu_{t,x}(\xi,\eta) = \nu^1_{t,x}(\xi) \otimes \nu^2_{t,x}(\eta).$$

PROOF. Take any $f \in C_c^\infty(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. By (3.2.12) and a trivial calculation, we find that

$$\partial_t f(R_\varepsilon) - \partial_x\big(c(u) f(R_\varepsilon)\big) = T_1^\varepsilon + T_2^\varepsilon, \tag{3.5.58}$$

where

$$\begin{aligned} T_1^\varepsilon &:= \partial_x\Big(\big(c(u_\varepsilon) - c(u)\big) f(R_\varepsilon)\Big) + \varepsilon\, \partial_x\big(f'(R_\varepsilon)\, \partial_x R_\varepsilon\big), \\ T_2^\varepsilon &:= 2\tilde{c}'(u_\varepsilon)(S_\varepsilon - R_\varepsilon) f(R_\varepsilon) \\ &\quad + \tilde{c}'(u_\varepsilon)\big(R_\varepsilon^2 - S_\varepsilon^2\big) f'(R_\varepsilon) - \varepsilon f''(R_\varepsilon)(\partial_x R_\varepsilon)^2. \end{aligned} \tag{3.5.59}$$

By Lemma 3.2.4, $(c(u_\varepsilon) - c(u)) f(R_\varepsilon) \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ for any $p < \infty$. By Lemma 3.2.3, $\varepsilon f'(R_\varepsilon)\, \partial_x R_\varepsilon \to 0$ in L^2, thus $\{T_1^\varepsilon\}$ is precompact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. By Lemma 3.2.3 again, we find that $\{T_2^\varepsilon\}$ is uniformly bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. Since $f(R_\varepsilon)$ and $c(u_\varepsilon)$ are uniformly bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$, thus by Murat lemma [53] (or Corollary 1 on p. 8 of [23]), we find that $\{T_2^\varepsilon\}$ is also a precompact subset of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. Summing up the above, we have proved the precompactness

$$\big\{\partial_t f(R_\varepsilon) - \partial_x\big(c(u) f(R_\varepsilon)\big)\big\} \subset\subset H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{R}\big). \tag{3.5.60}$$

Exactly as the proof of (3.5.60), we can also prove that

$$\big\{\partial_t g(S_\varepsilon) + \partial_x\big(c(u) g(S_\varepsilon)\big)\big\} \subset\subset H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{R}\big). \tag{3.5.61}$$

Hence, by (3.5.60), (3.5.61) and the generalized compensated-compactness theorem in [27], we find that

$$f(R_\varepsilon) g(S_\varepsilon) \rightharpoonup \overline{f(R)} \cdot \overline{g(S)} \quad \text{as } \varepsilon \to 0, \tag{3.5.62}$$

where $(\overline{f(R)}, \overline{g(S)})$ is the weak limit of $(f(R_\varepsilon), g(S_\varepsilon))$. Thus, by Lemma 3.1.6, we have proved that for any $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$,

$$\begin{aligned} &\iiiint \varphi(t,x) f(\xi) g(\eta)\, \mathrm{d}\mu_{t,x}(\xi,\eta)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad = \lim_{\varepsilon \to 0} \iint \varphi(t,x) f(R_\varepsilon) g(S_\varepsilon)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad = \iint \varphi(t,x) \overline{f(R)} \cdot \overline{g(S)}\, \mathrm{d}x\, \mathrm{d}t \end{aligned}$$

$$= \iint \varphi(t,x) \int f(\xi) g(\eta)\, \mathrm{d}\nu^1_{t,x}\, \mathrm{d}\nu^2_{t,x}\, \mathrm{d}x\, \mathrm{d}t$$
$$= \iiiint \varphi(t,x) f(\xi) g(\eta)\, \mathrm{d}\nu^1_{t,x}(\xi) \otimes \nu^2_{t,x}(\eta)\, \mathrm{d}x\, \mathrm{d}t. \tag{3.5.63}$$

Since the above equality holds for any $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, $f(\xi), g(\eta) \in C_c^\infty(\mathbb{R})$, the proof of Lemma 3.5.2 is complete. □

We prove in the next lemma that the two single-variable Young measures are Dirac measures provided that $p > 3$.

LEMMA 3.5.3 (Strong precompactness). *Assume* $R_0 \leqslant 0$, $S_0 \leqslant 0$, $(R_0, S_0) \in L^p(\mathbb{R})$ *with* $p > 3$ *and* $c' \geqslant 0$. *Let* $(\overline{R}, \overline{S})$ *be the weak-star limit of* $\{(R_\varepsilon, S_\varepsilon)\}$ *in* $L^\infty(\mathbb{R}^+, L^p(\mathbb{R}))$. *Then* $\nu^1_{t,x}(\xi) = \delta_{\overline{R}(t,x)}(\xi)$ *and* $\nu^2_{t,x}(\eta) = \delta_{\overline{S}(t,x)}(\eta)$.

PROOF.
Step 1. Firstly, as in [48], we introduce the notation

$$T_\lambda(\xi) = \begin{cases} \xi, & -\lambda \leqslant \xi \leqslant 0, \\ -\lambda, & \xi \leqslant -\lambda, \\ 0, & \text{otherwise}, \end{cases}$$
$$S_\lambda(\xi) = \begin{cases} \frac{1}{2}\xi^2, & -\lambda \leqslant \xi \leqslant 0, \\ -\lambda\xi - \frac{1}{2}\lambda^2, & \xi \leqslant -\lambda, \\ 0, & \text{otherwise}. \end{cases} \tag{3.5.64}$$

That is, $T_\lambda(\xi)$ is a cut-off function of

$$\xi^- = \begin{cases} \xi, & \xi \leqslant 0, \\ 0, & \xi \geqslant 0, \end{cases}$$

and $S_\lambda(\xi)$ is an antiderivative of $T_\lambda(\xi)$. Next, we multiply $T_\lambda(R_\varepsilon)$ to both sides of the first equation of (3.1.6) to obtain

$$\begin{aligned} &\partial_t S_\lambda(R_\varepsilon) - \partial_x\big(c(u_\varepsilon) S_\lambda(R_\varepsilon)\big) \\ &\quad = -\tilde{c}'(u_\varepsilon)\big[2(R_\varepsilon - S_\varepsilon) S_\lambda(R_\varepsilon) - T_\lambda(R_\varepsilon)\big(R_\varepsilon^2 - S_\varepsilon^2\big)\big] \\ &\qquad + \varepsilon T_\lambda(R_\varepsilon)\, \partial_{xx} R_\varepsilon. \end{aligned} \tag{3.5.65}$$

Now, we claim that

$$\lim_{\varepsilon \to 0} \varepsilon \iint_{\mathbb{R}^+ \times \mathbb{R}} \varphi T_\lambda(R_\varepsilon)\, \partial_{xx} R_\varepsilon\, \mathrm{d}x\, \mathrm{d}t \leqslant 0 \quad \forall 0 \leqslant \varphi \in C_c^\infty\big(\mathbb{R}^+ \times \mathbb{R}\big). \tag{3.5.66}$$

In fact, we have

$$
\begin{aligned}
&-\iint_{\mathbb{R}^+\times\mathbb{R}} \varphi T_\lambda(R_\varepsilon)\,\partial_{xx}R_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\iint_{\mathbb{R}^+\times\mathbb{R}} \varphi T_\lambda'(R_\varepsilon)(\partial_x R_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{\mathbb{R}^+\times\mathbb{R}} \partial_x\varphi\, T_\lambda(R_\varepsilon)\,\partial_x R_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.5.67}
$$

Then claim (3.5.66) follows from this (3.5.67), the monotonicity $T_\lambda'(\xi)\geqslant 0,$ the energy bound (3.2.14), and the fact that $|T_\lambda(\xi)|\leqslant\lambda$. Hence by taking $\varepsilon\to 0$ in (3.5.65) and applying Lemma 3.5.1 and (3.5.66), we find

$$
\begin{aligned}
&\partial_t\overline{S_\lambda(R)}-\partial_x\big(c(u)\overline{S_\lambda(R)}\big)\\
&\quad\leqslant\tilde{c}'(u)\iint_{\mathbb{R}^2} 2(\eta-\xi)S_\lambda(\xi)+T_\lambda(\xi)\big(\xi^2-\eta^2\big)\,\mathrm{d}\mu_{t,x}(\xi,\eta).
\end{aligned}
\tag{3.5.68}
$$

Step 2. On the other hand, by applying Lemma 3.5.1 again and taking the limit for the first equation of (3.1.6) as $\varepsilon\to 0,$ we find

$$
\partial_t\overline{R}-\partial_x\big(c(u)\overline{R}\big)=-\tilde{c}'(u)\overline{(R-S)^2}. \tag{3.5.69}
$$

Convolving with standard Friedrichs' mollifier j_ε, we find

$$
\partial_t\overline{R}^\varepsilon-\partial_x\big(c(u)\overline{R}^\varepsilon\big)=-\big(\tilde{c}'(u)\overline{(R-S)^2}\big)*j_\varepsilon+r_\varepsilon, \tag{3.5.70}
$$

where $\overline{R}^\varepsilon=\int_{\mathbb{R}}\overline{R}(t,y)j_\varepsilon(x-y)\,\mathrm{d}y$ and $r_\varepsilon=j_\varepsilon*\partial_x(c(u)\overline{R})-\partial_x(c(u)\overline{R}^\varepsilon)$. By DiPerna–Lions folklore Lemma 2.3 of Lions [48] and Lebesgue dominated convergence theorem in the time direction, we have $r_\varepsilon\to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$ (or see Lemma II.1 of [19]). Again, we multiply $T_\lambda(\overline{R}^\varepsilon)$ to both sides of (3.5.70), we find

$$
\begin{aligned}
&\partial_t S_\lambda\big(\overline{R}^\varepsilon\big)-\partial_x\big(c(u)S_\lambda\big(\overline{R}^\varepsilon\big)\big)\\
&\quad=\big[-\big(\tilde{c}'(u)\overline{(R-S)^2}\big)*j_\varepsilon+r_\varepsilon+2\tilde{c}'(u)\big(\overline{R}-\overline{S}\big)\overline{R}^\varepsilon\big]T_\lambda\big(\overline{R}^\varepsilon\big)\\
&\qquad-2\tilde{c}'(u)\big(\overline{R}-\overline{S}\big)S_\lambda\big(\overline{R}^\varepsilon\big).
\end{aligned}
\tag{3.5.71}
$$

Thus, by taking $\varepsilon\to 0$ in (3.5.71), we find

$$
\begin{aligned}
&\partial_t S_\lambda\big(\overline{R}\big)-\partial_x\big(c(u)S_\lambda\big(\overline{R}\big)\big)\\
&\quad=-\tilde{c}'(u)\big[\,\overline{(R-S)^2}-2\big(\overline{R}-\overline{S}\big)\overline{R}\,\big]T_\lambda\big(\overline{R}\big)\\
&\qquad-2\tilde{c}'(u)\big(\overline{R}-\overline{S}\big)S_\lambda\big(\overline{R}\big).
\end{aligned}
\tag{3.5.72}
$$

While by Lemma 3.5.2,

$$\begin{aligned}
&\overline{(R-S)^2} \\
&\quad = \iint_{\mathbb{R}^2} (\xi-\eta)^2 \, \mathrm{d}\mu_{t,x}(\xi,\eta) \\
&\quad = \int_{\mathbb{R}} \xi^2 \, \mathrm{d}\nu^1_{t,x}(\xi) + \int_{\mathbb{R}} \eta^2 \, \mathrm{d}\nu^2_{t,x}(\eta) - 2\int_{\mathbb{R}} \xi \, \mathrm{d}\nu^1_{t,x}(\xi) \int_{\mathbb{R}} \eta \, \mathrm{d}\nu^2_{t,x}(\eta) \\
&\quad = \overline{R^2} + \overline{S^2} - 2\overline{R}\,\overline{S}.
\end{aligned} \tag{3.5.73}$$

By summing up (3.5.72) and (3.5.73), we find

$$\begin{aligned}
&\partial_t S_\lambda\big(\overline{R}\big) - \partial_x\big(c(u)S_\lambda\big(\overline{R}\big)\big) \\
&\quad = \tilde{c}'(u)\big\{-2\big(\overline{R}-\overline{S}\big)S_\lambda\big(\overline{R}\big) + 2T_\lambda\big(\overline{R}\big)\overline{R}^2 - T_\lambda\big(\overline{R}\big)\big(\overline{R^2}+\overline{S^2}\big)\big\}.
\end{aligned} \tag{3.5.74}$$

Step 3. Now, we subtract (3.5.74) from (3.5.68) to obtain

$$\begin{aligned}
&\partial_t\big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big) - \partial_x\big(c(u)\big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big)\big) \\
&\quad \leqslant \tilde{c}'(u)\bigg\{\int_{\mathbb{R}} \big[-2\xi S_\lambda(\xi) + T_\lambda(\xi)\xi^2\big] \, \mathrm{d}\nu^1_{t,x}(\xi) + 2\overline{R}S_\lambda\big(\overline{R}\big) - T_\lambda\big(\overline{R}\big)\overline{R}^2 \\
&\qquad\qquad + \iint_{\mathbb{R}^2} 2\eta S_\lambda(\xi) \, \mathrm{d}\mu_{t,x}(\xi,\eta) - 2\overline{S}S_\lambda\big(\overline{R}\big) \\
&\qquad\qquad + \iint_{\mathbb{R}^2} -T_\lambda(\xi)\eta^2 \, \mathrm{d}\mu_{t,x}(\xi,\eta) + T_\lambda\big(\overline{R}\big)\overline{S^2} + T_\lambda\big(\overline{R}\big)\big(\overline{R^2} - \overline{R}^2\big)\bigg\}.
\end{aligned} \tag{3.5.75}$$

Trivially, one has

$$T_\lambda\big(\overline{R}\big)\big(\overline{R^2} - \overline{R}^2\big) \leqslant 0 \tag{3.5.76}$$

since $T_\lambda(\overline{R}) \leqslant 0$ and $\overline{R^2} - \overline{R}^2 \geqslant 0$. Next, by Lemma 3.5.2, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \eta S_\lambda(\xi) \, \mathrm{d}\mu_{t,x}(\xi,\eta) - \overline{S}S_\lambda\big(\overline{R}\big) = \overline{S}\big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big) \leqslant 0, \\
&\iint_{\mathbb{R}^2} -T_\lambda(\xi)\eta^2 \, \mathrm{d}\mu_{t,x}(\xi,\eta) + T_\lambda\big(\overline{R}\big)\overline{S^2} = -\big(\overline{T_\lambda(R)} - T_\lambda\big(\overline{R}\big)\big)\overline{S^2} \leqslant 0,
\end{aligned} \tag{3.5.77}$$

due to the fact that $S_\lambda(\xi)$ is a convex function of ξ, $\overline{S} \leqslant 0$, $T_\lambda(\xi)$ is convex in $(-\infty, 0)$ and $R_\varepsilon \leqslant 0$ (by Lemma 3.3.1). Further, we observe from the explicit structures of S_λ and T_λ

that

$$
\begin{aligned}
&-2\xi S_\lambda(\xi) + T_\lambda(\xi)\xi^2 = \lambda\xi(\xi+\lambda)|_{\xi\leqslant-\lambda},\\
&2\overline{R}S_\lambda\big(\overline{R}\big) - T_\lambda\big(\overline{R}\big)\overline{R}^2 = -\lambda\overline{R}\big(\overline{R}+\lambda\big)\big|_{\overline{R}\leqslant-\lambda} \leqslant 0.
\end{aligned}
\tag{3.5.78}
$$

We thus have from (3.5.75) to (3.5.78) that

$$
\begin{aligned}
&\partial_t\big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big) - \partial_x\big(c(u)\big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big)\big)\\
&\quad\leqslant \tilde{c}'(u)\int_{\mathbb{R}} \lambda\xi(\xi+\lambda)|_{\xi\leqslant-\lambda}\,\mathrm{d}\nu^1_{t,x}(\xi)\\
&\quad\leqslant \tilde{c}'(u)\int_{\mathbb{R}} |\xi|^3\big|_{\xi\leqslant-\lambda}\,\mathrm{d}\nu^1_{t,x}(\xi).
\end{aligned}
\tag{3.5.79}
$$

Step 4. Next we prove that

$$
\lim_{\lambda\to\infty}\sup_{t\geqslant0}\left\|\int_{\mathbb{R}}|\xi|^3\big|_{\xi\leqslant-\lambda}\,\mathrm{d}\nu^1_{t,x}(\xi)\right\|_{L^1(\mathbb{R})} = 0 \quad \text{provided } p>3. \tag{3.5.80}
$$

In fact,

$$
\left\|\int_{\mathbb{R}}|\xi|^3\big|_{\xi\leqslant-\lambda}\,\mathrm{d}\nu^1_{t,x}(\xi)\right\|_{L^1(\mathbb{R})} \leqslant \frac{1}{\lambda^{p-3}}\sup_{\varepsilon>0}\int_{\mathbb{R}}|R_\varepsilon|^p\mathbb{1}_{R_\varepsilon\leqslant-\lambda/2}(t,x)\,\mathrm{d}x. \tag{3.5.81}
$$

Thus, Lemma 3.3.2 and (3.5.81) implies (3.5.80).

Step 5. So, by Lemma 3.5.1 and integrating (3.5.79) over $\mathbb{R}$, we find

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(\overline{S_\lambda(\mathbb{R})} - S_\lambda\big(\overline{R}\big)\big)(t,x)\,\mathrm{d}x \leqslant \iint_{\mathbb{R}^2}\tilde{c}'(u)|\xi|^3\mathbb{1}_{\xi\leqslant-\lambda}\,\mathrm{d}\nu^1_{t,x}\,\mathrm{d}x. \tag{3.5.82}
$$

Hence by (3.5.80), Lebesgue dominated convergence theorem, and taking $\lambda\to\infty$ in (3.5.82), we find

$$
\int_{\mathbb{R}}\big(\overline{R^2} - \overline{R}^2\big)(t,x)\,\mathrm{d}x = 0.
$$

This implies that $\nu^1_{t,x}(\xi) = \delta_{\overline{R}}(\xi)$. Exactly as the above procedure, we can prove that $\nu^2_{t,x}(\eta) = \delta_{\overline{S}}(\eta)$. This completes the proof of Lemma 3.5.3. □

REMARK 3.5.4. We remark that the cut-off step of T_λ in the proof of Lemma 3.5.3 is not needed for $p>3$, although we choose to include it both to show that the condition $p>3$ is used only in one essential step and to use it for the proof of Theorem 3.1.6.

PROOF OF THEOREM 3.1.5. Now, let $R_0, S_0 \leqslant 0$, $(R_0, S_0) \in L^p(\mathbb{R})$ for $p > 3$ and $\operatorname{supp}(R_0, S_0) \subset [-a, a]$. We have two cases.

(a) *Data mollification approximation*. We mollify (R_0, S_0) to $(R_0^\varepsilon, S_0^\varepsilon)$ with the standard Friedrichs' mollifier. By Lemmas 3.4.1–3.4.3 and the proofs of Theorems 3.1.2 and 3.1.3, we find that (3.4.39) has a global smooth solution $(R_\varepsilon, S_\varepsilon)$ which is uniformly bounded in $L^\infty(\mathbb{R}^+, L^p(\mathbb{R}))$ with $\operatorname{supp}(R_\varepsilon, S_\varepsilon) \subset (-a - C_2 t, a + C_2 t)$. Thus, by following the proof of Lemma 3.3.1 and that of Lemmas 3.5.1–3.5.3, we can find a subsequence of $\{R_\varepsilon, S_\varepsilon\}$ (for simplicity, we still denote them by $\{R_\varepsilon, S_\varepsilon\}$), and two functions $(R, S) \in L^\infty(\mathbb{R}^+, L^p(\mathbb{R}))$, such that $(R_\varepsilon, S_\varepsilon) \to (R, S)$ in $L^q(\mathbb{R}^+, L^r(\mathbb{R}))$ for any $q < \infty, r < p$. Thus

$$\operatorname{supp}\big(R(t,\cdot), S(t,\cdot)\big) \subset [-a - C_2 t - 1, a + C_2 t + 1]. \tag{3.5.83}$$

Then repeating the support argument in the proofs of Theorems 3.1.2 and 3.1.3, we can prove that (3.4.53) holds.

(b) *Equation regularization approximation*. We need only to show that the vanishing viscosity limit has compact support. Let $\varphi(y)$ be a monotone decreasing smooth function so that it is equal to 1 for $y < -1$ and equal to 0 for $y > 0$. We multiply the energy density equation

$$\partial_t\big(R^2 + S^2\big) + \partial_x\big[c\big(S^2 - R^2\big)\big] = 2\varepsilon\big(R\,\partial_x^2 R + S\,\partial_x^2 S\big)$$

with the test function $\varphi(x + C_2 t + a)$, where the superscript ε is omitted for simplicity, to yield

$$\begin{aligned}
&\partial_t\big[\big(R^2 + S^2\big)\varphi\big] + \partial_x\big[\big(S^2 - R^2\big)c\varphi\big] \\
&\quad = 2\varepsilon\big(R\,\partial_x^2 R + S\,\partial_x^2 S\big)\varphi + \big[C_2\big(R^2 + S^2\big) + \big(S^2 - R^2\big)c\big]\varphi'.
\end{aligned}$$

Integrating it over $[0, t] \times \mathbb{R}$ and using the support property of the initial data and $\varphi' \leqslant 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \big(R^2 + S^2\big)\varphi\,\mathrm{d}x + 2\varepsilon\int_0^t\int_{\mathbb{R}} \varphi\big((\partial_x R)^2 + (\partial_x S)^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \varepsilon\int_0^t\int_{\mathbb{R}} \varphi''\big(R^2 + S^2\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Sending $\varepsilon \to 0$, we find that (R, S) is supported on $[-C_2 t - a - 1, \infty)$ at each $t \geqslant 0$. Similarly we can prove that (R, S) is supported on $(-\infty, C_2 t + a + 1]$. Thus (R, S) is supported on $[-C_2 t - a - 1, C_2 t + a + 1]$.

Thus by summing up the first two equations of (3.4.39) and using (3.4.53), we find that $u(t, x)$ satisfies (3.1.3). Moreover, our solution u is finite in any finite time interval $[0, T]$ from the compact support property, the third and fourth equations of (3.2.12), and the energy bound. Then from condition (3.1.10), Lemma 3.3.2, and convexity of L^{p+1}, we have $\partial_x u \in L^{p+1}([0, T] \times \mathbb{R})$ for any $T > 0$ provided that (3.1.10) holds. This completes the proof Theorem 3.1.5. $\square$

3.6. *Square integrable data*

We prove Theorem 3.1.6 in this subsection. First we establish a lemma.

LEMMA 3.6.1 (Local space–time higher integrability estimate). *Let* $(R_0, S_0) \in L^2 \cap L^1(\mathbb{R})$. *Let* c *satisfies* (3.1.8). *Let* $\alpha = d_2/d_1 \in (0,1)$ *with* d_2 *an even positive integer and* d_1 *an odd positive integer. Let* $a < b$ *and* $\chi(\cdot) \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}\chi \subset (a,b)$. *Then for the viscous solution* $(R_\varepsilon, S_\varepsilon, u_\varepsilon)$ *to* (3.2.12), *we have*

$$\begin{aligned}\bigg|\int_0^T\int_a^b & \chi(x)\tilde{c}'(u_\varepsilon)\\ &\times\left[(1-\alpha)\left(R_\varepsilon^{1+\alpha}-S_\varepsilon^{1+\alpha}\right)+(1+\alpha)R_\varepsilon^\alpha S_\varepsilon^\alpha\left(R_\varepsilon^{1-\alpha}-S_\varepsilon^{1-\alpha}\right)\right]\\ &\times(R_\varepsilon-S_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\bigg|\leqslant C_{T,a,b},\end{aligned}\tag{3.6.84}$$

where $C_{T,a,b}$ *depends only on* $(T, a, b, \chi, \|R_0\|_{L^2}, \|S_0\|_{L^2})$ *(but not* ε *or* α*).*

PROOF. Take an odd function $f(\xi) \in C^\infty(\mathbb{R})$, $f(\xi) = 0$ in $[-\frac{1}{2}, \frac{1}{2}]$, with

$$f'(\xi) = \begin{cases} \xi^\alpha, & |\xi| \geqslant 1,\\ 0, & |\xi| \leqslant \frac{1}{2}.\end{cases}$$

Then

$$f(\xi) = \int_0^\xi f'(\zeta)\,\mathrm{d}\zeta = \begin{cases} \frac{1}{1+\alpha}\xi^{1+\alpha} + C_1 & \text{if } |\xi| \geqslant 1,\\ 0 & \text{if } |\xi| \leqslant \frac{1}{2}.\end{cases}\tag{3.6.85}$$

Now, we multiply $f'(R_\varepsilon)\chi(x)$ to the first equation of (3.2.12) to obtain

$$\partial_t\big(f(R_\varepsilon)\chi\big) - c\chi\,\partial_x f(R_\varepsilon) = \tilde{c}'\big(R_\varepsilon^2 - S_\varepsilon^2\big)\chi f'(R_\varepsilon) + \varepsilon\chi f'(R_\varepsilon)\,\partial_{xx}R_\varepsilon,$$

and integrate the above equation over $[0, T] \times [a, b]$ to find

$$\begin{aligned}&\int_a^b \big(f(R_\varepsilon)\chi\big)(T,\cdot) - \big(f(R_0)\chi\big)\,\mathrm{d}x\\ &\quad + \int_0^T\int_a^b \big[2\tilde{c}'(R_\varepsilon - S_\varepsilon)\chi f(R_\varepsilon) + c\chi' f(R_\varepsilon)\big]\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_a^b \tilde{c}'\big(R_\varepsilon^2 - S_\varepsilon^2\big)\chi f'(R_\varepsilon) - \varepsilon\,\partial_x\big(\chi f'(R_\varepsilon)\big)\,\partial_x R_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.6.86}$$

Then, by (3.6.85) and a trivial rearrangement of (3.6.86), we obtain

$$\begin{aligned}
&\int_0^T\int_a^b \left(2\tilde{c}'(R_\varepsilon - S_\varepsilon)\chi\left(\frac{1}{1+\alpha}R_\varepsilon^{1+\alpha} + C_1\right) - \tilde{c}'\big(R_\varepsilon^2 - S_\varepsilon^2\big)\chi R_\varepsilon^\alpha\right)\mathbb{1}_{|R_\varepsilon|\geqslant 1}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T\int_a^b \big(\tilde{c}'\big(R_\varepsilon^2 - S_\varepsilon^2\big)\chi f'(R_\varepsilon) - 2\tilde{c}'(R_\varepsilon - S_\varepsilon)\chi f(R_\varepsilon)\big)\mathbb{1}_{|R_\varepsilon|<1}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \varepsilon\int_0^T\int_a^b \big(\chi f''(R_\varepsilon)(\partial_x R_\varepsilon)^2 + \chi' f'(R_\varepsilon)\,\partial_x R_\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_a^b \big(f(R_0)\chi\big)\,\mathrm{d}x - \int_a^b \big(f(R_\varepsilon)\chi\big)(T,\cdot)\,\mathrm{d}x - \int_0^T\int_a^b c\chi' f(R_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.6.87}$$

Using the energy (3.2.14), we see that the absolute value of the left-hand side of (3.6.87) is bounded. Again by the energy estimate, we see that the restriction to $|R_\varepsilon| \geqslant 1$ on the right-hand side can be removed. So

$$\left|\int_0^T\int_a^b \chi\tilde{c}'\left((R_\varepsilon - S_\varepsilon)\frac{1}{1+\alpha}R_\varepsilon^{1+\alpha} - \frac{1}{2}\big(R_\varepsilon^2 - S_\varepsilon^2\big)R_\varepsilon^\alpha\right)\mathrm{d}x\,\mathrm{d}t\right| \leqslant C_{T,a,b}. \tag{3.6.88}$$

Exactly as the proof of (3.6.88), we find

$$\left|\int_0^T\int_a^b \chi\tilde{c}'\left(-(R_\varepsilon - S_\varepsilon)\frac{1}{1+\alpha}S_\varepsilon^{1+\alpha} - \frac{1}{2}\big(S_\varepsilon^2 - R_\varepsilon^2\big)S_\varepsilon^\alpha\right)\mathrm{d}x\,\mathrm{d}t\right| \leqslant C_{T,a,b}. \tag{3.6.89}$$

By summing (3.6.88) and (3.6.89), we obtain

$$\begin{aligned}
&\Bigg|\int_0^T\int_a^b \tilde{c}'\chi\Bigg(\frac{1}{1+\alpha}\big(R_\varepsilon^{1+\alpha} - S_\varepsilon^{1+\alpha}\big)(R_\varepsilon - S_\varepsilon) \\
&\qquad\qquad - \frac{1}{2}\big(R_\varepsilon^2 - S_\varepsilon^2\big)\big(R_\varepsilon^\alpha - S_\varepsilon^\alpha\big)\Bigg)\mathrm{d}x\,\mathrm{d}t\Bigg| \leqslant C_{T,a,b}.
\end{aligned} \tag{3.6.90}$$

While a very simple calculation shows that

$$\begin{aligned}
&\frac{1}{1+\alpha}\big(R_\varepsilon^{1+\alpha} - S_\varepsilon^{1+\alpha}\big)(R_\varepsilon - S_\varepsilon) - \frac{1}{2}\big(R_\varepsilon^2 - S_\varepsilon^2\big)\big(R_\varepsilon^\alpha - S_\varepsilon^\alpha\big) \\
&\quad = \left(\frac{1}{1+\alpha} - \frac{1}{2}\right)(R_\varepsilon - S_\varepsilon)\big(R_\varepsilon^{1+\alpha} - S_\varepsilon^{1+\alpha}\big) \\
&\qquad + \frac{1}{2}R_\varepsilon^\alpha S_\varepsilon^\alpha(R_\varepsilon - S_\varepsilon)\big(R_\varepsilon^{1-\alpha} - S_\varepsilon^{1-\alpha}\big).
\end{aligned} \tag{3.6.91}$$

Thus, we obtain

$$\left|\int_0^T\int_a^b \chi\tilde{c}'(R_\varepsilon - S_\varepsilon)\left[\left(\frac{1}{1+\alpha}-\frac{1}{2}\right)\left(R_\varepsilon^{1+\alpha}-S_\varepsilon^{1+\alpha}\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}R_\varepsilon^\alpha S_\varepsilon^\alpha\left(R_\varepsilon^{1-\alpha}-S_\varepsilon^{1-\alpha}\right)\right]\mathrm{d}x\,\mathrm{d}t\right| \leqslant C_{T,a,b}. \tag{3.6.92}$$

Multiplying (3.6.92) by $2(1+\alpha)$, we obtain (3.6.84). This completes the proof of (3.6.84). □

From Lemma 3.6.1 and its proof, we have the corollary.

COROLLARY 3.6.2 ($L^{2+\alpha}$ estimate). *Let* $c'(\cdot) \geqslant 0$. *Let* $(R_0, S_0) \in L^2(\mathbb{R})$ *with compact support. Then for* $0<\alpha<1$, *there holds the estimate*

$$\int_0^T\int_a^b c'(u_\varepsilon)|\partial_x u_\varepsilon|^{2+\alpha}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,T,a,b}, \tag{3.6.93}$$

where the constant $C_{\alpha,T,a,b}\to\infty$ *as* $\alpha\to 1$.

PROOF. First take $\alpha = d_2/d_1$ with d_2 an even natural number and d_1 an odd natural number. We use (3.6.84) on the interval $[a-1, b+1]$. Since each term in the integral (3.6.84) is nonnegative, we immediately obtain

$$\int_0^T\int_{a-1}^{b+1}\chi(1-\alpha)c'(u_\varepsilon)(R_\varepsilon - S_\varepsilon)^2\left(R_\varepsilon^\alpha + S_\varepsilon^\alpha\right)\mathrm{d}x\,\mathrm{d}t \leqslant C_{T,a,b}. \tag{3.6.94}$$

By the third equation of (3.2.12), (3.6.94) and choosing $\chi = 1$ on $[a,b]$, we immediately obtain (3.6.93). For any other $\alpha\in(0,1)$, we can use interpolation. This completes the proof of the corollary. □

Now, let $R_0\leqslant 0$, $S_0\leqslant 0$, $(R_0,S_0)\in L^2(\mathbb{R})$ with $\mathrm{supp}(R_0,S_0)\subset(-a,a)$. Let $c'\geqslant 0$. We mollify (R_0,S_0) to $(R_0^\varepsilon, S_0^\varepsilon)$ with the standard Friedrichs' mollifier. Then by Theorem 3.1.2 and its proof on the compactness of the support of solutions, we find that (3.1.5) has a global smooth solution $(R_\varepsilon, S_\varepsilon)$ which is uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ with $\mathrm{supp}(R_\varepsilon, S_\varepsilon)\subset(-a-C_2t, a+C_2t)$, $R_\varepsilon\leqslant 0, S_\varepsilon\leqslant 0$, and satisfies the estimates in Lemma 3.6.1 and Corollary 3.6.2. By Lemmas 3.5.1 and 3.5.2, we find, similar as before, that

$$\mu_{t,x}(\xi,\eta) = \nu_{t,x}^1(\xi)\otimes\nu_{t,x}^2(\eta), \tag{3.6.95}$$

where $\nu_{t,x}^1(\xi)$, $\nu_{t,x}^2(\eta)$ and $\mu_{t,x}(\xi,\eta)$ are the Young measures associated with $\{R_\varepsilon\}$, $\{S_\varepsilon\}$ and $\{R_\varepsilon, S_\varepsilon\}$, respectively. We establish short-time decay on R and S to control the quadratic terms.

LEMMA 3.6.3 (Local decay). *Assume that c has strict monotonicity* (3.1.10). *Let $R_0 \leqslant 0$, $S_0 \leqslant 0$, $(R_0, S_0) \in L^2(\mathbb{R})$ with compact support. For the smooth solutions $(R_\varepsilon, S_\varepsilon)$ with mollified (R_0, S_0) and any $T > 0$, there exist a C and M_0 (both independent of $\varepsilon \in (0,1]$) such that for all $M \geqslant M_0$ there hold*

$$-2M \leqslant R_\varepsilon(t,x) \leqslant 0, \qquad -2M \leqslant S_\varepsilon(t,x) \leqslant 0, \quad t \in \left[\frac{C}{M}, T\right]. \tag{3.6.96}$$

PROOF. For simplicity, we omit the subscript ε in the proof of (3.6.96).

Step 1. Taking any $b \in \mathbb{R}$, we introduce the plus and minus characteristics $\Phi_t^{\pm}(b)$ as

$$\begin{cases} \frac{\mathrm{d}\Phi_t^{\pm}}{\mathrm{d}t} = \pm c\big(u\big(t, \Phi_t^{\pm}\big)\big), \\ \Phi_t^{\pm}\big|_{t=0} = b. \end{cases} \tag{3.6.97}$$

For the characteristic curve $x = \Phi_t^-(b)$, $t > 0$, we find by (3.1.8) that there exists the inverse function $t = t^-(x)$ defined for all $x < b$. Similarly, we have the inverse function $t = t^+(y)$ defined for all $y > d$ ($d \in \mathbb{R}$) for the plus characteristic $x = \Phi_t^+(d)$, $t > 0$. By the proof on p. 56 of [29], we have the energy conservation in a characteristic cone

$$\begin{aligned} &\int_d^{\Phi_t^+(d)} R^2\big(t^+(y), y\big)\,\mathrm{d}y + \int_{\Phi_t^+(d)}^b S^2\big(t^-(y), y\big)\,\mathrm{d}y \\ &\quad = \frac{1}{2}\int_d^b \big(R_0^2(x) + S_0^2(x)\big)\,\mathrm{d}x, \end{aligned} \tag{3.6.98}$$

where $d < b$ and $t > 0$ are such that $\Phi_t^+(d) = \Phi_t^-(b)$. In particular, (3.6.97) and (3.6.98) imply that

$$\int_0^t c\big(u\big(s, \Phi_s^-(b)\big)\big) S^2\big(s, \Phi_s^-(b)\big)\,\mathrm{d}s \leqslant \|R_0\|_{L^2}^2 + \|S_0\|_{L^2}^2 \quad \forall b \in \mathbb{R}. \tag{3.6.99}$$

Step 2. On the other hand, by (3.1.5), (3.6.97) and the fact that $R_\varepsilon \leqslant 0$, we find

$$\frac{\mathrm{d}|R(t, \Phi_t^-)|}{\mathrm{d}t} = -\tilde{c}'(u) R^2\big(t, \Phi_t^-\big) + \tilde{c}'(u) S^2\big(t, \Phi_t^-\big), \tag{3.6.100}$$

where Φ_t^- is the abbreviation of $\Phi_t^-(b)$. Our idea is to use the bound (3.6.99) of S^2 and utilize the negative quadratic term $-\tilde{c}'(u)R^2$ to show decay of $|R|$. We already know for fixed $T > 0$ that u_ε has a uniform bound for $\varepsilon \in (0,1]$ and $t \in [0,T]$. We choose C_0 to be the minimum of $\tilde{c}'(u_\varepsilon(x,t))$ in $\varepsilon \in (0,1]$, $t \in [0,T]$ and $x \in \mathbb{R}$. Then

$$\frac{\mathrm{d}|R(t, \Phi_t^-)|}{\mathrm{d}t} \leqslant -C_0 R^2\big(t, \Phi_t^-\big) + \tilde{c}'(u) S^2\big(t, \Phi_t^-\big). \tag{3.6.101}$$

Thus, by (3.6.99) and (3.6.101), for any $t_2 > t_1$, we have

$$\begin{aligned} \left|R\left(t_2, \Phi_{t_2}^-\right)\right| &\leqslant \left|R\left(t_1, \Phi_{t_1}^-\right)\right| + \int_{t_1}^{t_2} \left(\tilde{c}'(u)S^2\right)\left(s, \Phi_s^-\right) \mathrm{d}s \\ &\leqslant \left|R\left(t_1, \Phi_{t_1}^-\right)\right| + C', \end{aligned} \tag{3.6.102}$$

where C' depends on the maximum of $\tilde{c}'(\cdot)$, the minimum of $c(\cdot)$ and the total initial energy, but independent of (ε, b, t). We also choose $C' \geqslant 1$ for later convenience. Now, we take $M_0 = C' + 4$ and any $M \geqslant M_0$. The first case is $|R_0(b)| \leqslant M$, then by directly applying (3.6.102), we find

$$\left|R\left(t, \Phi_t^-\right)\right| \leqslant M + C' \leqslant 2M \quad \forall t \in \mathbb{R}^+. \tag{3.6.103}$$

Step 3. The second case is when $|R_0(b)| \geqslant M$. We have two subcases here. First subcase is when

$$\left|R\left(t, \Phi_t^-\right)\right| \geqslant \frac{M}{2} \quad \text{for all } t \in \left[0, \frac{C}{M}\right], \tag{3.6.104}$$

where $C := 4C'/C_0$. Then by (3.6.101) again, we find

$$\begin{aligned} \frac{1}{|R(t, \Phi_t^-)|} &\geqslant \frac{1}{|R_0(b)|} + C_0 t - \int_0^t \left(\tilde{c}'(u)\frac{S^2}{R^2}\right)\left(s, \Phi_s^-\right) \mathrm{d}s \\ &\geqslant C_0 t - \frac{4C'}{M^2}, \quad t \in \left[0, \frac{C}{M}\right]. \end{aligned} \tag{3.6.105}$$

Let $t_0 := C/M$. We find by (3.6.105) that

$$\left|R\left(t_0, \Phi_{t_0}^-\right)\right| \leqslant \frac{M}{3C'} \leqslant M, \tag{3.6.106}$$

since $M > 4$ and $C' \geqslant 1$. Hence, by summing up (3.6.102) and (3.6.106), we find that

$$\left|R\left(t, \Phi_t^-\right)\right| \leqslant M + C' \leqslant 2M, \quad \text{if } t \geqslant t_0 \tag{3.6.107}$$

since $M > C'$. The second subcase is when there exists some $t' \in [0, \frac{C}{M}]$ such that $|R(t', \Phi_{t'}^-)| < M/2$. Then, by (3.6.102), we have

$$\left|R\left(t, \Phi_t^-\right)\right| \leqslant \frac{M}{2} + C' \leqslant 2M \quad \text{if } t \geqslant t'. \tag{3.6.108}$$

By summing up (3.6.107), (3.6.108) and (3.6.103), we have

$$\left|R(t, y)\right| \leqslant 2M \quad \text{for all } t \in \left[\frac{C}{M}, T\right]. \tag{3.6.109}$$

We can estimate $|S|$ similarly. The proof of Lemma 3.6.3 is complete. $\square$

We now prove that the Young measures are Dirac measures.

LEMMA 3.6.4. *Suppose* (3.1.10) *hold and* $R_0 \leqslant 0$, $S_0 \leqslant 0$, $(R_0, S_0) \in L^2(\mathbb{R})$ *with compact support. Then the approximate solutions through initial data mollification have*

$$\nu^1_{t,x}(\xi) = \delta_{\overline{R}}(\xi), \qquad \nu^2_{t,x}(\eta) = \delta_{\overline{S}}(\eta).$$

PROOF.

Step 1. By the proof of Lemma 3.5.3 up to the first inequality of (3.5.79), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big)(t,x)\,\mathrm{d}x \leqslant C\lambda \iint_{\mathbb{R}^2} \xi^2\big|_{\xi \leqslant -\lambda}\,\mathrm{d}\nu^1_{t,x}\,\mathrm{d}x, \tag{3.6.110}$$

provided that we can prove that

$$\begin{aligned} &(R_\varepsilon - S_\varepsilon) S_\lambda(R_\varepsilon) \rightharpoonup \iint (\xi - \eta) S_\lambda(\xi)\,\mathrm{d}\mu_{t,x}(\xi,\eta),\\ &T_\lambda(R_\varepsilon)\big(R_\varepsilon^2 - S_\varepsilon^2\big) \rightharpoonup \iint T_\lambda(\xi)\big(\xi^2 - \eta^2\big)\,\mathrm{d}\mu_{t,x}(\xi,\eta),\\ &(R_\varepsilon - S_\varepsilon)^2 \rightharpoonup \iint (\xi - \eta)^2\,\mathrm{d}\mu_{t,x}(\xi,\eta) \end{aligned} \tag{3.6.111}$$

in the sense of distributions. But, by Corollary 3.6.2, we find that $\{R_\varepsilon - S_\varepsilon\}$ is uniformly bounded in $L^{2+\alpha}([0,T] \times \mathbb{R})$ for any $T < \infty$ and $\alpha < 1$ due to the fact that $\operatorname{supp}(R_\varepsilon, S_\varepsilon) \subset (-a - C_2 t, a + C_2 t)$. Thus, by Lemma 1.3.1 and a diagonal process for the time T, we can always find a subsequence of $\{R_\varepsilon, S_\varepsilon\}$ such that (3.6.111) holds.

Step 2. Now, fixing a $T > 0$ for Lemma 3.6.3, noticing that $\operatorname{supp} \nu^1_{t,x}(\cdot) \subset [-2M, 0]$ if $t \geqslant C/M$, by (3.6.96) for any large M, we have

$$\int_{C/M}^t \iint_{\mathbb{R}^2} \xi^2\big|_{\xi \leqslant -\lambda}\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \lambda \geqslant 2M. \tag{3.6.112}$$

Thus, for $\lambda \geqslant 2M$ and by (3.6.110), we have

$$\begin{aligned} &\int \big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big)(t,x)\,\mathrm{d}x\\ &\quad\leqslant \int \big(\overline{S_\lambda(R)} - S_\lambda\big(\overline{R}\big)\big)\left(\frac{C}{M}, x\right)\mathrm{d}x, \quad t \geqslant \frac{C}{M}. \end{aligned} \tag{3.6.113}$$

Thus again by (1.3.12), (3.6.113) and Lebesgue dominated convergence theorem, we find by passing $\lambda \to \infty$ in (3.6.113) that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \xi^2 \,\mathrm{d}\nu^1_{t,x}(\xi) - \overline{R}^2 \right)(t,x)\,\mathrm{d}x \\
&\quad \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \xi^2 \,\mathrm{d}\nu^1_{t,x}(\xi) - \overline{R}^2 \right)\left(\frac{C}{M}, x \right)\mathrm{d}x, \quad t \geqslant \frac{C}{M}.
\end{aligned} \tag{3.6.114}$$

Exactly as the proof of (3.6.114), we can prove that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \eta^2 \,\mathrm{d}\nu^2_{t,x}(\eta) - \overline{S}^2 \right)(t,x)\,\mathrm{d}x \\
&\quad \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \eta^2 \,\mathrm{d}\nu^2_{t,x}(\eta) - \overline{S}^2 \right)\left(\frac{C}{M}, x \right)\mathrm{d}x, \quad t \geqslant \frac{C}{M}.
\end{aligned} \tag{3.6.115}$$

Step 3. Next, we claim that

$$\lim_{t\to 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \xi^2 \,\mathrm{d}\nu^1_{t,x} + \int_{\mathbb{R}} \eta^2 \,\mathrm{d}\nu^2_{t,x} \right)\mathrm{d}x = \int_{\mathbb{R}} \left(R_0^2 + S_0^2 \right)\mathrm{d}x. \tag{3.6.116}$$

In fact, by the convexity properties of $S_\lambda(\cdot)$, Lemma 3.5.1, and the compact support and energy bound properties of $(R_\varepsilon, S_\varepsilon)$, we find that

$$\begin{aligned}
&\int \left(S_\lambda(\overline{R}) + S_\lambda(\overline{S}) \right)(t,x)\,\mathrm{d}x \\
&\quad \leqslant \int \left[\int S_\lambda(\xi)\,\mathrm{d}\nu^1_{t,x}(\xi) + \int S_\lambda(\eta)\,\mathrm{d}\nu^2_{t,x}(\eta) \right]\mathrm{d}x \\
&\quad = \lim_{\varepsilon\to 0} \int \left(S_\lambda(R_\varepsilon) + S_\lambda(S_\varepsilon) \right)(t,x)\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2} \int_{\mathbb{R}} \left(R_0^2 + S_0^2 \right)\mathrm{d}x.
\end{aligned} \tag{3.6.117}$$

Thus, by (1.3.12) and Lebesgue dominated convergence theorem, we find by tending $\lambda \to \infty$ in (3.6.117) that

$$\begin{aligned}
\int \left(\overline{R}^2 + \overline{S}^2 \right)(t,x)\,\mathrm{d}x &\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \xi^2 \,\mathrm{d}\nu^1_{t,x} + \int_{\mathbb{R}} \eta^2 \,\mathrm{d}\nu^2_{t,x} \right)\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}} \left(R_0^2 + S_0^2 \right)\mathrm{d}x.
\end{aligned} \tag{3.6.118}$$

While by (3.5.69) satisfied by $\overline{R}$ and a similar equation for $\overline{S}$ and Lemma C.1 of [48], p. 177, we find that

$$\left(\overline{R}(t,x), \overline{S}(t,x) \right) \rightharpoonup \left(R_0(x), S_0(x) \right) \quad \text{weakly in } L^2(\mathbb{R}) \text{ as } t \to 0. \tag{3.6.119}$$

Hence by (3.6.118) and (3.6.119), we have

$$\lim_{t\to 0}\int \left(\overline{R}^2+\overline{S}^2\right)(t,x)\,\mathrm{d}x=\int_{\mathbb{R}}\left(R_0^2+S_0^2\right)\mathrm{d}x. \tag{3.6.120}$$

Summing up (3.6.118) and (3.6.120), we prove (3.6.116). Incidentally, by (3.6.119), (3.6.120) and Theorem 1 on p. 4 of [23], we have

$$\left(\overline{R}(t,x),\overline{S}(t,x)\right)\to\left(R_0(x),S_0(x)\right)\quad\text{strongly in } L^2(\mathbb{R}) \text{ as } t\to 0. \tag{3.6.121}$$

Step 4. Let us take $M\to\infty$. Summing up (3.6.114)–(3.6.116), we find

$$\int_{\mathbb{R}}\left\{\left(\int_{\mathbb{R}}\xi^2\,\mathrm{d}\nu_{t,x}^1(\xi)-\overline{R}^2\right)+\left(\int_{\mathbb{R}}\eta^2\,\mathrm{d}\nu_{t,x}^1(\eta)-\overline{S}^2\right)\right\}(t,x)\,\mathrm{d}x\leqslant 0,\quad t>0.$$

That is,

$$\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\left(\xi-\overline{R}\right)^2\mathrm{d}\nu_{t,x}^1(\xi)+\int_{\mathbb{R}}\left(\eta-\overline{S}\right)^2\mathrm{d}\nu_{t,x}^2(\eta)\right\}\mathrm{d}x=0. \tag{3.6.122}$$

Thus, $\nu_{t,x}^1(\xi)=\delta_{\bar{R}(t,x)}(\xi)$ and $\nu_{t,x}^2(\eta)=\delta_{\bar{S}(t,x)}(\eta)$. The proof of Lemma 3.6.4 is complete. $\square$

PROOF OF THEOREM 3.1.6. By Lemma 3.6.4, we have

$$R_\varepsilon-S_\varepsilon\to\overline{R}-\overline{S}\quad\text{in } L^1_{\mathrm{loc}}\left(\mathbb{R}^+\times\mathbb{R}\right). \tag{3.6.123}$$

But since by Corollary 3.6.2, $\{\partial_x u_\varepsilon\}$ is uniformly bounded in $L^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$, we in fact prove that

$$R_\varepsilon-S_\varepsilon\to\overline{R}-\overline{S}\quad\text{in } L^{2+\alpha}_{\mathrm{loc}}\left(\mathbb{R}^+\times\mathbb{R}\right)\quad\forall\alpha<1. \tag{3.6.124}$$

Thus, $(\overline{R},\overline{S})$ is a weak solution of (3.1.5) and $\partial_x u(t,x)\in L^{2+\alpha}([0,T]\times\mathbb{R})$ for any $T<\infty$ and $\alpha<1$, due to the fact that both $\overline{R}(t,\cdot)$ and $\overline{S}(t,\cdot)$ have compact supports for any $t\leqslant T$. To establish the local decay (3.1.11), we let $T=2$ in Lemma 3.6.3 and use (3.6.96) at $t=C/M$ for $t\in(0,\frac{C}{M_0}]$. The boundedness for $t\geqslant 1$ in (3.1.11) follows from (3.6.102). The remaining of the proof of Theorem 3.1.6 is exactly as that of Theorem 3.1.5, which we omit here. This completes the proof of Theorem 3.1.6. $\square$

4. Weak solutions to (0.0.1)

4.1. *Introduction*

In this section we study the existence and regularity properties of weak solutions to (0.0.1), when $c(\cdot)$ is a given smooth, bounded, and positive function, $u_0(x)\in\mathrm{Lip}(\mathbb{R})$, and $u_1(x)\in L^\infty(\mathbb{R})$.

In the sequel, we always assume that there exist two positive constants C_1, C_2 such that

$$0 < C_1 \leqslant c(\cdot) \leqslant C_2 \quad \text{and} \quad \left|c^{(l)}(\cdot)\right| \leqslant M_l, \quad l \geqslant 1, \tag{4.1.1}$$

for some positive constants M_l.

THEOREM 4.1.1 (Global weak solutions). *Assume $c' \geqslant 0$ and $(R_0, S_0) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Then* (0.0.1) *has a global weak solution u in the sense of Definition* 3.1.1. *Moreover, there hold $(R, S) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ and $c'(u)|\partial_x u|^{2+\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ for any $\alpha \in (0, 1)$. Furthermore, the characteristics exist*; *i.e., the following ordinary differential equations have global solutions $\Phi_t^\pm(x) \in C([0,\infty) \times \mathbb{R})$ with $\partial_x \Phi_t^\pm(x) \in L^\infty([0,\infty) \times \mathbb{R})$*:

$$\begin{cases} \frac{\mathrm{d}\Phi_t^\pm(x)}{\mathrm{d}t} = \pm c\big(u\big(t, \Phi_t^\pm(x)\big)\big), \\ \Phi_0^\pm(x) = x. \end{cases} \tag{4.1.2}$$

In particular, if $S_0 \leqslant 0$, then $S(t,x) \leqslant 0$, $S(t,x) \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$, and for any $T > 0$, there exist two positive constants $M_1(T)$ and $M_2(T)$ such that

$$M_1(T) \leqslant \partial_x \Phi_t^+(x) \leqslant M_2(T), \quad 0 \leqslant t \leqslant T. \tag{4.1.3}$$

In Theorem 4.1.1, if $R_0 \leqslant 0$ then $R(t,x) \leqslant 0$ for all time $t > 0$ and the minus characteristics satisfy (4.1.3). We also note that similar results hold if we assume $c'(\cdot) \leqslant 0$. Finally, it is possible to reduce the L^∞ condition on the initial data to L^p, $p > 3$, but we choose not to present the tedious details.

Compared to the previous results in the last section and [74], our result here allows for blow-up of solutions in both characteristic families. As for the method, we still use the Young measure approach [27,39,49]. The difficulty is that potential oscillations, in terms of DiPerna and Majda [20], get amplified unboundedly by quadratic growth terms of the equation. We use cut-off functions to overcome the amplifying. The main work is establishing uncommon estimates and handling the cut-off error terms through various newly discovered renormalizations.

We point out that we expect multiple weak solutions to problem (0.0.1). Our weak solutions in Theorem 4.1.1 are solutions of the dissipative type. We plan to explore the uniqueness issue in future work.

4.2. *Approximate solutions and uniform estimates*

Similar to [48] and [72,74], let us define for $\varepsilon > 0$

$$Q_\varepsilon(\xi) := \begin{cases} \frac{1}{\varepsilon}\big(\xi - \frac{1}{2\varepsilon}\big), & \xi \geqslant \frac{1}{\varepsilon}, \\ \frac{1}{2}\xi^2, & -\infty < \xi < \frac{1}{\varepsilon}. \end{cases} \tag{4.2.4}$$

Let us now define the approximate solution sequence by the equations

$$
\begin{cases}
\partial_t R^\varepsilon - c(u^\varepsilon)\,\partial_x R^\varepsilon = \tilde{c}'(u^\varepsilon)\big(2Q_\varepsilon(R^\varepsilon) - (S^\varepsilon)^2\big),\\
\partial_t S^\varepsilon + c(u^\varepsilon)\,\partial_x S^\varepsilon = \tilde{c}'(u^\varepsilon)\big(2Q_\varepsilon(S^\varepsilon) - (R^\varepsilon)^2\big),\\
\partial_x u^\varepsilon = \frac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)},\\
\lim_{x\to-\infty} u^\varepsilon(t,x) = 0,\\
(R^\varepsilon, S^\varepsilon)\big|_{t=0} = (R_0, S_0)(x).
\end{cases}
\tag{4.2.5}
$$

For convenience, we sometimes omit the superscript ε in the approximate solution sequence $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$.

LEMMA 4.2.1 (Solution of (4.2.5) with smooth data). *Let $(R_0, S_0)(x) \in C_c^\infty(\mathbb{R})$. Then problem* (4.2.5) *has a global smooth solution $(R, S)(t,x) \in L^\infty(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}))$, $u(t,x) \in L^\infty(\mathbb{R}^+, W^{3,\infty}(\mathbb{R}))$, which satisfies the energy inequalities*

$$
\int \big(R^2 + S^2\big)(t,x)\,\mathrm{d}x \leqslant \int \big(R_0^2 + S_0^2\big)(x)\,\mathrm{d}x \tag{4.2.6}
$$

and

$$
\int_0^\infty \int_{\mathbb{R}} c'(u^\varepsilon) G_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqslant \int \big(R_0^2 + S_0^2\big)(x)\,\mathrm{d}x, \tag{4.2.7}
$$

where

$$
G_\varepsilon := R\big(R^2 - 2Q_\varepsilon(R)\big) + S\big(S^2 - 2Q_\varepsilon(S)\big) \geqslant 0.
$$

Moreover, if we introduce the plus and minus characteristics $\Phi_t^\pm(b)$ as

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\pm = \pm c\big(u(t, \Phi_t^\pm)\big),\\
\Phi_t^\pm\big|_{t=0} = b,
\end{cases}
\tag{4.2.8}
$$

then we have the energy inequality in a characteristic cone

$$
\int_a^d R^2\big(t_a^+(y), y\big)\,\mathrm{d}y + \int_d^b S^2\big(t_b^-(y), y\big)\,\mathrm{d}y \leqslant \frac{1}{2}\int_a^b \big(R_0^2 + S_0^2\big)(x)\,\mathrm{d}x, \tag{4.2.9}
$$

where $a < b$ and d is where the two characteristics $\Phi_t^+(a)$ and $\Phi_t^-(b)$ meet at some positive time, and $t = t_a^+(y)$ is the inverse of $y = \Phi_t^+(a)$, etc. Besides, we have

$$
R(t,x) \geqslant -M, \qquad S(t,x) \geqslant -M, \quad t \geqslant 0, \tag{4.2.10}
$$

where $M = \|(R_0, S_0)\|_{L^\infty} + \frac{1}{8} M C_1^{-1} \|(R_0, S_0)\|_{L^2}^2$. Finally, there holds $S(t,x) \leqslant 0$ provided that $S_0(x) \leqslant 0$.

PROOF. It is standard to prove the local existence of smooth solutions to (4.2.5) with smooth initial data. Now, we let T^* be the lifespan of a smooth solution to (4.2.5). It can be proved exactly as that in the proof of Lemma 3.4.1 that $\|R(t,\cdot)\|_{L^\infty} + \|S(t,\cdot)\|_{L^\infty}$ controls T^*; that is, to say that $T^* < +\infty$ implies

$$\lim_{t\to T^*}\left(\left\|R(t,\cdot)\right\|_{L^\infty} + \left\|S(t,\cdot)\right\|_{L^\infty}\right) = +\infty. \tag{4.2.11}$$

Hence, in order to establish the global existence, it suffices to show that $\|R(t,\cdot)\|_{L^\infty} + \|S(t,\cdot)\|_{L^\infty} < +\infty$ for any $t < +\infty$.

By a simple comparison principle, it is trivial to observe that $S(t,x) \leqslant 0$ for $t < T^*$ if $S_0 \leqslant 0$.

Next, by multiplying $R(t,x)$ to the first equation of (4.2.5), we find

$$\partial_t R^2 - \partial_x\big(c(u)R^2\big) = 2\tilde{c}'(u)\big\{-R\big(R^2 - 2Q_\varepsilon(R)\big) + R^2 S - RS^2\big\}. \tag{4.2.12}$$

Similarly, we find

$$\partial_t S^2 + \partial_x\big(c(u)S^2\big) = 2\tilde{c}'(u)\big\{-S\big(S^2 - 2Q_\varepsilon(S)\big) + RS^2 - SR^2\big\}. \tag{4.2.13}$$

Adding (4.2.12) and (4.2.13), we find

$$\begin{aligned} &\partial_t\big(R^2 + S^2\big) - \partial_x\big(c(u)\big(R^2 - S^2\big)\big) \\ &\quad = -2\tilde{c}'(u)\big\{R\big(R^2 - 2Q_\varepsilon(R)\big) + S\big(S^2 - 2Q_\varepsilon(S)\big)\big\} \leqslant 0, \end{aligned} \tag{4.2.14}$$

as

$$\begin{aligned} R\big(R^2 - 2Q_\varepsilon(R)\big) &= R\left(R - \frac{1}{\varepsilon}\right)^2 \mathbb{1}_{R\geqslant 1/\varepsilon} \geqslant 0, \\ S\big(S^2 - 2Q_\varepsilon(S)\big) &= S\left(S - \frac{1}{\varepsilon}\right)^2 \mathbb{1}_{S\geqslant 1/\varepsilon} \geqslant 0. \end{aligned} \tag{4.2.15}$$

By integrating (4.2.14) over $\mathbb{R}$ with respect to x, we deduce (4.2.6) and (4.2.7). Integrating (4.2.14) over the characteristic cone $\Delta := \{(t,x) \mid \Phi_t^+(a) \leqslant x \leqslant \Phi_t^-(b), 0 \leqslant t < T^*\}$, we deduce (4.2.9). By the first equation of (4.2.5) and equation (4.2.9), we have

$$\begin{aligned} R\big(t, \Phi_t^-(x)\big) &\geqslant R_0(x) - \int_0^t \big(\tilde{c}'(u)S^2\big)\big(s, \Phi_s^-(x)\big)\,\mathrm{d}s \\ &\geqslant -\|R_0\|_{L^\infty} - \frac{M_1}{8C_1}\int \big(R_0^2 + S_0^2\big)(x)\,\mathrm{d}x. \end{aligned} \tag{4.2.16}$$

This proves the first inequality of (4.2.10). Similarly, we can prove the second one.

On the other hand, since both $Q_\varepsilon(\cdot)$ and $\partial_\xi Q_\varepsilon(\cdot)$ are Lipschitz continuous with respect to ξ for any fixed ε, we can easily prove an ε-dependent upper bound for $(R^\varepsilon, S^\varepsilon)$. This completes the proof of Lemma 4.2.1. $\square$

LEMMA 4.2.2 ($L^{2+\alpha}$ estimate). *Let $c'(\cdot) \geqslant 0$, $(R_0, S_0) \in L^\infty \cap L^2$, $\alpha \in (0,1)$, $T > 0$, $a < b$. Then for the solutions $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ of* (4.2.5) *there holds the estimate*

$$\int_0^T \int_a^b c'\big(u^\varepsilon\big)\big|\partial_x u^\varepsilon\big|^{2+\alpha}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,T,a,b}, \tag{4.2.17}$$

where the constant $C_{\alpha,T,a,b}$ depends only on the $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$ norms of (R_0, S_0) and the listed variables.

PROOF. We assume that $\alpha = d_2/d_1 \in (0,1)$, where d_2 is an even positive integer and d_1 is an odd positive integer. We then multiply $R^\alpha(t,x)$ to the first equation of (4.2.5) to yield

$$\begin{aligned} &\frac{1}{1+\alpha}\big\{\partial_t R^{1+\alpha} - \partial_x\big(c(u)R^{1+\alpha}\big)\big\} + \frac{2}{1+\alpha}\tilde{c}'(u)(R-S)R^{1+\alpha} \\ &\quad = \tilde{c}'(u)\big(2R^\alpha Q_\varepsilon(R) - R^\alpha S^2\big). \end{aligned} \tag{4.2.18}$$

We note that

$$\frac{2}{1+\alpha}R^{2+\alpha} - 2R^\alpha Q_\varepsilon(R) = \frac{1-\alpha}{1+\alpha}R^{2+\alpha} + R^\alpha\left(R - \frac{1}{\varepsilon}\right)^2\bigg|_{R\geqslant 1/\varepsilon}. \tag{4.2.19}$$

By summing up (4.2.18) and (4.2.19), we obtain

$$\begin{aligned} &\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)R^{1+\alpha} + \tilde{c}'(u)\big(R^\alpha S^2 - SR^{1+\alpha}\big) \\ &\quad = -\frac{1}{1+\alpha}\,\partial_t R^{1+\alpha} + \frac{1}{1+\alpha}\,\partial_x\big(c(u)R^{1+\alpha}\big) \\ &\qquad - \tilde{c}'(u)R^\alpha\left(R - \frac{1}{\varepsilon}\right)^2 \mathbb{1}_{R\geqslant 1/\varepsilon}. \end{aligned} \tag{4.2.20}$$

Similar to the proof of (4.2.20), we can obtain for S the equation

$$\begin{aligned} &\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(S-R)S^{1+\alpha} + \tilde{c}'(u)\big(S^\alpha R^2 - RS^{1+\alpha}\big) \\ &\quad = -\frac{1}{1+\alpha}\,\partial_t S^{1+\alpha} - \frac{1}{1+\alpha}\,\partial_x\big(c(u)S^{1+\alpha}\big) \\ &\qquad - \tilde{c}'(u)S^\alpha\left(S - \frac{1}{\varepsilon}\right)^2 \mathbb{1}_{S\geqslant 1/\varepsilon}. \end{aligned} \tag{4.2.21}$$

For the specific choice of α, we have

$$\begin{aligned} &R^\alpha S^2 - SR^{1+\alpha} + S^\alpha R^2 - RS^{1+\alpha} \\ &\quad = R^\alpha S^\alpha(R-S)\big(R^{1-\alpha} - S^{1-\alpha}\big) \geqslant 0. \end{aligned} \tag{4.2.22}$$

Summing up (4.2.20)–(4.2.22), we obtain

$$\begin{aligned}
&\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)\big(R^{1+\alpha}-S^{1+\alpha}\big)\\
&\qquad+\tilde{c}'(u)R^{\alpha}S^{\alpha}(R-S)\big(R^{1-\alpha}-S^{1-\alpha}\big)\\
&\quad\leqslant \frac{1}{1+\alpha}\big\{-\partial_t\big(R^{1+\alpha}+S^{1+\alpha}\big)+\partial_x\big(c(u)\big(R^{1+\alpha}-S^{1+\alpha}\big)\big)\big\}.
\end{aligned}\tag{4.2.23}$$

Note that our solutions of (4.2.5) have finite speed of propagation due to condition (4.1.1). We can cut off our initial data (R_0, S_0) to make them compactly supported without changing the solutions in the domain $[0, T] \times [a, b]$. So we assume that $\operatorname{supp} R_0$, $\operatorname{supp} S_0 \subset [a, b]$. By (4.1.1), (4.2.8) and (4.2.9), we have that $\operatorname{supp} R(t, \cdot)$ and $\operatorname{supp} S(t, \cdot)$ are contained in $[a - C_2T, b + C_2T]$ for $t \leqslant T$. Thus, integrating (4.2.23) over $[0, T] \times \mathbb{R}$, we find

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\bigg\{\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)\big(R^{1+\alpha}-S^{1+\alpha}\big)\\
&\qquad\qquad+\tilde{c}'(u)R^{\alpha}S^{\alpha}(R-S)\big(R^{1-\alpha}-S^{1-\alpha}\big)\bigg\}\,\mathrm{d}x\,\mathrm{d}s\leqslant C_{\alpha,T,a,b}.
\end{aligned}\tag{4.2.24}$$

By (4.2.24), we immediately obtain

$$\frac{1-\alpha}{1+\alpha}\int_0^T\int_a^b \tilde{c}'(u)(R-S)^2\big(R^{\alpha}+S^{\alpha}\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant C_{\alpha,T,a,b}. \tag{4.2.25}$$

This implies (4.2.17), which completes the proof of Lemma 4.2.2. □

We note that the constant $C_{\alpha,T,a,b}$ in Lemma 4.2.2 tends to infinity as $\alpha \to 1$. The vanishing viscosity approximation to (0.0.1), as used in the last chapter, does not yield estimates (4.2.14) or (4.2.9), which are crucial to our later steps.

4.3. *Precompactness*

Let $(R_0, S_0) \in L^\infty \cap L^2(\mathbb{R})$. Let $j_\varepsilon(x)$ be standard Friedrichs' mollifier. We denote $R_0^\varepsilon = R_0 * j_\varepsilon$ and $S_0^\varepsilon = S_0 * j_\varepsilon$. Then by Lemma 4.2.1, problem (4.2.5) has a global smooth solution $(R^\varepsilon, S^\varepsilon, u^\varepsilon)$ with the initial data $(R_0^\varepsilon, S_0^\varepsilon)$. Moreover, by summing up (4.2.6), (4.2.7) and (4.2.10), we have

$$\int\big(\big(R^\varepsilon\big)^2+\big(S^\varepsilon\big)^2\big)(t,x)\,\mathrm{d}x\leqslant\int\big(R_0^2+S_0^2\big)(x)\,\mathrm{d}x \tag{4.3.26}$$

and

$$R^\varepsilon(t,x)\geqslant -M,\qquad S^\varepsilon(t,x)\geqslant -M,\quad t\geqslant 0. \tag{4.3.27}$$

We shall also use energy estimate (4.2.9) in this new setting.

We establish the precompactness of $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)(t,x)\}$ in this subsection.

Firstly, by copying the proof of Lemma 3.2.4, we can prove up to a subsequence of ε_j that

$$u^{\varepsilon_j}(t,x) \text{ converges uniformly to a continuous function } u(t,x) \tag{4.3.28}$$

on every compact subset of $[0,\infty)\times\mathbb{R}$.

We remark that by Proposition 3.13 of [39] and Lemma 3.5.1, 3.5.2, we have

$$\begin{aligned} &\xi \in L^\infty\big(\mathbb{R}^+, L^2\big(\mathbb{R}^+\times\mathbb{R}, \mathrm{d}x\otimes \mathrm{d}\nu^1_{tx}(\xi)\big)\big),\\ &\eta \in L^\infty\big(\mathbb{R}^+, L^2\big(\mathbb{R}^+\times\mathbb{R}, \mathrm{d}x\otimes \mathrm{d}\nu^2_{tx}(\eta)\big)\big), \end{aligned} \tag{4.3.29}$$

and

$$R^\varepsilon S^\varepsilon \rightharpoonup RS \quad \text{as } \varepsilon\to 0 \tag{4.3.30}$$

in the sense of distributions. Glassey, Hunter and Zheng [30] have derived (4.3.30) by applying the div–curl lemma for a sequence of energy conservative weak solutions $\{u^\varepsilon(t,x)\}$ of (0.0.1), assuming that $\{u^\varepsilon(t,x)\}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^+\times\mathbb{R})$ for some $p>2$. Here we have provided the uniform estimate.

In the sequel, we use the notation

$$\overline{g(R,S)} = \int_{\mathbb{R}} g(\xi,\eta)\,\mathrm{d}\mu_{tx}(\xi,\eta).$$

Thus, $(\overline{R},\overline{S})$ represents the weak-star limit of $\{R^\varepsilon, S^\varepsilon\}$ in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ or the weak limit in $L^2((0,T)\times\mathbb{R})$ for all $T<\infty$.

With the above preparation, we can now prove the precompactness of $\{R^\varepsilon, S^\varepsilon\}$.

LEMMA 4.3.1 (Precompactness of $\{(R^\varepsilon, S^\varepsilon)\}$). *Assume $c'\geqslant 0$ and $(R_0(x), S_0(x))\in L^\infty\cap L^2(\mathbb{R})$. Then $\nu^1_{tx}(\xi)=\delta_{\overline{R}(t,x)}(\xi)$ and $\nu^2_{tx}(\eta)=\delta_{\overline{S}(t,x)}(\eta)$.*

PROOF. The idea is to derive an evolution equation (inequality) for the quantity $\overline{R^2}-\overline{R}^2$, so that it is zero for all positive time if it is zero at time zero which is true in our case. In the derivation of the evolution equation we need to cut off desired multipliers and mollify various equations that are true only in the weak sense.

Since the proof of $\nu^1_{tx}(\xi)=\delta_{\overline{R}(t,x)}(\xi)$ is the same as that of $\nu^2_{tx}(\eta)=\delta_{\overline{S}(t,x)}(\eta)$, we present only the proof for the former.

Step 1. *Derivation of the equation for $\overline{R}$.* We write the first equation of (4.2.5) in the form

$$\partial_t R^\varepsilon - \partial_x\big(c\big(u^\varepsilon\big)R^\varepsilon\big) = -\tilde{c}'\big(u^\varepsilon\big)\big\{\big(R^\varepsilon - S^\varepsilon\big)^2 + \big[\big(R^\varepsilon\big)^2 - 2Q_\varepsilon\big(R^\varepsilon\big)\big]\big\}. \tag{4.3.31}$$

We claim that there holds in the sense of distributions

$$\tilde{c}'(u^\varepsilon)(R^\varepsilon - S^\varepsilon)^2 \rightharpoonup \tilde{c}'(u) \iint_{\mathbb{R}^2} (\xi - \eta)^2 \, \mathrm{d}\mu_{tx}(\xi, \eta) \\ = \tilde{c}'(u)\big(\overline{R^2} + \overline{S^2} - 2\overline{R}\,\overline{S}\big). \tag{4.3.32}$$

The claim will follow from Lemmas 3.5.1 and 3.5.2 and estimate (4.2.17). We take a smooth cut-off function $\psi(\xi)$ with $\psi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\operatorname{supp}\psi \subset \{\xi \mid |\xi| \leqslant 2\}$. Then, by (4.2.17), we have

$$\left| \int_0^T \int_{\mathbb{R}} R^\varepsilon (R^\varepsilon - S^\varepsilon) \left(1 - \psi\left(\frac{R^\varepsilon}{k}\right)\right) \mathrm{d}x \, \mathrm{d}t \right| \\ \leqslant C \operatorname{meas}\big\{(t, x) \bigm| 0 \leqslant t \leqslant T,\, R^\varepsilon \geqslant k\big\}^{1/p} \leqslant C k^{-2/p}, \tag{4.3.33}$$

where $1/p = \alpha/(2(2+\alpha))$ for any $0 < \alpha < 1$. While by Lemma 3.5.1, for any test function $\varphi(t, x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, there holds

$$\lim_{\varepsilon \to 0} \int_0^\infty \int_{\mathbb{R}} \varphi \tilde{c}'(u^\varepsilon) R^\varepsilon (R^\varepsilon - S^\varepsilon) \psi\left(\frac{R^\varepsilon}{k}\right) \mathrm{d}x \, \mathrm{d}t \\ = \iint_{\mathbb{R}^+ \times \mathbb{R}} \varphi \tilde{c}'(u) \iint_{\mathbb{R}^2} \xi(\xi - \eta) \psi\left(\frac{\xi}{k}\right) \mathrm{d}\mu_{t,x}(\xi, \eta) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.34}$$

By summing up (4.3.29), (4.3.33), (4.3.34) and Lebesgue dominated convergence theorem as $k \to \infty$ we find

$$\lim_{\varepsilon \to 0} \int_0^\infty \int_{\mathbb{R}} \varphi \tilde{c}'(u^\varepsilon) R^\varepsilon (R^\varepsilon - S^\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\ = \iint_{\mathbb{R}^+ \times \mathbb{R}} \varphi \tilde{c}'(u) \iint_{\mathbb{R}^2} \xi(\xi - \eta) \, \mathrm{d}\mu_{t,x}(\xi, \eta) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.35}$$

A similar proof yields that

$$\lim_{\varepsilon \to 0} \int_0^\infty \int_{\mathbb{R}} \varphi \tilde{c}'(u^\varepsilon) S^\varepsilon (R^\varepsilon - S^\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\ = \iint_{\mathbb{R}^+ \times \mathbb{R}} \varphi \tilde{c}'(u) \iint_{\mathbb{R}^2} \eta(\xi - \eta) \, \mathrm{d}\mu_{t,x}(\xi, \eta) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.36}$$

Claim (4.3.32) follows from (4.3.35) and (4.3.36).

Now, from (4.2.7), we have easily that

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}} \tilde{c}'(u^\varepsilon) \big[(R^\varepsilon)^2 - 2Q_\varepsilon(R^\varepsilon)\big] \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{4.3.37}$$

Thus, again by (4.3.28), Lemma 3.5.1, (4.3.32) and (4.3.37), we have

$$\partial_t \overline{R} - \partial_x\big(c(u)\overline{R}\big) = -\tilde{c}'(u)\big(\overline{R^2} - 2\overline{R}\,\overline{S} + \overline{S^2}\big). \tag{4.3.38}$$

Step 2. *Cut-off of* $(R^\varepsilon)^2$. Similar to [72], let us define for $\lambda > 0$

$$T_\lambda^+(\xi) = \begin{cases} \xi, & \xi \leqslant \lambda, \\ \lambda, & \xi \geqslant \lambda, \end{cases} \qquad S_\lambda^+(\xi) = \begin{cases} \frac{1}{2}\xi^2, & \xi \leqslant \lambda, \\ \lambda\big(\xi - \frac{\lambda}{2}\big), & \xi \geqslant \lambda. \end{cases} \tag{4.3.39}$$

We multiply the first equation of (4.2.5) with $T_\lambda^+(R^\varepsilon)$ to obtain

$$\begin{aligned} &\partial_t S_\lambda^+\big(R^\varepsilon\big) - \partial_x\big(c\big(u^\varepsilon\big)S_\lambda^+\big(R^\varepsilon\big)\big) \\ &\quad = -2\tilde{c}'\big(u^\varepsilon\big)\big(R^\varepsilon - S^\varepsilon\big)S_\lambda^+\big(R^\varepsilon\big) \\ &\qquad + \tilde{c}'\big(u^\varepsilon\big)T_\lambda^+\big(R^\varepsilon\big)\big(2Q_\varepsilon\big(R^\varepsilon\big) - \big(S^\varepsilon\big)^2\big). \end{aligned} \tag{4.3.40}$$

By Lemma 3.5.1, a similar proof of (4.3.32) and (4.3.37), we find that

$$\begin{aligned} &\tilde{c}'\big(u^\varepsilon\big)\big(R^\varepsilon - S^\varepsilon\big)S_\lambda^+\big(R^\varepsilon\big) \rightharpoonup \tilde{c}'(u)\overline{(R-S)S_\lambda^+(R)}, \\ &\tilde{c}'\big(u^\varepsilon\big)T_\lambda^+\big(R^\varepsilon\big)\big(2Q_\varepsilon\big(R^\varepsilon\big) - \big(S^\varepsilon\big)^2\big) \rightharpoonup \tilde{c}'(u)\overline{T_\lambda^+(R)\big(R^2 - S^2\big)}. \end{aligned}$$

Taking $\varepsilon \to 0$ in (4.3.40), we obtain

$$\begin{aligned} &\partial_t \overline{S_\lambda^+(R)} - \partial_x\big(c(u)\overline{S_\lambda^+(R)}\big) \\ &\quad = \tilde{c}'(u)\big\{-2\overline{RS_\lambda^+(R)} + \overline{T_\lambda^+(R)R^2} + 2\overline{S}\,\overline{S_\lambda^+(R)} - \overline{T_\lambda^+(R)}\,\overline{S^2}\big\}. \end{aligned} \tag{4.3.41}$$

Step 3. *Cut-off of* $\overline{R}^2$. Convolving (4.3.38) with the standard Friedrichs' mollifier j_ε, we find

$$\partial_t \overline{R}^\varepsilon - \partial_x\big(c(u)\overline{R}^\varepsilon\big) = -\big(\tilde{c}'(u)\overline{(R-S)^2}\big) * j\varepsilon + \gamma_\varepsilon, \tag{4.3.42}$$

where $\overline{R}^\varepsilon = \int_{\mathbb{R}} \overline{R}(t,y)j_\varepsilon(x-y)\,\mathrm{d}y$ and $\gamma_\varepsilon = j_\varepsilon * \partial_x(c(u)\overline{R}) - \partial_x(c(u)\overline{R}_\varepsilon)$. By DiPerna–Lions folklore Lemma 2.3 of Lions [48] and Lebesgue dominated convergence theorem in the time direction, we have $\gamma_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ (or see Lemma 2.1 of [19]). Again, we multiply (4.3.42) with $T_\lambda^+(\overline{R}^\varepsilon)$ to find

$$\begin{aligned} &\partial_t S_\lambda^+\big(\overline{R}^\varepsilon\big) - \partial_x\big(c(u)S_\lambda^+\big(\overline{R}^\varepsilon\big)\big) \\ &\quad = \big[-\big(\tilde{c}'(u)\overline{(R-S)^2}\big) * j_\varepsilon + \gamma_\varepsilon + 2\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)\overline{R}^\varepsilon\big]T_\lambda^+\big(\overline{R}^\varepsilon\big) \\ &\qquad - 2\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)S_\lambda^+\big(\overline{R}^\varepsilon\big). \end{aligned} \tag{4.3.43}$$

Taking $\varepsilon \to 0$ in (4.3.43), we find

$$\begin{aligned}
&\partial_t S_\lambda^+(\overline{R}) - \partial_x\big(c(u)S_\lambda^+(\overline{R})\big)\\
&\quad = \big[-\tilde{c}'(u)\overline{(R-S)^2} + 2\tilde{c}'(u)\big(\overline{R}-\overline{S}\big)\overline{R}\big]T_\lambda^+(\overline{R}) - 2\tilde{c}'(u)\big(\overline{R}-\overline{S}\big)S_\lambda^+(\overline{R})
\end{aligned}$$

or

$$\begin{aligned}
&\partial_t S_\lambda^+(\overline{R}) - \partial_x\big(c(u)S_\lambda^+(\overline{R})\big)\\
&\quad = \tilde{c}'(u)\big\{-2\overline{R}S_\lambda^+(\overline{R}) + T_\lambda^+(\overline{R})\overline{R}^2 + 2\overline{S}S_\lambda^+(\overline{R})\\
&\qquad\qquad - T_\lambda^+(\overline{R})\overline{S^2} - T_\lambda^+(\overline{R})\big(\overline{R^2}-\overline{R}^2\big)\big\}. \qquad (4.3.44)
\end{aligned}$$

Step 4. *Evolution equation for "$\overline{R^2} - \overline{R}^2$".* By substracting (4.3.44) from (4.3.41), we find

$$\begin{aligned}
&\partial_t\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big) - \partial_x\big(c(u)\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big)\big)\\
&\quad = \tilde{c}'(u)\big\{-2\overline{RS_\lambda^+(R)} + \overline{T_\lambda^+(R)R^2} + 2\overline{R}S_\lambda^+(\overline{R}) - T_\lambda^+(\overline{R})\overline{R}^2\\
&\qquad\qquad + 2\overline{S}\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big) - \big(\overline{T_\lambda^+(R)} - T_\lambda^+(\overline{R})\big)\overline{S^2}\\
&\qquad\qquad + T_\lambda^+(\overline{R})\big(\overline{R^2}-\overline{R}^2\big)\big\}. \qquad (4.3.45)
\end{aligned}$$

But, by the explicit structures of $S_\lambda^+(\cdot)$ and $T_\lambda^+(\cdot)$, we find

$$\begin{aligned}
&-2\xi S_\lambda^+(\xi) + T_\lambda^+(\xi)\xi^2 = -\lambda(\xi-\lambda)^2\mathbb{1}_{\xi\geqslant\lambda} - \lambda^2\big(\xi - T_\lambda^+(\xi)\big),\\
&2\overline{R}S_\lambda^+(\overline{R}) - T_\lambda^+(\overline{R})\overline{R}^2 = \lambda\big(\overline{R}-\lambda\big)^2\mathbb{1}_{\overline{R}\geqslant\lambda} + \lambda^2\big(\overline{R} - T_\lambda^+(\overline{R})\big).
\end{aligned} \qquad (4.3.46)$$

Since

$$\xi^2 = 2S_\lambda^+(\xi) + (\xi-\lambda)^2\mathbb{1}_{\xi\geqslant\lambda}, \qquad (4.3.47)$$

we have

$$\begin{aligned}
&\overline{R^2} - \big(\overline{R}\big)^2\\
&\quad = 2\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big) + \overline{(R-\lambda)^2\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)^2\mathbb{1}_{\overline{R}\geqslant\lambda}. \qquad (4.3.48)
\end{aligned}$$

Summing up (4.3.45)–(4.3.48), we find that

$$\begin{aligned}
&\partial_t\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big) - \partial_x\big(c(u)\big(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})\big)\big)\\
&\quad = \tilde{c}'(u)\big\{-\big(\lambda - T_\lambda^+(\overline{R})\big)\big[\overline{(R-\lambda)^2\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)^2\mathbb{1}_{\overline{R}\geqslant\lambda}\big]
\end{aligned}$$

$$
\begin{aligned}
&- \lambda^2\big[T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\,\big] - \big(\overline{T_\lambda^+(R)} - T_\lambda^+\big(\overline{R}\big)\big)\overline{S^2}\\
&+ 2\big(\overline{S} + T_\lambda^+\big(\overline{R}\big)]\big)\big(\overline{S_\lambda^+(R)} - S_\lambda^+\big(\overline{R}\big)\big)\big\}
\end{aligned}
$$

$$
\leqslant \tilde{c}'(u)\big\{\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\,\big)\overline{S^2} + 2\big(\overline{S} + T_\lambda^+\big(\overline{R}\big)\big)\big(\overline{S_\lambda^+(R)} - S_\lambda^+\big(\overline{R}\big)\big)\big\}, \tag{4.3.49}
$$

since $T_\lambda^+(\xi)$ is concave and $(\xi-\lambda)^2\mathbb{1}_{\xi\geqslant\lambda}$ is convex in ξ. We note in passing that we could save $-\lambda^2$ to reduce the term $\overline{S^2}$ by $-\lambda^2$, as we have done in paper [74]. But that is not enough when S is unbounded, so we choose the new path – renormalization.

Step 5. *Renormalization*. We let $f_\lambda(t,x) := \overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})$. Rewriting (4.3.49), we find

$$
\begin{aligned}
&\partial_t f_\lambda - \partial_x\big(c(u) f_\lambda\big)\\
&\quad\leqslant \tilde{c}'(u)\big\{2\big(\overline{S} + T_\lambda^+\big(\overline{R}\big)\big) f_\lambda + \overline{S^2}\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\,\big)\big\}.
\end{aligned} \tag{4.3.50}
$$

Next, we claim that

$$
\frac{1}{2}\big(\overline{T_\lambda^+(R)} - T_\lambda^+\big(\overline{R}\big)\big)^2 \leqslant \overline{S_\lambda^+(R)} - S_\lambda^+\big(\overline{R}\big). \tag{4.3.51}
$$

In fact, by Cauchy inequality, we have

$$
\big(\overline{T_\lambda^+(R)}\big)^2 = \left(\int T_\lambda^+(\xi)\,\mathrm{d}\nu_{tx}^1(\xi)\right)^2 \leqslant \overline{\big(T_\lambda^+(R)\big)^2}. \tag{4.3.52}
$$

Using the identities

$$
\xi = T_\lambda^+(\xi) + (\xi-\lambda)\mathbb{1}_{\xi\geqslant\lambda}, \qquad \overline{R} = T_\lambda^+\big(\overline{R}\big) + \big(\overline{R}-\lambda\big)\mathbb{1}_{\overline{R}\geqslant\lambda} \tag{4.3.53}
$$

we have

$$
\begin{aligned}
T_\lambda^+\big(\overline{R}\big)\overline{T_\lambda^+(R)} &= T_\lambda^+\big(\overline{R}\big)\overline{R} - T_\lambda^+\big(\overline{R}\big)\overline{(R-\lambda)\mathbb{1}_{R\geqslant\lambda}}\\
&= \big(T_\lambda^+\big(\overline{R}\big)\big)^2 - T_\lambda^+\big(\overline{R}\big)\big(\overline{(R-\lambda)\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)\mathbb{1}_{\overline{R}\geqslant\lambda}\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\big(\overline{T_\lambda^+(R)} - T_\lambda^+\big(\overline{R}\big)\big)^2\\
&\quad= \big(\overline{T_\lambda^+(R)}\big)^2 + \big(T_\lambda^+\big(\overline{R}\big)\big)^2 - 2T_\lambda^+\big(\overline{R}\big)\overline{T_\lambda^+(R)}\\
&\quad\leqslant \overline{\big(T_\lambda^+(R)\big)^2} - \big(T_\lambda^+\big(\overline{R}\big)\big)^2 + 2T_\lambda^+\big(\overline{R}\big)\big(\overline{(R-\lambda)\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)\mathbb{1}_{\overline{R}\geqslant\lambda}\big).
\end{aligned} \tag{4.3.54}
$$

Using

$$S_\lambda^+(\xi) - \frac{1}{2}\big(T_\lambda^+(\xi)\big)^2 = \lambda(\xi-\lambda)\mathbb{1}_{\xi\geqslant\lambda}$$

we then have

$$\begin{aligned}
&\overline{S_\lambda^+(R)} - S_\lambda^+\big(\overline{R}\big)\\
&\quad= \frac{1}{2}\Big(\overline{\big(T_\lambda^+(R)\big)^2} - \big(T_\lambda^+\big(\overline{R}\big)\big)^2\Big) + \lambda\Big(\overline{(R-\lambda)\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)\mathbb{1}_{\overline{R}\geqslant\lambda}\Big)\\
&\quad\geqslant \frac{1}{2}\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\big)^2 + \big(\lambda - T_\lambda^+\big(\overline{R}\big)\big)\Big(\overline{(R-\lambda)\mathbb{1}_{R\geqslant\lambda}} - \big(\overline{R}-\lambda\big)\mathbb{1}_{\overline{R}\geqslant\lambda}\Big)\\
&\quad\geqslant \frac{1}{2}\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\big)^2.
\end{aligned}$$

This result proves (4.3.51).

Notice that $f_\lambda(t,x) \in L^\infty(R^+, L^2(\mathbb{R}))$ for any fixed λ. Thus by DiPerna–Lions folklore Lemma 2.3 [48] and Lebesgue dominated convergence theorem in the time direction again, we have

$$\begin{aligned}
&\partial_t f_\lambda^\varepsilon - \partial_x\big(c(u) f_\lambda^\varepsilon\big)\\
&\quad\leqslant \tilde{c}'(u)\big\{2\big(\overline{S} + T_\lambda^+\big(\overline{R}\big)\big) f_\lambda^\varepsilon + \overline{S^2}\big[T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\big]\big\} + \gamma_\varepsilon,
\end{aligned}\tag{4.3.55}$$

where $f_\lambda^\varepsilon(t,x) := \int_{\mathbb{R}} f_\lambda(t,y) j_\varepsilon(x-y)\,\mathrm{d}y$ and $\gamma_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$. For any $\eta > 0$, we multiply (4.3.55) with $\frac{1}{2}(f_\lambda^\varepsilon + \eta)^{-1/2}$ to yield

$$\begin{aligned}
&\partial_t\big(f_\lambda^\varepsilon + \eta\big)^{1/2} - \partial_x\big(c(u)\big(f_\lambda^\varepsilon + \eta\big)^{1/2}\big)\\
&\quad\leqslant \tilde{c}'(u)\big(\overline{R} + T_\lambda^+\big(\overline{R}\big)\big) f_\lambda^\varepsilon\big(f_\lambda^\varepsilon + \eta\big)^{-1/2} - 2\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)\big(f_\lambda^\varepsilon + \eta\big)^{1/2}\\
&\qquad+ \frac{1}{2}\tilde{c}'(u)\overline{S^2}\big(f_\lambda^\varepsilon + \eta\big)^{-1/2}\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\big)\\
&\qquad+ \frac{1}{2}\big(f_\lambda^\varepsilon + \eta\big)^{-1/2}\gamma_\varepsilon.
\end{aligned}\tag{4.3.56}$$

By taking $\varepsilon \to 0$ in (4.3.56), we find

$$\begin{aligned}
&\partial_t(f_\lambda + \eta)^{1/2} - \partial_x\big(c(u)(f_\lambda + \eta)^{1/2}\big)\\
&\quad\leqslant \tilde{c}'(u)\big(\overline{R} + T_\lambda^+\big(\overline{R}\big)\big) f_\lambda(f_\lambda + \eta)^{-1/2} - 2\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)(f_\lambda + \eta)^{1/2}\\
&\qquad+ \frac{1}{2}\tilde{c}'(u)\overline{S^2}(f_\lambda + \eta)^{-1/2}\big(T_\lambda^+\big(\overline{R}\big) - \overline{T_\lambda^+(R)}\big).
\end{aligned}\tag{4.3.57}$$

Moreover, by (4.3.51), we find that

$$\overline{S^2}(f_\lambda+\eta)^{-1/2}\big(T_\lambda^+\big(\overline{R}\big)-\overline{T_\lambda^+(R)}\big)\leqslant 2\overline{S^2}.$$

To establish almost everywhere convergence, we first have by Cauchy–Schwarz inequality that

$$\big|\overline{R}-\overline{T_\lambda(R)}\big|=\int(\xi-\lambda)\mathbb{1}_{\xi\geqslant\lambda}\,\mathrm{d}\nu_{t,x}^1(\xi)\leqslant\frac{1}{\lambda}\int\xi^2\,\mathrm{d}\nu_{t,x}^1(\xi).$$

This together with (4.3.29) yields that

$$\lim_{\lambda\to\infty}\big\|\overline{R}-\overline{T_\lambda^+(R)}\big\|_{L^1([0,T]\times\mathbb{R})}=0\quad\forall T<\infty.$$

Similarly, we can prove that $\lim_{\lambda\to\infty}\|\overline{R}-T_\lambda^+(\overline{R})\|_{L^1([0,T]\times\mathbb{R})}=0$. Then by triangle inequality, we obtain

$$\lim_{\lambda\to\infty}\big\|\overline{T_\lambda^+(R)}-T_\lambda^+\big(\overline{R}\big)\big\|_{L^1([0,T]\times\mathbb{R})}=0\quad\forall T<\infty.$$

Thus, by Lebesgue dominated convergence theorem, we find for any $T>0$ that

$$\lim_{\lambda\to\infty}\big\|\overline{S^2}(f_\lambda+\eta)^{-1/2}\big(T_\lambda^+\big(\overline{R}\big)-\overline{T_\lambda^+(R)}\big)\big\|_{L^1([0,T]\times\mathbb{R})}=0. \tag{4.3.58}$$

Trivially, by (4.3.29) and Lebesgue dominated convergence theorem, we have

$$\lim_{\lambda\to\infty}f_\lambda(t,x)=\frac{1}{2}\big(\overline{R^2}-\overline{R}^2\big)=:f(t,x). \tag{4.3.59}$$

Summing up (4.3.57)–(4.3.59), we obtain

$$\begin{aligned}&\partial_t(f+\eta)^{1/2}-\partial_x\big(c(u)(f+\eta)^{1/2}\big)\\&\quad\leqslant 2\tilde{c}'(u)\big(\overline{R}f(f+\eta)^{-1/2}-\big(\overline{R}-\overline{S}\big)(f+\eta)^{1/2}\big).\end{aligned} \tag{4.3.60}$$

We let $\sqrt{f(t,x)}=:g$. Then, by taking $\eta\to0$ in (4.3.60), we find

$$\partial_t g-\partial_x\big(c(u)g\big)\leqslant 2\tilde{c}'(u)\overline{S}g. \tag{4.3.61}$$

Step 6. *The proof of the precompactness* (*re-renormalization*). Notice that $g(t,x)\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))$. Thus again by DiPerna–Lions folklore Lemma 2.3, [48], we obtain

$$\partial_t g^\varepsilon-\partial_x\big(c(u)g^\varepsilon\big)\leqslant 2\tilde{c}'(u)\overline{S}g^\varepsilon+\gamma_\varepsilon, \tag{4.3.62}$$

where $0\leqslant g^\varepsilon(t,x):=\int g(t,y)j_\varepsilon(x-y)\,\mathrm{d}y$ and $\gamma_\varepsilon(t,x)\to0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$.

On the other hand, parallel to the proof of (4.3.38), we can prove that

$$\partial_t \overline{S} + \partial_x\big(c(u)\overline{S}\big) = -\tilde{c}'(u)\overline{(R-S)^2}. \tag{4.3.63}$$

Moreover, by the third equation of (4.2.5) and (4.3.28), there holds

$$2c(u)u_x = \overline{R} - \overline{S}. \tag{4.3.64}$$

Substracting (4.3.63) from (4.3.38) we obtain

$$\partial_t\big(\overline{R}-\overline{S}\big) - \partial_x\big(c(u)\big(\overline{R}+\overline{S}\big)\big) = 0.$$

Substituting (4.3.64) into the above equation we find

$$\partial_x\big(c(u)\big(2u_t - \big(\overline{R}+\overline{S}\big)\big)\big) = 0,$$

that is,

$$u_t = \frac{1}{2}\big(\overline{R}+\overline{S}\big). \tag{4.3.65}$$

Dividing (4.3.62) with $c(u)$ we obtain

$$\partial_t\left(\frac{g^\varepsilon}{c(u)}\right) - \partial_x\big(g^\varepsilon\big) \leqslant -2\tilde{c}'(u)\overline{S}\frac{g^\varepsilon}{c(u)} + \frac{\gamma_\varepsilon}{c(u)}. \tag{4.3.66}$$

Taking $\varepsilon \to 0$ in (4.3.66) we obtain

$$\partial_t\left(\frac{g}{c(u)}\right) - \partial_x g \leqslant -2\tilde{c}'(u)\overline{S}\frac{g}{c(u)}. \tag{4.3.67}$$

We next claim that

$$g(t,x) \in L^\infty_{\text{loc}}\big(\mathbb{R}^+, L^1(\mathbb{R})\big). \tag{4.3.68}$$

Firstly, by the definition of $g(t,x)$, we have $g(t,x) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$. Now, let us take $\varphi(x) \in C_c^\infty(\mathbb{R})$ with $\varphi(x) = 1$ for $|x| \leqslant 1$, $\operatorname{supp}\varphi \subset \{x \mid |x| \leqslant 2\}$. Then by multiplying (4.3.67) with $\varphi(x/n)$ and integrating it over $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{C_2}\int g(t,x)\varphi\left(\frac{x}{n}\right)\mathrm{d}x \\
&\quad\leqslant \int \frac{g(0,x)}{c(u)}\varphi\left(\frac{x}{n}\right)\mathrm{d}x + \frac{1}{n}\int_0^t\!\!\int g\left|\varphi'\left(\frac{x}{n}\right)\right|\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int 2\tilde{c}'(u)\overline{S}\frac{g}{c(u)}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant \frac{t}{\sqrt{n}}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|\partial_x\varphi\|_{L^2} + Ct\big\|\overline{S}\big\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}.
\end{aligned} \tag{4.3.69}$$

Thus, Fatou's lemma yields that

$$\int g(t,x)\,\mathrm{d}x \leqslant Ct\|\overline{S}\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}. \tag{4.3.70}$$

This proves claim (4.3.68).

On the other hand, it follows from (4.3.27) that there is a constant C such that $-\overline{S} \leqslant C$. Thus by (4.3.68), we can integrate (4.3.67) over $\mathbb{R}$ to obtain

$$\int_{\mathbb{R}} \frac{g}{c(u)}(t,x)\,\mathrm{d}x \leqslant C\int_0^t\int_{\mathbb{R}} \frac{g}{c(u)}(s,x)\,\mathrm{d}x\,\mathrm{d}s. \tag{4.3.71}$$

Applying Gronwall's inequality to (4.3.71) we have

$$g(t,x) = 0 \quad \text{a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.3.72}$$

Hence, $f(t,x) = 0$ a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and therefore $\nu^1_{t,x}(\xi) = \delta_{\overline{R(t,x)}}(\xi)$. Similarly, we can prove that $\nu^2_{t,x}(\eta) = \delta_{\overline{S(t,x)}}(\eta)$. This completes the proof of Lemma 4.3.1. □

LEMMA 4.3.2 (Flow regularity). *Let u be the limit of $\{u^{\varepsilon_j}\}$. Then the flows $\Phi^\pm_t(x)$,*

$$\begin{cases} \frac{\mathrm{d}\Phi^\pm_t(x)}{\mathrm{d}t} = \pm c\big(u\big(t,\Phi^\pm_t(x)\big)\big), \\ \Phi_t(x)|_{t=0} = x, \end{cases} \tag{4.3.73}$$

are Lipschitz continuous with respect to x. Moreover, if $S_0 \leqslant 0$, then for any $T > 0$ there exist two positive constants $M_1(T)$ and $M_2(T)$ such that

$$M_2(T) \leqslant \partial_x\Phi^+_t(x) \leqslant M_1(T), \quad 0 \leqslant t \leqslant T. \tag{4.3.74}$$

PROOF. For simplicity, we deal with only the positive characteristics of (4.3.73). Consider the approximate solutions u^ε and their flows $\Phi^\varepsilon_t(x)$

$$\frac{\mathrm{d}\Phi^\varepsilon_t(x)}{\mathrm{d}t} = c\big(u^\varepsilon\big(t,\Phi^\varepsilon_t(x)\big)\big), \quad \Phi^\varepsilon_t(x)\big|_{t=0} = x. \tag{4.3.75}$$

Taking ∂_x on both sides of the above equation, we find

$$\begin{aligned} \partial_x\Phi^\varepsilon_t(x) &= \exp\left[\int_0^t \big(c'\big(u^\varepsilon\big)\,\partial_y u^\varepsilon\big)\big(s,\Phi^\varepsilon_s(x)\big)\,\mathrm{d}s\right] \\ &= \exp\left[\int_0^t c'\big(u^\varepsilon\big)\left(\frac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)}\right)\big(s,\Phi^\varepsilon_s(x)\big)\,\mathrm{d}s\right]. \end{aligned} \tag{4.3.76}$$

Now, $-S^\varepsilon$ has a uniform upper bound and R^ε is uniformly bounded in L^2 along plus characteristics from Lemma 4.2.1, therefore the exponent in (4.3.76) is bounded from above. This shows that $\Phi^+_t(x)$ is Lipschitz continuous with respect to x.

Furthermore, if $S_0 \leqslant 0$, then S^{ε} is uniformly bounded, hence (4.3.74) follows directly from (4.3.76). This completes the proof of Lemma 4.3.2. □

Now, we prove Theorem 4.1.1.

PROOF OF THEOREM 4.1.1. Firstly, by (4.3.64) and (4.3.65), we find

$$\overline{R} = \partial_t u + c(u)\,\partial_x u, \qquad \overline{S} = \partial_t u - c(u)\,\partial_x u. \tag{4.3.77}$$

Secondly, by (4.3.38), (4.3.63) and Lemma 4.3.1, we find that

$$\begin{cases} \partial_t \overline{R} - \partial_x\big(c(u)\overline{R}\big) = -\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)^2, \\ \partial_t \overline{S} + \partial_x\big(c(u)\overline{S}\big) = -\tilde{c}'(u)\big(\overline{R} - \overline{S}\big)^2 \end{cases} \tag{4.3.78}$$

hold in the sense of distributions. Summing up the two equations of (4.3.78) and using (4.3.77), we find that there holds (3.1.3). Moreover, by (4.3.26) and (4.3.77), there holds (3.1.2). By Lemma 4.2.2 and (4.3.64), there holds $c'(u)|u_x|^{2+\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. Finally Lemma 4.3.2 shows that the last part of Theorem 4.1.1 holds. This completes the proof of Theorem 4.1.1. □

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